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ABSTRACT

This two volume collection of proceedings contains working group reports, research reports, oral reports, poster session reports and discussion group reports presented at PME-NA 21. Only the plenary and research reports are full reports; the others are brief abstracts. Full reports include: (1) "Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning" (Raymond Duval); (2) "On the Development of Human Representational Competence from an Evolutionary Point of View: From Episodic to Virtual Culture" (James J. Kaput); (3) "Representation and Evolution: A Discussion of Duval's and Kaput's Papers" (Patrick Thompson); (4) "The Role of Visual Representations in the Learning of Mathematics" (Abraham Arcavi); (5) "Concept and Representation in the Research on Probability Education" (Ana Marza Ojeda Salazar); (6) "On Representations and Situated Tools" (Luis Moreno Armella); (7) "Language Use in Embodied Action and Interaction in Knowing Fractions" (Thomas Kieren); (8) "The Use of Technology as a Means to Explore Mathematics Qualities in Proposed Problems" (Manuel Santos Trigo); (9) "Rethinking Representations" (Luis Radford); (10) "On Visualization and Generalization in Mathematics" (Norma C. Presmeg); (11) "School Algebra: Theory and Practice" (Eugenio Filloy, Teresa Rojano, James Kaput, Carolyn Kieran, Luis Puig, and Tenoch Cedillo); (12) "Social and Cultural Contexts in Mathematics Pedagogy (Yolanda de la Cruz); (13) "Determining Linearity: The Use of Visualization in Problem Solving" (Despina A. Stylianou and Ed Dubinsky); (14) "Mathematical Representation of Money Interest" (Youngyul Oh); (15) "The Notation Students Develop to Express a Rule" (Carol W. Bellisio and Alice S. Alston); (16) "Professionals Read Graphs (Imperfectly?)" (Wolff-Michael Roth); (17) "Spreadsheet and Composition of Functions" (J. Armando Landa H. and Sonia Ursini); (18) "Investigating Students' Developing Concept of Rate" (Susan Nickerson and Janet Bowers); (19) "A Conceptual Network for the Teaching-Learning Process of the Concept of Volume" (Mariana Saiz and Olympia Figueras); (20) "The Teacher's Role in Supporting Students' Development of

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Statistical Reasoning" (Kay McClain); (21) "Cross-National Comparisons of Representational Skill for Problem Solving" (Mary E. Brenner); (22) "Variations in Preference for Visualization among Mathematics Students and Teachers" (Norma C. Presmeg); (23) "The Little Teacher: A Student Role in Mathematics Small Groups" (Kathy M.C. Ivey and Sharon B. Walen); (24) "Narration as a Tool for Analyzing Beliefs on Calculus -- A Case Study" (Guenter Toerner); (25) "Writing about Mathematics in a Secondary Methods Course (Kate Masarik); (26) "A Web-Based Database of Problems and Practices and Real Communities of ESP Teachers and Students" (Eric Hsu). (AA)

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Proceedings of the
Twenty First Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 1

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Centro de Investigación y de Estudios Avanzados-IPN
Universidad Autónoma del Estado de Morelos
Cuernavaca, Morelos, México

October 23 – 26, 1999

Editors
Fernando Hitt
Manuel Santos

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Proceedings of the
Twenty First Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education
PME-NA XXI

Volume 1

Centro de Investigación y de Estudios Avanzados-IPN
Universidad Autónoma del Estado de Morelos
Cuernavaca, Morelos, México

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

Hosting the PME-NA meeting involves a great responsibility for the Program Committee. The Local Committee took the challenge to organize it because we thought that our mathematics education community would gain not only by the variety of research ideas presented during the conference, but also by the direct interaction with other researchers.

The theme of the conference is "The importance and the role of representation and mathematics visualization in the learning of mathematics"; this area has been part of the research agenda of many scholars around the world. Thus, the conference becomes a natural forum to reflect on the diversity of research that the mathematics education community has pursued recently. Noted Scholars were invited to address fundamental issues regarding this theme. Raymond Duval, from University of Littoral Côte-d'Opale, France, and James Kaput, from University of Massachusetts-Dartmouth, USA, have worked extensively in this area, and they present two visions that include theoretical foundations and current research directions. The corresponding titles of these presentations are "Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning" and "On the development of human representational competence from an evolutionary point of view: From episodic to virtual culture". Patrick Thompson, from Vanderbilt University, USA, reacts to these presentations.

Abraham Arcavi, from Weizmann Institute of Science, Israel, in his plenary, highlights diverse examples showing "The role of visual representation in the learning of mathematics". Sunday Ajose, from East Carolina University, USA, comments on the ideas presented by Arcavi.

A plenary panel addressing the topic of representation from a disciplinary perspective includes the participation of Ana María Ojeda and Luis Moreno, both members of the Mathematics Education Department, Cinvestav-IPN, Mexico. The corresponding titles of their presentations are: "Concept and representation in the research on probability education", and "Situated tools and situated proofs". Ed Dubinsky, from Georgia State University, USA, reacts to these presentations.

Thomas Kieren, from University of Alberta, Canada, addresses a plenary on "Language use in embodied action and interaction in knowing fractions" and Jere Confrey, from University of Texas at Austin, USA, comments on this presentation.

The working and discussion groups proved to be an efficient mechanism to promote discussions among participants at the conference held in North Caroline State University. To follow up the groups agenda, the

PMENA XXI program includes eight Working Groups and two Discussion Groups which will meet three times (4 hours), during the development of the conference. These groups are:

Working Groups

Geometry and Technology

Representations and Mathematics Visualization

School Algebra: Theory and Practice

The Role of Advanced Mathematical Thinking in Mathematics Education Reform

Gender and Mathematics: Exploring the Absences in Research

Rational Numbers > Representational Fluency > 21st Century Conceptual Tools. "Going Beyond Constructivism"

The Complexity of Learning to Reason Probabilistically

Using Socio-Cultural Theories in Mathematics Education Research

Discussion Groups

Social and Cultural Context in Mathematics Pedagogy

Performance Assessment of K-12. Preservice Teachers' Mathematical Content Knowledge

The *Proceedings* include the papers of the plenary speakers and the discussant reactions; 8 reports of the working groups and 2 of the discussion groups, 83 research reports, 37 short oral reports and 13 poster reports. There were 166 proposals submitted for review. For the first time some guidelines for evaluation were sent to all reviewers to assist them in their process of evaluation.

The editors wish to express thanks to all those who submitted proposals, the reviewers of proposals, the PME-NA XXI Steering Committee, and the PME-NA XXI Local Committee and especially we would like to acknowledge the support received by the University of Morelos and the Department of Mathematics Education of Cinvestav-IPN.

Fernando Hitt
Program Chair

Manuel Santos
Conference Coordinator

Mexico, July, 1999

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REPRESENTATION, VISION AND VISUALIZATION: COGNITIVE FUNCTIONS IN MATHEMATICAL THINKING. BASIC ISSUES FOR LEARNING

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Mathematics education has been very sensitive to needed changes over the past fifty years. Researches in developmental psychology, new technologies, andnew requirements in assessment have supported them. But their impact has been more effective on mathematics curriculum and on means of teaching than on the explanations of the deep processes of understanding and learning in mathematics. Difficulties of such research stem from the necessity of defining a framework within the epistemological constraints specific to mathematical activity and the cognitive functions of thought which it involves are not separated. That requires going beyond local studies of concept acquiring at each level of the curriculum and beyond mere reference to very general theories of learning and even beyond global description of student's activity in classroom.

Representation and visualization are at the core of understanding in mathematics. But in which framework can their role in mathematical thinking and in learning of mathematics be analyzed? Already in 1961, Piaget admitted the difficulty to understand what mathematicians call "intuition", a way of understanding which has close links with representation and visualization: "rien n'est plus difficile à comprendre pour un psychologue que ce que les mathematiciens entendent par intuition". He distinguished "many forms of mathematical intuition" (1961, pp. 223-241) from the empirical ones to the symbolizing ones. From a cognitive viewpoint, the question is not easier. Representation refers to a large range of meaning activities: steady and holistic beliefs about something, various ways to evoke and to denote objects, how information is coded. On the contrary, visualization seems to emphasize images and empirical intuition of physical objects and actions. Which ones are relevant to analyze the understanding in mathematics in order to bring out conditions of learning?

Our purpose in this panel is to focus on some main distinctions which are necessary to analyze the mathematical knowledge from a learning point of view and to explain how many students come up against difficulties at each level of curriculum and very often cannot go beyond. Studies about reasoning, proving, using geometrical figures in problem solving, reading of graphs... have made these distinctions necessary. They lead not only to emphasize semiotic representations as an intrinsic process of thinking but also to relativize some other ones as the distinction between internal and external representations. They lead also to point out the gap between vision and visualization. And from a learning point of view, visualization, the only relevant cognitive modality in mathematics, cannot be used as an immediate and obvious support for understanding. All these distinctions find accurate expression in different sets of cognitive variables. Within the compass of this panel we shall confine ourselves to sketching the complex cognitive architecture that any subject must develop because it underlies the use of representations and visualization in mathematics.

I. Three key ideas to define a framework to analyze the conditions of learning

1. The first one is the paradoxical character of mathematical knowledge

On the one hand, the use of systems of semiotic representation for mathematical thinking is essential because, unlike the other fields of knowledge (botany, geology, astronomy, physics), there is no other ways of gaining access to the **mathematical objects** but to produce **some semiotic representations**. In the other fields of knowledge, semiotic representations are images or descriptions about some phenomena of the real external world, to which we can gain a perceptual and instrumental access without these representations. In mathematics it is not the case.

On the other hand, the understanding of mathematics requires not confusing the mathematical objects with the used representations. This begins early with numbers, which have not to be identified with digits, numeral systems (roman, binary, decimal). And figures in geometry, even when they are constructed with accuracy, are just representations with particular values that are not relevant. And they cannot be taken as proofs.

2. The second one is the ambiguous meaning of the term "representation"

This term is often used to refer to mental entities: image, something away or missing that is evoked and, finally, what subjects understand. In this context, "mental" representation is considered as the opposite of signs which should be only "material" or "external" signs. Semiotic, and therefore external representations, would be at first necessary for the communication between the subjects. But this is a misleading division (Duval, 1995b, pp. 24-32) which brings about two very damaging confusions.

When it is applied to the representations, the distinction mental/external refers to their mode of production and not to their nature or to their form. In that sense, signs are neither mental nor physical or external entities. More specifically, there is not a term to term correspondence between the distinction mental/material and the distinction signified/signifiant, because the signifiant of any sign is not determined by its material realization but only by its opposite relations to the other signs: it is the number of possible choices what matters, as Saussure explained it. The binary system and decimal systems are very trivial examples of this semiotic determination of significance: the significance of any digit depends not only on its position but also on the number of possible choices per position. And, as for language, any use of a semiotic system can be mental or written (that is external). Thus, mental arithmetic uses the same decimal system like written calculation but not the same strategies because of the cognitive cost.

There are two kinds of cognitive representations. Those that are intentionally produced by using any semiotic system: sentences, graphs, diagrams, drawings... Their production can be either mental or external. And there are those which are causally and automatically produced either by an organic system (dream or memory visual images) or by a physical device (reflections, photographs). In one case, the content of the representations denotes the represented object: it is an explicit selection because each significant unit results from a choice. In the other case, the content of representations is the outcome of a physical action of the represented object on some organic system or on some physical device (Duval *et al.*, 1999, pp. 32-46). In other words, the basic division is not the one between mental representation and external representation, which is often used in cognitive sciences as though it was evident and primary, but the other one between semiotic representation and physical/organic representation. We cannot deal anyway with a representation without taking into account the system in which it is produced.

3. The third one is about the need of various semiotic systems for mathematical thinking

History shows that progress in mathematics has been linked to the development of several semiotic systems from the primitive duality of cognitive modes which are based on different sensory systems: language and image. For example, symbolic notations stemmed from written language have led to the algebraic writing and, since the nineteenth century, to the creation of formal languages. For imagery, there was the construction of plane figures with tools, then that in perspective, then the graphs in order to "translate" curves into equations. Each new semiotic system provided specific means of representation and processing for mathematical thinking. For that reason, we have called them "register of representation" (Duval, 1995b). Thus, we have several registers for discursive representation and several systems for visualization. That entails a complex cognitive interplay underlying any mathematical activity.

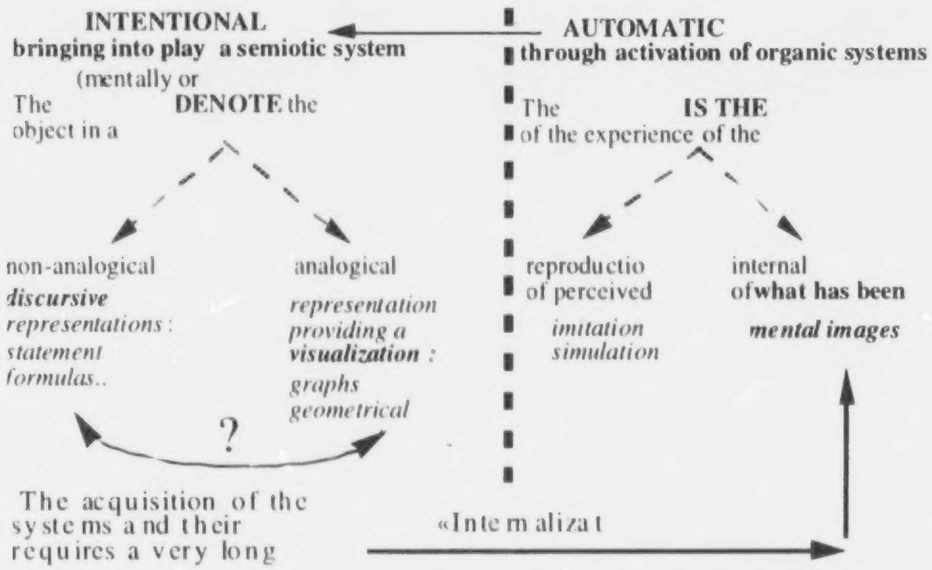
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Figure 1. Cognitive classification of conscious representations. This classification can be expanded more and includes all kinds of representations. We can notice the existence of two heterogeneous kinds of "mental images": the "quasi-percepts" which are an extension of perception (on the right) and the internalized semiotic visualizations (on the left). Actions like the physical ones (rotation, displacement, separation) can still be performed on some quasi-percepts and their time cost can be measured by reaction times to comparison tasks.

Firstly, as well as for discourse (description, explanation, reasoning, computation) as for visualization, we have two kinds of registers: the registers with a triadic structure of significance (natural language, 2D or 3D shapes representation) and registers with a dyadic structure of significance (symbolic notations, formal languages, diagrams) (Duval, 1995b, pp. 63-64). Within a dyadic structure any meaning is reduced to an

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explicitly defined denotation of objects. Within a triadic structure, we have meanings playing independently of any explicit denotation of objects and one must take into account their interplay. We can even fall into a cognitive conflict between the meaning game, which is proper to the register, and the denotation set for the representation. For example, the complexity of geometrical figures stems from their triadic structure of significance. Secondly, mathematical thinking often requires to activate in parallel two or three registers, even when only one is externally used, or seems sufficient, from a mathematical point of view.

This need of various registers of representation gives rise to several questions that are important in order to understand the real conditions of learning mathematics. First of all, there is a question about the specific way of working in each register: what operations are favored, or are only possible, within each register? This question is not trivial, because there are several registers for visualization and because they cannot be the same. Then, there are questions about the change from one register to another one. Are these changes very frequent or necessary? Are they always easy or evident to make? At last, is there a register more convenient or more intrinsically suitable for the mathematical thinking than others? It is obvious that registers with dyadic structure are technically more useful and more powerful than registers with triadic structure. But natural language remains essential for a cognitive control and for understanding within any mathematical activity. These questions may appear unimportant from a mathematical point of view. Even more, very often a mathematician cannot see why these questions arise. But from a didactical point of view, they are those questions that the difficulties of learning pose.

II. How the problems of mathematics learning come to light in this framework

1. No learning in mathematics can progress without understanding how the registers work

Cartesian graphs are very common examples because they look visually easy to grasp. But many observations have shown that most 15-17 year old students cannot discriminate the equations $y = x + 2$ and $y = 2x$ when looking at the two graphs presented in Figure 2. Notwithstanding this kind of failure, students succeed in the standard tasks such as constructing the graph from a given equation or reading the coordinates of a point! This kind of failure means that graphs cannot be useful

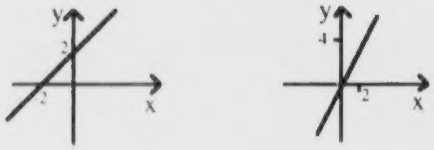


Figure 2. Visual discrimination of two elementary linear functions. This kind of discrimination presupposes that the qualitative values of two visual variables be distinguished: comparison between the angle with x-axis and the one obtained by bisection of xy angle, and position of crossing point with y-axis. Most often students confine themselves to the visual variable which is not relevant: how far some points are above x-axis (Duval, 1988).

representations neither to control intuitively some calculations nor to organize and to interpret data in other fields. And we have similar observations for each register of representation, even those which look more natural, like geometrical figures, or which are very utilized, like the decimal system in which the position of digits determines the operative meaning (French National Assessment, 1992, 1996).

All these repeated observations show that semiotic representations constitute an irreducible aspect of mathematical knowledge and that wanting to subordinate them to concepts leads to false issues in learning. That amounts to forget the paradox of mathematical knowledge: mathematics objects, even the more elementary objects in arithmetic and geometry, are not directly accessible like the physical objects. Each semiotic register of representation has a specific way of working, of which students must become aware.

2. We must distinguish two kinds of cognitive operations in mathematics thinking: "processing" and conversion

Mathematical processes are composed of two kinds of transformations of representations. There are transformations that are made within the same register of representation, like arithmetical or algebraic computation. The semiotic possibilities of generating a new representation from a given representation are exploited. With the dyadic structure, these possibilities depend both on the semiotic system and on mathematical rules. The geometrical figures give also rise to the intrinsic gestalt transformations of configurations apart from any previous consideration of mathematical properties. These gestalt transformations are like the visual transformations that anamorphoses or jigsaws lead to bring into play. We have called "processing" this kind of transformation.

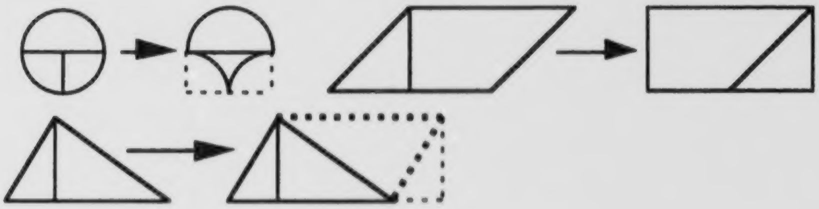


Figure 3. Visual change of configuration. The figurative units of any figure can be "reconfigured", mentally or materially, in another figure. For this kind of merely figurative transformation, neither hypotheses nor mathematical justification are required. Very often problem solving or explanations meant to convince students to resort to such transformations as if they were immediate and obvious for every student. Many observations show that this is not the case. There are factors that inhibit or trigger the «visibility» of such transformations. We can study them experimentally (Duval, 1995a).

And there are transformations that lay on a change of register: the representation of an object is "translated" into a different representation of the same object in another register. For example, when we go from a statement in native language to a literal expression. The transformation of equations into Cartesian graphs is another example. We have called "conversion" this kind of transformation.

One does not pay very close attention to the gap between these two kinds of cognitive operations that underlie mathematical processes. Nevertheless, if most students can learn some processing, very few of them can really convert representations. Much misunderstanding stems from this inability. But, very often, teachers attach more importance to the mathematical processes than to their application to daily life problems or to physical, or economic problems.

3. Conversion of representations is crucial problem in the learning of mathematics

Mathematical activity, in problem solving situations, requires the ability to change of register, either because another presentation of data, which fits better an already known model, is required, or because two registers must be brought together into play, like figures and natural language or symbolic notations in geometry. From a didactical point of view, only students who can perform register change do not confuse a mathematical object with its representation and they can transfer their mathematical knowledge to other contexts different from the one of learning. Two facts show the great complexity of conversion operation.

- Any conversion can be congruent or non-congruent. When a conversion is congruent the representation of the starting register is transparent to the representation of the target register. In other words, conversion can be seen like an easy translation unit to unit. Very accurate analyses of the congruent or non-congruent character of the conversion of a representation into another one can be systematically done. And they explain in a very accurate way many errors, failures, misunderstandings or mental blocks (Duval, 1995b, pp. 45-59; 1996, pp. 366-367).
- The congruence or the non-congruence of any conversion depends on its direction. A conversion can be congruent in one way and non-congruent in the opposite way. That leads to striking contrasts in the performances of students, such as those summarized in Figure 4.

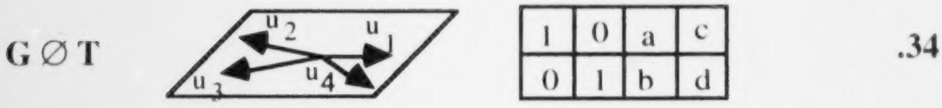


Figure 4. Elementary task of conversion (Pavlopoulou, 1993, p. 84)

Of course, the contrasts caused by the non congruence can be observed in a systematic way at all stages of the curriculum, from the more elementary verbal problems at primary school (Damm, 1992), to the university level.

It is surprising to see that this wide-ranging phenomenon is always ignored in the teaching of mathematics. Most teachers, mathematicians and even psychologists pay little attention to the difference of nature between processing and conversion. These two kinds of cognitive operations are grouped together in the unity of mathematical processes to solve a problem. And when a change of register must be introduced in the learning, one generally chooses one direction and the cases that are congruent. From a cognitive point of view, it is frequently a one-sided activity, which is proposed to students! There is something like an instinct to avoid the non-congruence situations that lead to real difficulties. But they are impossible to avoid especially when transfer of knowledge is required. Then failures and blocks are explained as conceptual misunderstanding, what is not a right explanation, since we have a contrast of successes and failures for the same mathematical objects in very similar situations. In reality the fact that students don't recognize anymore, when direction of conversion is changed, reveals a lack of co-ordination between the registers that have to bring into play together. The coordination of registers is not the consequence of understanding mathematics; on the contrary, it is an essential condition.

4. The learning of mathematics and the progressive coordination between registers

All these various and continual observations point out to a basic requirement that is specific for any progress in the learning of mathematics: the coordination between the registers of representation. This basic requirement is not fulfilled for most students, what is noticed in a global way often at the end of learning. For example, many teachers have, in one way or another, experienced what Schoenfeld (1986) described after a one yearlong study:

> [S]tudents may make virtually no connections between reference domains and symbols systems that we would expect them to think of as being nearly identical... the interplay occurs far more rarely than one would like (pp.239-242)

> [T]he students did not see any connection between the deductive mathematics of theorem proving and the inductive mathematics of doing constructions... they fail to see the connections or dismiss the proofs as being irrelevant (pp.243-244) If students fail to see such obvious connections, they are missing what lies at the core of mathematics (p.260)

Schoenfeld characterized this splitting rightly like an "inappropriate compartmentalization" (p. 226). But, unlike Schoenfeld's analysis, the kind of operative connections we expect to be made when learning is not between deductive and empirical mathematics, proofs and constructions, nor between mathematical structures and symbol structures, but between the different registers of semiotic representation. These connections between registers make up the cognitive architecture by which the students can recognize the same object through different representations and can make objective connections between deductive and empirical mathematics. Learning mathematics implies the construction of this cognitive architecture. It always begins with the coordination of a register providing visualization and a register performing one of the four discursive functions (Duval, 1995b, pp. 88-94).

III. Vision and visualization

From a psychological point of view, "vision" refers to visual perception and, by extension, to visual imagery. As perception, vision involves two essential cognitive functions.

- The first one consists in giving **direct access** to any physical object "in person". That is the reason why visual perception is always taken as a model for the epistemological notion of intuition. Nothing is more convincing than what is seen. In that sense, vision is the opposite of representation, even of the "mental images", because representation is something which stands instead of something else (Peirce). We shall call this function the ***epistemological function***.
- The second one is quite different. Vision consists of **apprehending simultaneously several objects** or a whole field. In other words, vision seems to give immediately a complete apprehension of any object or situation. In that sense, vision is the opposite of discourse, of deduction, which requires a sequence of focusing acts on a string of statements. We shall call it the ***synoptic function.***

In fact, visual perception performs in a very imperfect way the synoptic function. Firstly, because we are inside a three dimensional world: just one side of things can be seen, and complete apprehension requires movement, either of the one who is looking at it or of what is seen. In any case, this movement is a transformation of the perceived content: we have just a juxtaposition of successive sights which can be full-face, in profile, from above...Secondly, because visual perception always focalizes on a particular part of the field and can jump from one part to another one. There is no visual perception without such an exploration.

Now we can ask the following question that is decisive in the perspective of learning: are there cognitive structures that can perform both the epistemological and the synoptic function for the mathematical knowledge? The previous remarks lead us to answer this question negatively. More precisely, they lead to distinguish visualization from vision. Unlike vision, which provides a direct access to the object, visualization is based on the production of a semiotic representation. As Piaget, who has highlighted the synthetic inability of 3-5 year old children for the drawing of geometrical gestalts, explained it:

> **Le dessin est une représentation,** c'est-à-dire qu'**il suppose la construction d'une image bien distincte de la perception elle-même**, et rien ne prouve que les rapports spatiaux dont cette image est faite soient du même niveau que ceux dont témoigne la perception correspondante (1972, p.65).

We have here the breaking point between visual perception and visualization. A semiotic representation does not show things as they are in the 3D environment or as they can be physically projected on a small 2D material support. That is the matter of visual perception. **A semiotic representation shows relations or, better, organization of relations between representational units.** These representational units can be 1D or 2D shapes (geometrical figures), coordinates (Cartesian graphs), propositions (propositional deductive graphs or "proof graph"), or words (semantic networks). And these units must be **bi-dimensionally connected, because any organization requires at least two dimensions to become obvious**. In a string of discrete units (words, symbols, propositions) not any organization can be displayed. Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization. And that is why visualization should not be reduced to vision, that is to say: visualization makes visible all that is not accessible to vision. We can see now the gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary, visualization can get at once a complete apprehension of any organization of relations. We say "can get" and "cannot get" because visualization requires a long training, as we shall prove it below. However, what visualization apprehends can be the start of a series of transformations, that makes its inventive power. This difference between visual perception and visualization entails two consequences for the learning of mathematics.

Visualization refers to a cognitive activity that is intrinsically semiotic, that is, neither mental nor physical. Also such expressions as "mental image", "mental representation", ""mental imagery", are equivocal. They can only be the extension of visual perception. Accordingly, Neisser wrote:

> "[V]isual image" is a partly undefined term for something *seen* somewhat in the way real objects are seen when little or nothing in the immediate or very sensory input appears to justify it. Imagery ranges from the extremely vivid and externally localized images of the eidetiker to the relatively hazy and unlocalized images of visual memory.
>
> (Neisser, 1967; p.146)

Experiments on mental rotation of three-dimensional objects, since Shepard and Meztler (1971), are in the line of this conception of mental image as an extension of visual perception. But "mental imagery" can also be a mere visualization, that is, the mental production of semiotic representations as in mental calculation. Thus in "mind", we find the split into two kinds of representation back (*Figure*1). By resorting to mental images one does not avoid the difficulties arising from the paradoxical character of mathematics.

The way of watching is not the same in vision than in visualization. Two phenomena are confusing this issue. First, when they are graphically produced, semiotic representations are subject to visual perceptive apprehension. In that sense, visualization is always displayed within visual perception or within its mental extension. Second, some semiotic representations, like drawings, aim at being "iconic" representations: there is a relating likeness between the representation content and the represented object, so that one recognizes it (a tree, a car, a house) at once, without further information. Iconic representations refer to a previous perception of the represented object, from which to their concrete character. In mathematics, visualization does not work with such iconic representations: to look at them is not enough to see, that is, to notice and understand what is really represented.

The use of visualization requires a specific training, specific to visualize each register. Geometrical figures or Cartesian graphs are not directly available as iconic representations can be. And their learning cannot be reduced to training to construct them. This is due to the simple reason that construction makes attention to focus successively on some units and properties, whereas visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it. Most frequently, students go no further than to a local apprehension and do not see the relevant global organization but an iconic representation.

To sum up, visualization, which performs only the synoptic function, is not intuition but representation. In that sense, there are several possible geometrical registers for visualization. Visualization in mathematics is needed because it displays organization of relations, but it is not primitive, because it is not mere visual perception. In this respect, there is learning from the geometrical registers. Is there any vision that could perform the epistemological function? That is a philosophical question. From a cognitive view, the essential fact is the paradoxical character of the mathematical knowledge, which excludes any resort to mental representations as direct grasping of mathematical objects, at least in the didactical context.

IV. How visualization works toward understanding

We have characterized visualization as a bi-dimensional organization of relations between some kinds of units. Through visualization, any organization can be synoptically grasped as a configuration. In this way, we have as many kinds of visualization as kinds of units: geometrical configurations where units are 1D or 2D shapes or Gestalts, Cartesian graphs where units are couples {point, coordinates}, propositional graphs where units are statements... For the visualization of each register of visualization there are some rules or some intrinsic constraints to produce units and to form their relations. Thus, geometrical configurations can be constructed with tools and according to mathematical properties of the represented objects. One does not draw a pentagon as an oak-leaf or as a flower. There lies the point where visualization leads away from any iconic representation of a material object. In the perspective of learning, three problems have to be taken into account about visualization: the problem of discrimination, the problem of processing and the problem of coordination with a discursive register.

1. How can the relevant visual features be discriminated?

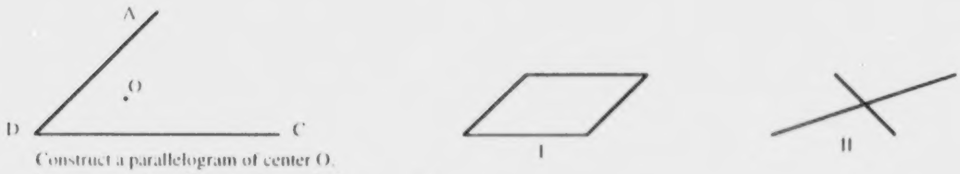
Unlike iconic representations, visualizations used in mathematics are not sufficient to know what are the denoted objects. Very early, young children learn quickly to recognize by themselves images of physical objects, perhaps because schematizations of frequently perceived outlines are automatically developed. But learning visualization in mathematics is not quite so easy and successful as it is for physical objects and real environment.

In front of simple Cartesian graphs, most students only have a local apprehension confined to the associations of points with coordinates. They do not get a global apprehension of all visual variables, which enables them to discriminate visually between the different graphs of functions such as

$$y = 2 - x, \quad y = 2x, \quad y = x + 2.$$

In other words, Cartesian graphs do not work visually for most students except for giving the naïve holistic information: the line goes up or down, like a mountain road. But that can be misleading when they have to compare the graphs of two series of observations. And Cartesian graphs can perform anyway a checking or a heuristic function in the tasks of formulae computation or interpretation. No connections can be made between the different graphs and the definitions, descriptions or explanations that are displayed in other registers.

Some simple 2D geometrical figures are taught at the primary level: triangle, circle, different quadrilateral polygons. But all these geometrical figures are equivocal representations. They can be hard iconic representations and they are nothing further than an herbarium of mathematical Gestalts. Or they can work as representations of geometrical objects and, in this case, they must appear as 2D organizations of 1D figural units. In other words, there are quite different apprehensions of the most elementary geometrical figures; the one which is according only to the spontaneous perceptive work, and the other which is "discursive" or anchored in some statements (definitions, theorems), (Duval, 1998, pp. 39-40). Thus, with the discursive apprehension, we can have several figures for the same geometrical object: for example, there are two typical figures to represent a parallelogram (I and II in Figure 5).



Construct a parallelogram of center O, having one side on DA, another on DC

Figure 5. Which of the two figures, I or II, can be useful to solve the problem? With the visual help of *Figure* I, one can only roughly make the drawing by successive attempts of measurements on DA and DC. With the visual help of *Figure* II, one easily succeeds by drawing the diagonal DOD'. Although they knew all the properties of parallelogram, most students failed as if they were confined themselves to visualization I (Dupuis, 1978, pp.79-81). In fact, I and II give a visual help only when one works with configurations of 1D figural units.

Such observations have been made many times for very simple problems (Schoenfeld, 1986, pp. 243-244). And these phenomena are all the stronger the geometrical figure appears as a joint of several Gestalts (triangles, parallelograms, circles, straight lines). For most students, there is like a heuristic deficiency of geometrical interpretation to visualization. But the equivocal character of geometrical figures appears also when a figure is directly taken for proof and leads to reject any resort to deductive reasoning. In that case, the figure works as a true iconic representation which makes discursive apprehension meaningless.

All these observations, which can be made anytime and anywhere in curricula, reveal the intrinsic difficulties of mathematical visualization. The intricacy of mathematical visualization does not consist in its visual units—they are fewer and more homogeneous than for the images—but in the implicit selection of which visual contrast values within the configurations of units are relevant and which are not. Here is the representation barrier specific to learn visualization in mathematics. Is it really taken into account in teaching?

Very often one believes that to learn how to construct graphs or geometrical figures is enough to learn visualization in mathematics. Moreover, in this kind of task students get satisfactory results. But any such a task of construction requires only a succession of local apprehensions: one needs to focus on units and not on the final configuration. In other words, a student can succeed in constructing a graph or a geometrical figure and being unable to look at the final configurations other than as iconic representations. That is easy to observe and to explain.

Constructing a graph requires only to compute some coordinates and to plot a straight line, a curve: one goes ever from data tables, or from equations, to graduated axis. But visualization requires the opposite change: one must go from the whole graph to some visual values that point to the characteristic features of the represented phenomenon or that correspond to a kind of equation and to some characteristic values within the equation. Therefore, visualization causes the anticipation of the kind of equation to find out. And this gap between local apprehension and global apprehension that can exist to the end of the construction is more important for geometrical figures than for graphs. The reason is that from a geometrical figure we have not one but many possible configurations or subconfigurations. And the relevant configurations or subconfigurations in the context of a problem are not always those highlighted at first glance. What we called above a heuristic deficiency is like an inability to go further from this first glance. What reason is it due to - teaching or some cognitive way of working?

2. Visualization and figural processing

In order to analyze any form of visualization there is a key idea: the existence of several registers of representation provides specific ways to process each register. Thus, if geometrical figures depend on a register, that is, on a system of representation, we must obtain specific visual operations that are peculiar to this register and that allow to change any initial geometrical figure into another one, while keeping the properties of the initial figure. What are these visual operations?

Three kinds of operations can be distinguished according to the way of modifying a given figure (Duval, 1988, pp. 61-63; 1995a, p.147):

(a) The mereologic way: you can divide the whole given figure into parts of various shapes (bands, rectangles) and you can combine these parts in another whole figure or you can make appear new subfigures. In this way, you change the shapes that appeared at the first glance: a parallelogram is changed into a rectangle, or a parallelogram can appear by combining triangles. We call "reconfiguration" the most typical operation.

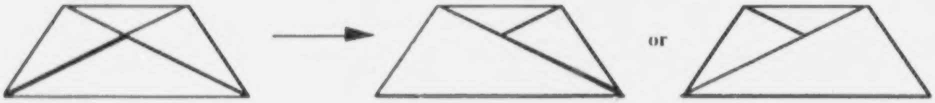


Figure 6. Figural processing by reconfiguration. Apprehension of this transformation within the only starting figure can be inhibited by the visual difficulty of double use of one sub-figure. But the starting figural frame is not changed like in the examples in Figure 3.

(b) The optic way: you can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently. Plane figures are seen as if they were located in a 3D space. The typical operation is to make two similar figures overlap in depth (Duval, 1995 b, p.187): the smaller one is seen as it was the bigger one at the distance (See Figure 7).

(c) The place way: you can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and horizontal lines.

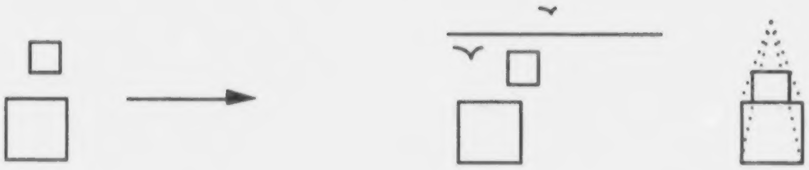
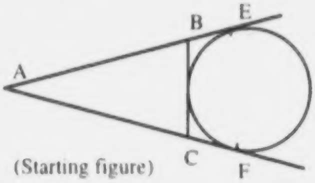


Figure 7. Figural processing by overlapping in depth of two similar figures

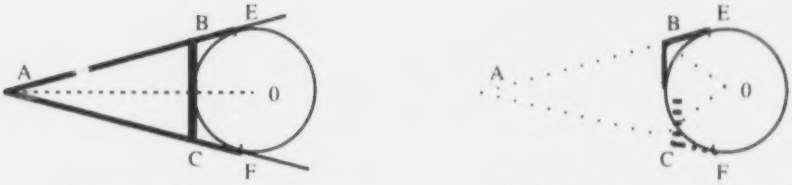
These various operations constitute a specific figural processing which provides figures with a heuristic function. One of these operations can give an insight to the solution of a problem. We call it the operative apprehension of a given figure. It is different both from perceptual and discursive operation.

Operative apprehension is different from perceptual apprehension because perception fixes at the first glance the vision of some shapes and this evidence makes them steady.



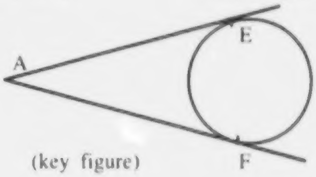
(Starting figure)

Comparison problem : is the perimeter of the triangle ABC greater, equal or smaller than the length of the two segments EA and AF ?



subconfiguration or shape organisation (I)

subconfiguration or shape organisation (II)



(key figure)

Figure 8. The perception of the starting figure highlights the shape organization (I) and makes it steady. But solving the problem requires the apprehension of the shape organization (II). Changing the perceptual apprehension of (I) into a perceptual apprehension

> of (II) constitutes a non-natural jump, because the symmetry axis AO sets forward the triangle within which the side BC is like an indivisible visual unit. Changing (I) into (II) requires looking at BC as a configuration of two segments! Moreover, the starting figure can be constructed without having to take into account the shape organization (II) with BO and BC as symmetry axis. Less than fifty per cent of 14-year-old students succeeded such a jump. And key figure does not help them for that (Mesquita, 1989, pp. 40, 68-69; Pluvinage, 1990, p. 27). However, by changing just a little the problem statement, and therefore the starting figure, all students can succeed: by naming I the point of intersection between AO and BC and by asking them to compare BI and IC, students are led to look at BC as a configuration of two segments. In that case, the statement of the problem becomes a congruent description of the subconfiguration (II), and geometrical visualization is reduced to an illustration function (Duval, 1999). But the learning problem is bypassed. A true didactical approach requires to embrace the whole range of variations of the conditions of a problem and to bring out the various factors that make them clear. It is only on the basis of students' knowledge that teachers can organize learning sequences.

In operative apprehension, the given figure becomes a starting point in order to investigate others configurations that can be obtained by one of these visual operations. In this respect, operative apprehension can develop several strings of figures from a given figure. According to the stated problem, one string shows an insight to the solution. The ability to think of drawing some units more on the given figure is one of the outward sign of operative apprehension. Now we can pose well the problem of heuristic deficiency: why perceptual apprehension does not ever lead to operative apprehension? For each operation, we were able to identify visual variables that trigger or inhibit the visibility of the relevant subfigure and operation within a given figure. And we were able to define the conditions of their influence on operative apprehension. Even the use of key figures in problem solving depends on these visual variables. Therefore, it would be naïve to believe that providing students with key figures would help them in problem solving. At the least change in the starting figure, most students do not recognize the correspondence with the key figure anymore. The visual variables must be taken into account in teaching. Their study opens an important field of research in order to understand the way cognition works for visualization in geometry (Duval, 1995a, pp.148-154; 1998, pp. 41-46).

Operative apprehension is independent of discursive apprehension. Vision does not start from hypotheses and does not follow from mathematical deduction. Otherwise, geometrical figures would not perform a heuristic function but only an illustrative function (Duval, 1999). That is the blind spot of many didactical studies. They do not differentiate between visualization and hypotheses, which depend on two heterogeneous registers of representation, and they subordinate the way of working of visualization to the way of working of deduction or of computation. In fact, shape recognition is independent of shape size and of perimeter magnitude. For example, when hypotheses include numbers as measures of sides or segments, operative apprehension is neutralized and the figure fulfills only an illustrative or support function. We can have even a conflict between the figure and the measures leading to a paradox. The most well known case is the reconfiguration of an 8 x 8 square into a 5 x 13 rectangle, within which a parallelogram is perceptively reduced to a diagonal.

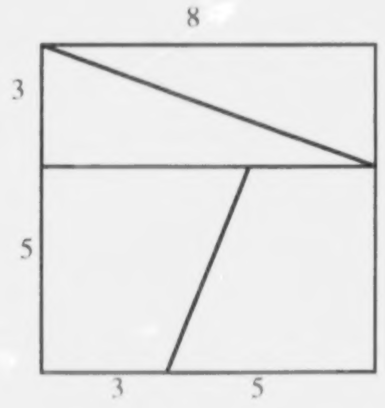


Figure 9.

Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization. Usually, the introduction of "geometrical figures" runs against this fact. Mathematical tasks are conceived as if the perceptual, discursive and operative apprehensions were inseparable! And the general outcome for most students is the inhibition of operative apprehension and a lack of interplay between perceptual and discursive apprehension.

3. Transitional visualization and development of the coronation of registers of representation

There is an introspective illusion that often distorts the analysis of mathematics learning processes. What is simple from a mathematical point of view appears also simple from a cognitive point of view when we are becoming experts. In fact, more often than not, what is taken as mathematically simple becomes cognitively evident only at the end of learning (Duval, 1998c). That is why assuming these simple-evident conditions cannot be taken as a starting point for learning and teaching. As I said above (II.4), learning mathematics implies the construction of this cognitive architecture that includes several registers of representation and their coordination. Thus geometrical figures used to solve problems involves some ability in operative apprehension and awareness of how deductive reasoning works. Students do not come into such apprehension and awareness by themselves. Moreover, some coordination is required between operative apprehension, discursive apprehension and deductive reasoning. In other words, geometrical activity requires continual shifts between visualization and discourse. In order to achieve such coordination another kind of visualization is required.

The introduction of graphs in proof learning is well known since their use with computer tutor (Anderson et al., 1987). This example is interesting because it shows the hidden cognitive complexity of any visualization. In front of that use we must ask two questions:

- Firstly, what can be visualize from any propositional graph?
- Secondly, what kind of task makes the students able to understand proof by means of visualization?

The answer to the first question seems easy. "Proof graphs" display the whole deductive organization of propositions like a tree structure. But from that, one does not visualize how such organization works. The essential point is not visible on a graph: each connection is only based on the status of the connected propositions, and we have three kinds of deductive status. And in order to be able to become aware of this point, one must succeed at least once in constructing a whole proof graph. That concers to the second question, we find two kinds of task: to construct oneself the whole graph or to find out forward and backward paths from hypotheses to the to-be-proven statement, which are already given at the top and at the bottom of screen.

In Anderson's research, proof graph was used to provide heuristic help "during problem-solving". Hence the second kind of task was chosen. As to what graph is expected to visualize, it is mainly "a hint in the form of suggesting the best nodes from which to infer" (Anderson et al., 1987, p.116). In other words, proof graph must focus attention on the new step to find out in order to progress. This way of using a graph turned out to be disappointing. And it is easy to know why. On the one hand, a graph cannot perform a heuristic function in geometry problem solving: that depends on figural processing. On the other hand, if the goal is to understand how deductive organization of propositions works, the task has the crucial point bypassed. In fact, proof graph becomes a helpful visualization for the students only when they have to construct it by themselves according to rules explaining how to shift the status of propositions into visual values. Then proof graph can visualize **not a particular proof of the to-be-proven statement, but how any proof works** (Duval, 1989, 1991). To understand how a mathematical proof works and why it does not work, as other language reasoning is the necessary condition for being convinced by a mathematical proof. We are there on the crucial threshold of learning in mathematics. Under very specific conditions, proof graph is a kind of visualization that allows one to explore and to check our own understanding of deductive reasoning. Once this threshold is crossed, proof graph becomes useless and interplay can start between deductive reasoning and geometrical figures. Proof graph is a transitional visualization that furthers register coordination.

It may more evident for proving than for any other mathematical activity, that what is mathematically simple is cognitively complex and can be understood only at the end of learning. Heterogeneous ways of working, specific to each register, must be first learnt in parallel. Is it possible to lead frontally all the training that this requires? For experimental reasons, our researches have aimed separately at each register and we have identified some conversion problems. But, recently, an attempt to join all the aspects involved in proof activity has been made within a computational environment (Luengo, 1997). And this attempt seems to be promising.

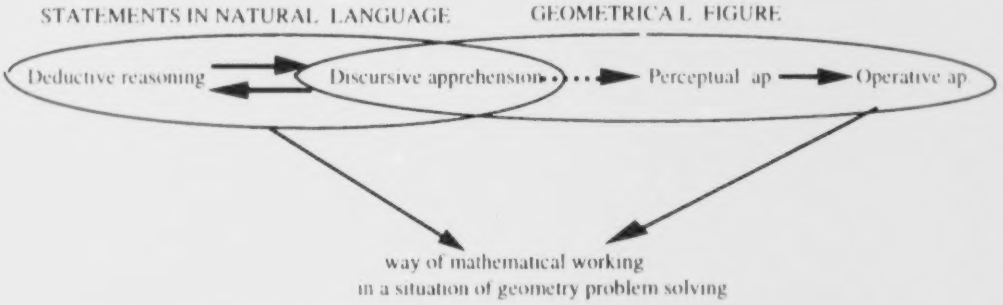

STATEMENTS IN NATURAL LANGUAGE
GEOMETRICAL FIGURE
Deductive reasoning
Discursive apprehension
Perceptual ap
Operative ap
way of mathematical working
in a situation of geometry problem solving

Figure 10. Skills and coordinations to be developed in mathematics education. Most often students confine to perceptual apprehension and reduce discursive apprehension to simple denomination of recognized shapes.

Conclusion

There is no direct access to mathematical objects but only to their representations. We cannot compare any mathematical object to its representations, as we can compare a model with its photo or its drawn image. This comparison remains attached to epistemological patterns to analyze knowledge (Platon, *Res Publica* VI 510 a-e, X 596 a-e), and it cannot be relevant in mathematics and in teaching of mathematics. We can only work on and from semiotic representations, because they are the means of processing. At the same time we have to be able to activate in parallel two or three registers of representations. That determines the three specific requirements in learning of mathematics: to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding, to convert a representation from a register to another one; and to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register. This is not the familiar way of thinking. And it is the reason why an anchorage in concrete manipulations or in applications to real situations is often pursued in order to make sense of the activity proposed. But that comes often to a sudden end, because it does not provide means of transfer to other contexts. Besides, representation becomes usable in mathematics only when it involves physical things or concrete situations. We find the same problems with visualization use, whatever the register be, it focuses on a synoptic way, organization of particular units and it does not show objects as any iconic representation. One does not look at mathematical visualization as one does at images.

Mathematical activity has two sides. The visible or conclusive side is the one of mathematical objects and valid processes used to solve a given problem. The hidden and crucial side is the one of cognitive operations by which anyone can perform the valid processes and gain access to a mathematical object. Registers of semiotic representation and their coordination set up the cognitive architecture which anyone can perform the cognitive operations underlying mathematical processes. That means that any cognitive operation, such as processing within a register or conversion of representation between two registers, depends on several variables. To find out what these variables are and how they interact is an important field of research for learning mathematics. Indeed, from a mathematical point of view only one side matters, from a didactical point of view the two sides are equally essential. In concrete terms, any task or any problem that the students are asked to solve requires a double analysis, mathematical and cognitive: the cognitive variables must be taken into account in the same way as the mathematical structure for "concept construction" (Duval, 1996). But for that, teachers must know themselves these variables and take them into account as didactical variables. They must be able to analyze the function that each visualization can perform in the context of a determined activity (Duval et al., 1999). We are here in front of an important field of research. But it seems still often neglected because most didactical studies are mainly centered on one side of the mathematical activity, as if mathematical processes were natural and cognitively transparent. There is no true understanding in mathematics for students who do not "incorporate" into their "cognitive architecture" the various registers of semiotic representations used to do mathematics, even those of visualization.

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ON THE DEVELOPMENT OF HUMAN REPRESENTATIONAL COMPETENCE FROM AN EVOLUTIONARY POINT OF VIEW: FROM EPISODIC TO VIRTUAL CULTURE[1]

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> *The modern human mind evolved from the primate mind through a series of major adaptations, each of which led to a new representational system. Each successive new representational system has remained intact within our current mental architecture, so that the modern mind is a mosaic structure of cognitive vestiges from earlier stages of human emergence* (Donald, 1991).

INTRODUCTION

Recent work by evolutionary psychologist Merlin Donald (1991) argues that human cognition has developed across evolutionary time through a series of four distinct stages, each growing out of its predecessor and yielding its own cultural form. They began with episodic (ape-like) memory and passed through mimetic (physical-action-based), mythic (spoken), and theoretical (written) transformations. David Williamson Shaffer and I have argued that we are entering, via computational media, a fifth stage of cognitive development leading to a virtual culture, which will replace the writing-based theoretic culture and which will support and be supported by a new hybrid mind, just as each of the predecessor stages subsumed its prior stage (Shaffer & Kaput, in press). I also draw upon recent work by Terrence Deacon (1997), who argues that the development of human linguistic competence needs to be viewed in a new way, through the co-evolution of brain and language, and where the major defining features of real human language are its embodiment of a relatively small number of recombinable (syntactical) elements and symbolic reference, features not shared by communication devices used by other species.

I suggest that the evolutionary perspective needs to complement mathematics educators' other ways of understanding the learning and use of mathematics, especially the semiotic side of the subject. It turns out that mathematics has played a critical role in the development of both writing and computational media, each the means by which a new stage of cognition was reached. Further, our understanding of language, especially its referential nature and its relationship to brain function, has implications for how we understand the symbolic aspects of mathematics and how they may be learned. I will recount first the Merlin Donald analyses and then move on to describe the new stage into which we are emerging.

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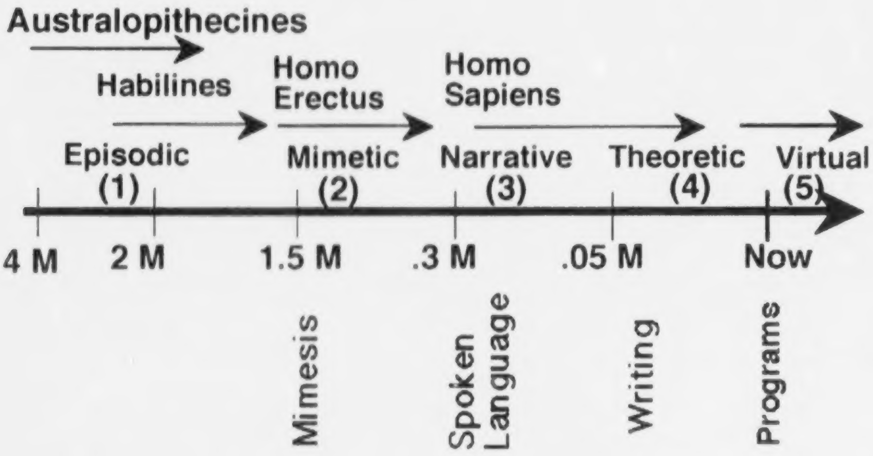
[1] This paper draws upon joint work with David Williamson Shaffer which appears in a recent issue of *Educational Studies in Mathematics*. My work in the paper was supported by Department of Education OERI grant

Four Stages of Mental Evolution: An Overview[2]

In *Origins of the Modern Mind* (Donald, 1991), Merlin Donald argues from anatomical, psychological, linguistic and archeological evidence of human evolutionary development that human culture has gone through four distinct stages of development. He suggests that each of these stages of *cultural* development was driven by a specific *cognitive* advance, and that these changes in cognition led to changes in brain development as well as new kinds of communication and social interaction. These assertions are consistent with those made by Deacon (1997). See Figure 1 for a timeline that situates the stages within our species' evolution.



Australopithecines
Habilines
Homo Erectus
Homo Sapiens
Episodic (1)
Mimetic (2)
Narrative (3)
Theoretic (4)
Virtual (5)
4 M
2 M
1.5 M
.3 M
.05 M
Now
Mimesis
Spoken Language
Writing
Programs

Figure 1: A Four Million Year Timeline

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[2] This section draws upon Shaffer & Kaput (in press)

Stage 1: Episodic Cognition

The *first stage* Donald outlines is essentially that of primate (ape-like) cognition with origins among early primates more than three million years ago. This stage is based on "episodic" thought, which Donald describes as thinking based on literal recall of events. Apes can remember details of, for example, a social interaction, and can even recall those details in context—thus an ape might "remember" that a larger male is dominant because he can recall a fight where the dominant male won. But, as Donald and many studies of primate behavior make clear, apes do not "represent" events in the sense of attaching labels to events or generalizing from events except in a straightforward associative way. They do not process events other than storing their images in episodic memory, apparently with acute event perception. Referential language as we know it does not play a role, because there is no substantive semantics that might relate situations or events beyond direct, conditioned associations—there is nothing for that kind of language to "be about," and there is no separation possible between event and cognitive replay of the event. Donald argues that apes who have learned rudimentary sign language are essentially storing and using the signs in much the same way as they would process any kind of conditioning—they "remember" signs as responses leading in certain circumstances toward pleasure or away from pain (p. 154). Deacon (1997) argues that this is not language in the general sense of embodying real (flexible) reference and real generative syntax. Nonetheless, it served primitive social and survival needs very well, for millions of years.

Stage 2: Mimesis—the Roots of Reference

Episodic cognition provided a basis for social interaction by giving early hominids the ability to recall previous events and respond accordingly. This rudimentary socialization was extended by the development of the fundamental ability to "represent" events physically dating from homo erectus about 1.5 million years ago (see Figure 1). Donald describes this as "mimesis," or "the ability to produce conscious, self-initiated, representational acts that are intentional but not linguistic" (p. 168). For example, following the gaze or pointing gesture of another requires an understanding that their gestures are referring to something of interest. Or, more dramatically, reenacting or replaying events using the body or objects shows a basic ability to process events and to communicate about them to oneself and to others—the beginnings of (1) creating an autonomously controllable self separate from the world and (2) a base for intentionality. This form of communication also helps explain social changes and other achievements such as increasingly elaborate tools, migration out of Africa, seasonal base camps, and the use of fire and shelters—all before spoken language would be physiologically possible. Hence much more is involved than what Piaget would call the development of the sensori-motor child.

> Even in modern humans, mimesis is usually an elaboration of or a summary of episodic experience ... The representation of skills, whether in crafts or athletics, involves an episodic re-enactment. In modeling social roles, events are assembled in sequences that convey relationships. They resemble the events as they occur in the real world; in fact they could be seen as an idealized template of those events. ... Episodic event registration continues to serve as the raw material of higher cognition in mimetic culture, but rather than serving as the peak of the cognitive hierarchy, it performs a subsidiary role. The highest level of processing in the mimetically skilled brain is no longer the analysis and breakdown of perceptual events; it is the modeling of these events in self-initiated motor acts. The consequence, on a larger scale, was a culture that could model its episodic predecessors. (pp. 197–8)

Donald argues that this ability to represent events was not (and is not) dependent on language. The morphological changes required for the development of speech are quite dramatic, and therefore unlikely to occur without some evolutionary pressure favoring the ability to communicate using language. Donald believes that the evolution of language was dependent on this prior cognitive development: namely, the development of crude symbolic reference usable in a voluntary way (as opposed to alarm calls, mating sounds, etc.) It also reflects neurological evolution, especially the substantial enlargement of the brain and changes in its structure as reflected in evidence from the available fossil record. It is also essential for the level of social attribution necessary for the social structures known to exist during this period. Finally, he argues, this form of communication is consistent with self-generated practice ("auto-cued rehearsal") and pedagogy based on mimicry.

Stage 3: The Emergence of Syntax and Real (Spoken) Language and the Mythic Culture

The development of language marked the arrival of a "mythic" culture based on narrative transmission of cultural understanding, comprising the *third stage* beginning about 300,000 years ago (see also Bruner, 1973, 1986, 1996). I will quote Donald directly and extensively.

> [Language's] function was evidently tied to the development of integrative thought — to the grand unifying synthesis of formerly disconnected, time-bound snippets of information. ... The myth is the prototypical, fundamental, integrative mind tool. It is inherently a modeling device, whose primary level of representation is thematic. The preeminence of myth in early human society is testimony that humans were using language for a totally new kind of integrative thought. Therefore, the possibility must be entertained that the primary human adaptation was not language qua language but rather integrative, initially mythical, thought. Modern humans developed language in response to pressure to improve their conceptual apparatus, not vice versa. (p. 215).

From Deacon's (1997) perspective, we need to be aware of another factor—the fact that the emergence of language and changes in the brain occurred in concert. That is to say, language evolved according to the young child's brain's ability to learn it—and vice-versa. The next quote helps set the scale of the changes we are concerned with as we contemplate the move to what we will term the "virtual culture." In particular, *the meaning of what it is to be human* was deeply transformed, anatomically and socially, as rapid and fluent spoken language emerged.

> Mythic culture, like all major hominid innovations before it, was a complete pattern of cultural adaptation, including some very complex anatomical adaptations....Changes occurred in most areas of the brain, as well as to many peripheral nerves and receptor surfaces. There was major muscular and skeletal redesign, including the face, body mass, cranial shape, respiration, and posture; there was a revolution in social structure; and there was a great change in the fundamental survival strategies of the human race. The entire nervous system had to adjust to its new selection pressures and changing conditions; it was not a simple matter of acquiring a new "language system" with a cleanly isolated cerebral region attached to a modified vocal tract. (p. 263)

Another important factor to be acknowledged is the new role of spoken language as a creator and organizer of human experience, and how this role was manifest both psychologically and culturally.

> Mythic integration was contingent on symbolic invention and on the deployment of a more efficient symbol-making apparatus. The phonological adaptation, with its articulatory buffer memory, provided this. Once the mechanism was in place for developing and rehearsing narrative commentaries

> on events, and expansion of semantic and propositional memory was inevitable... . At the same time, a major role in attentional control was assumed by the language system. The rehearsal loops of the verbal system allowed a rapid access and self-cueing of memory. Language thus provided a much improved means of conscious, volitional manipulation of the modeling process. (p. 268)

I should note that Donald is by no means a naïve realist in his use of the word "modeling" above. He well understands—and indeed this is one of his key lessons—that humans build worlds by building world-making tools on an evolutionary scale, not only on a developmental scale. Indeed, this is one of the reasons we need to attend to the evolutionary perspective. Relatedly, in her recent book on language and development, Katherine Nelson (1996) accepts Donald's categorization of stages of mental development, but argues that in individual (as opposed to evolutionary) development, the evolutionary relationship Donald describes between representation and language is reversed. That is, Nelson argues that culturally available language drives, or at least strongly influences, individual cognitive development (as well as symbolic competence). Language provides an external structure that scaffolds a child's ability both to represent events, and later to develop narrative and categorical understanding of its world, where its world is already richly structured linguistically. Papert has made a similar point about the development of mathematical understanding in the context of a mathematically-rich surrounding culture (Papert, 1980). In other words, it seems reasonable, probably obvious, that characterizations of evolutionary development of a cognitive ability and individual development of the same ability might differ—and that the evolutionary development of a new form of representation might have profound developmental consequences.

Stage 4: The Emergence of Writing Part 1: The Semiotic and Psychological Sides

The *fourth stage* Donald identifies is that of "theoretic culture," a culture based on written symbols and paradigmatic thought. Again, Donald argues that the principal driver here was in the needs for a new cognitive ability rather than a new means of expression. In this case, the need to work with complex phenomena drove the development of pictographic external representations beginning 30-50 thousand years ago. While these showed up earliest, and apparently in the service of mythic ritual (e.g., the many Ice-Age cave paintings in uninhabited ceremonial places), they use episodic reference (realism), grew out of mimetically organized and transmitted manufacturing skill, and drew upon the kind of conceptual skill that made and maintained the mythic stories. However, they seemed not to evolve into either ideographic or phonologically-based forms of expression, which appear in the historical record very late, about 6000 and 4000 years ago, respectively—at the emergence of cities and city-states and the associated commerce.

Many, but by no means all, recorded societies have developed pictographic competence, but only about 10% have developed some form of indigenous writing, and fewer still actually produced a body of written literature of any kind, so pictographic notation seems to be relatively independent as a means of expression.. The record-keeping needs of commerce and astronomy drove the creation of external symbol systems (p. 333ff), of which mathematical notations were probably the first as argued in great detail by Denise Schmandt-Besserat (1978, 1992, 1994). She provides detailed descriptions of how marked iconic clay tokens representing traded quantities (e.g., the number of containers of grain, or vessels of oil) were impressed on the outsides of the clay envelopes that contained them. These envelopes containing the tokens, with two dimensional impressions of their contents on the outside, were accounting records. Over two millennia or more the redundant tokens gradually were replaced by their descriptions on the outsides of the envelopes, which, in turn, became clay tablets with increasingly stylized cuneiform markings impressed on them.

Of special interest to mathematics educators is the matter of how quantities came to be expressed, how the new degrees of freedom available in visual (over oral) representation were employed to convey information and the intentions of the writer (who was usually a highly trained scribe), and the question of how phonetic writing (writing based on the representation of sounds—phonemes) related to the strictly visual starting points of writing. While space limitations prevent a full discussion (these issues are the subject of entire scholarly fields), we can summarize a few of the more salient findings.

Apparently, number symbols constituted the first purely visual, non-iconic and non-phonetic symbols. And in the various ways that larger numbers were represented, via embedding and grouping, we see the beginnings of systematic structure being imposed on the two dimensional space—driven by the need to be unambiguous in matters of trade and accounting. Of special interest is how the idea of representing a quantity efficiently and unambiguously seemed to emerge. According to Harris (1986), the essential step (which he identifies as the starting point of writing) was the invention of the "slotting" systems for accounting to overcome the inefficiencies of repetition required of iterative token systems. Previously, the accountant had to count every item, every token, or every token-symbol, when making up a total or determining a balance. Each item was individually represented in a kind of "count-all" system. As trade increased to involve thousands of items, this system was error-prone and inefficient, and led to the use of "slots" in lists to represent, for example, the kind of item in one slot , and the number of such items in another slot. Thus lists took on new form, with explicit places for such things as the properties of items (e.g., new, old, paid-for, owned-by), type of item (e.g., sheep, jar of oil), and the *number* of such item. Thus there were distinct places for different kinds of signs, with an implicit linguistic structure that was, in turn, not designed as a way of encoding speech, but rather as an independent visual expression of the mental models and intentions of the writer. The invention of writing and the invention of a way to represent quantities seem to coincide!

The resulting system that evolved over the millennium or more that followed was highly complex, required skilled interpretation, and used all kinds of different conventions, including mixes of phonetic, pictographic, spatial, and other grammatical markings intended to reduce ambiguity. This complexity evolved not only in cuneiform texts, but in Egyptian hieroglyphics (which tended more rapidly towards phonetic representation), and in Chinese ideographs (which did not) as well as in Mayan writing (which was less standardized and allowed the writer more flexibility). In all these systems, mapping onto a sound-stream was subsidiary to the expression of ideas. Indeed, the remarkable success of Chinese ideographic writing over several millennia, despite the complexity that prevented universal literacy, makes clear the functional independence of writing from speech —writing did not *arise* as the encoding of speech.

Nonetheless, over several millennia of evolution in the Mediterranean basin and the Middle East, apparently driven by the need to counter the pull towards complexity in expression, and the simultaneous need to support an ever wider literacy, scripts became ever more phonetic, with smaller clusters of signs (syllabaries) specifying individual sounds, leading to the Arabic, Hebrew, Aramaic and Phoenician alphabets about 3500 years ago—all of which had a few dozen or less of such sets of signs. And about 3000 years ago, the Phoenician alphabet was adapted by the Greeks to form what has become the basic alphabet of Indo European languages—about two dozen recombinable marks with which to create strings of visual marks that map onto a sound stream—the pre-existing speech system—and vice-versa. This solved the complexity problem by tapping into an existing powerful and flexible system while sacrificing some of the directness of purely visual systems. Both Deacon (1997) and Donald distinguish between communication and the use of specific language systems, and Donald points out that actual communication even today involves a mix of alphabet-based writing, ideograms (sometimes called "icons" nowadays), pictograms, and logograms—as well as gesture and various forms inherited from mimetic culture. This is especially true in mathematics, where a large variety of non-phonetic logographic signs are used (parentheses, bars, brackets, slashes, dots, operation signs, etc) as well as varieties of positional conventions, e.g., exponents, fractions. In some ways, mathematical writing, in its flexible exploitation of two dimensional space and non-phonetic character, shares features with the early writing forms. And it also shares the complexity problem that limits broad learnability—which keeps mathematics education researchers in business. It also lacks one of the strengths of alphabetic writing, which can draw upon acoustic memory ("sounds like ...").

Donald traces out the different neurophysiological changes in memory processing associated with the different kinds of external representation systems, including parallel visual and auditory processing associated with alphabetic systems. One basic point is that the nature and processing of the biological mind is changed, and changed in different ways, by the presence of different physical notation systems. Old neurological structures come to be used in new ways since there isn't time for biological evolution to have an effect.

Stage 4: The Emergence of Writing Part 2: The Theoretic Culture Side

Writing, and hence the existence of stable external representations, involved two profound changes: (1) a shift from auditory to visual modalities, and (2) a move to deeply engage non-biological means to support mental processes. But before writing, 4000 years ago, an enormous amount of practical knowledge had already been built at widely dispersed locations across Europe, Asia, and Central America, knowledge that did not require sophisticated writing—domestication of animals and plants, sewing, metallurgy of various kinds, sailing ships, beer and wine, baked bread, and so on. In the form of early astronomy, the beginnings of scientific thinking in the sense of selective observation, data collection and organization, and even prediction, were also in place, often using external measurement and data collection devices such as the specially organized sets of stones in Stonehenge. These kinds of invention had practical uses, both for agricultural and socio-cultural purposes, and, to a certain extent, amounted to working models. While intellectual theorizing had yet to begin, the practical progress created an increase in wealth that would (for the political elite) create room for a version of academic life in Greece about 2700 years ago.

In addition to the non-cognitive enabling practicalities, and a certain political openness to the exchange of ideas, the availability of alphabetic writing "eventually created the intellectual climate for fundamental change: the human mind began to reflect on the contents of its own representations, to modify and refine them" (Donald, 1991, p. 335). This lead to the birth and rapid growth of analytical philosophy and logic, mathematics (especially geometry and the idea of proof), biology (especially systematic taxonomy and embryology), geography, among other fields such as theater, politics, ethics and architecture, that began the "theoretic culture." Somehow, the structure of the human thought process had suddenly changed. How and why?

> The key discovery that the Greeks made seems to have been a combinatorial strategy, a specific approach to thought that might be called the theoretic attitude. The Greeks collectively, as a society, went beyond pragmatic or opportunistic science and had respect for speculative philosophy, that is, reflection for its own sake. ... In effect, the Greeks were the first to fully exploit the new cognitive architecture that had been made possible by visual symbolism. ... The critical innovation was the simple habit of recording speculative ideas—that is, of *externalizing the process* of oral commentary on events. Undoubtedly, the Greeks had brilliant forebears in Mesopotamia, China, and Egypt; but none of these civilizations developed the habit of recording the verbalizations and speculations, the oral discourses revealing the *process* in action. The great discovery here was that, by entering ideas, even incomplete ideas, into the public record, they could later be improved and refined. Written literature for the first time contained long tracts of speculation—often very loose speculation—on a variety of fundamental questions. The very existence of these books meant that ideas were being stored and transmitted in a more robust, permanent form than was possible in an oral tradition. Ideas on every subject, from law and morality to the structure of the universe, were written down, studied by generations of students, and debated, refined and modified. A collective process of examination, creation, and verification was founded. The process was taken out of biological memory and placed in the public arena, *out there* in the media and structures of the External Symbolic Storage System. ...

> They founded the *process* of externally encoded cognitive exchange and discovery. [italics in original] (p. 342)

Over the two millennia since this breakthrough, progress in the application of this evolutionary innovation has been slow and irregular. For the first thousand years, while thought and effective use of language were held in highest value across western civilization, the actual exercise of these values were primarily in the form of oral debate—although the rules of rhetoric and the various curricula intended to teach them were recorded in writing, with Aristotle's rhetoric being the foundation.. These values were also given expression in the core curriculum structures that were at the heart of the universities founded at the beginning of the next thousand years, especially in the Trvium, which focused on logic, grammar and debate and gradually shifted from oral towards written forms. But, of course, specialized knowledge exploiting externalized thought processes and specialized symbol systems , and their products, began to grow more rapidly in the past 400 years, a growth that is accelerating. Formal arguments, systematic taxonomies, induction, deduction, verification, differentiation, quantification, idealization, formal measurement, detailed, systematic analyses, all subject to continual iterative public scrutiny in a shared extra-cortical space that extends in time across generations yield systems of thought that feed recursively on themselves. And with the invention of the printing press, the number of participants could likewise grow. Indeed, because the central material object of the theoretic culture is the *book*, the printing press would have such a profound effect on the shape of our societies, at least western societies (McLuhan, 1962).

At the same time, as Donald suggests, the mythic forms of meaning-making and significance, continues to coexist with this theoretic one after tens of thousands of years.

> The first step in any new area of theory development is always anti-mythic: things and events must be stripped of their previous mythic significances before they can be subjected to what we call "objective" theoretical analysis ... "demythologized." ... Before the human body could be dissected and catalogued, it had to be demythologized. Before ritual or religion could be subjected to "objective" scholarly study, they had to be demythologized. Before nature could be classified and placed into a theoretical framework, it too had to be demythologized. Nothing illustrates the transition from mythic to theoretic culture better than the process of demythologization, which is still going on, thousands of years after it began. The switch from a predominantly narrative mode of

> thought to a predominantly theoretic mode apparently requires a wrenching cultural transformation. (p. 275)

The Hybrid Mind At Work in a PME Plenary

Donald argues that all of these ways of thinking—episodic, mimetic, narrative, and theoretic—exist simultaneously, and that we move among and use them in a fluid way. So, for example, a plenary PME lecture (spoken!) involves mimetic, mythic and theoretic written representation. Episodically, you are likely to recall whether the speaker perspired, or seemed engaging, or sneezed, and you see the speaker's inevitable mimetic gestures and motions, perhaps accompanied by non-written graphics. We cannot ignore the mythic context, which serves to define the social, political and participation structures of the event. One might even characterize the almost (but not entirely) ritualistic repetition of the "history and aims of PME," as well as the honor given to its founders in our shared documents, as residual mythic elements. But at the same time we attempt to build science within the theoretic culture. And, of course, the entire event has at its core the "paper," stored in the proceedings that are laboriously constructed and that we happily carry home with us. Donald refers to the "hybrid mind" as our means of actively and generatively embodying all the cultural and representational forms that preceded us.

A Fifth Stage of Cognitive Development: Autonomous, External Processing Leading to a Virtual Culture

The Externalization of Computation

I can type the following two-variable function into my computer and see the surface that constitutes its graph, as in Figure 2, within a fraction of a second:

$$z = [\sin xy + 1/2 \cos 2x + 1/3 \sin 3y + 1/4 \cos 4(x+y)]/[1 + |\sin 5y + 1/2 \cos 6x + 1/3 \sin 7y + 1/4 \cos 8x|]$$

Moreover, I can then use my mouse to manipulate that graph as if it were a physical object—turn it on its side, rotate it, etc.[3] Even more significantly, any constant in the function can be treated as a parameter and allowed to range over whatever domain I choose to define. In other words, this can be experienced as a class of functions, not a single function.

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[3] This was taken from the standard desk accessory Macintosh graphing calculator demonstration written by Ronald Avitsur (and included on every Power PC Macintosh—without Apple Computer Company's official knowledge!)

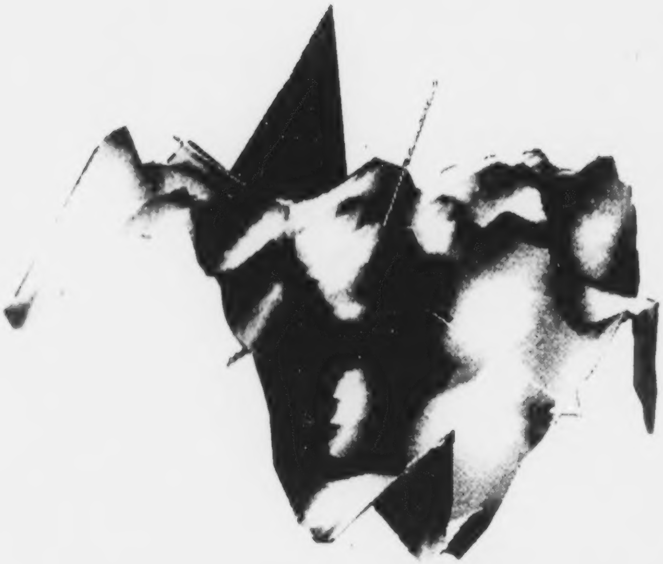

Figure 2: Graph of z

As I type, my computer is automatically checking my spelling and underlining in red all words not appearing in its dictionaries. Indeed, literally millions of computations are taking place in this box on my lap during the writing of this paper. You, on the other hand, are reading a static, inert (black & white) printed document, an item and an activity from the theoretic culture, an external, physical record of my work. Figure 2 is an external record of computations done elsewhere.

As you drive your car, many different microprocessors are computing such things as the fuel/air mixture being injected into the cylinders based on data continuously drawn off the physical vehicle. Any passenger airplane has many such processors of varying complexity, for example, taking weight distribution data for the plane before take-off and outputting settings for the wing and tail flaps, lift-off speed, attack-angle for lift, and so on. Abstract and highly complex representations of chemical and microbiological entities, particularly genomes and proteins, can be treated as formal systems subject to algebra-like manipulation, and then manipulated by computers to examine new possibilities for drugs and therapies—the sciences have assumed new computational forms with new intermediate objects. While the designs of these processors and the computations they are performing are the products of human minds, the computations they are performing are occurring *outside* human minds, autonomously and, in some cases, almost invisibly. Indeed, many millions of computations at many different locations across and above the continent were required to send this paper to the editors, and many more millions to print and copy it. All of these took place outside human heads.

Much could be said about what makes these externally executed computations different from those that we actively perform with our minds, usually in tight loops of interaction with physical material. For our purposes here it suffices to remind ourselves that the traditional numeric or algebraic computations that dominate school curricula are comprised of highly organized productions of physical character strings on paper by following certain rules. These rules, in turn, are executed in concert with highly organized semiotic space in very physical ways that involve much more than knowing the rules in an abstract sense. In addition to "mathematical mental actions" involving some level of understanding of the rules, they involve varying levels of perceptual processing, fine motor skill, and so on, just as with the abacus—although the abacus involves different actions on different physical material. Our typical characterizations of school algorithms tend to underplay their physicality, their dependence on actions both structured by and that structure physical material. This tendency to underplay the material side of algorithms in practice may work to underplay their difference from machine executed algorithms and cause us to overlook the significance of what has changed now that computation can be executed autonomously without direct human facilitation.

Returning to Donald, the development of an ability to represent events created a "mimetic" culture based on communication mediated by the exchange of physical gestures, actions, postures, etc. The addition of language made possible a "mythic" culture based on the exchange of narrative stories—the great stories that embodied, enriched and organized human experience within and across generations before the dawn of writing. The creation of written symbols led to a "theoretical" culture based on external symbolic storage, and led to an entirely new means of organizing and enriching human experience that led, in the west, to science, and to logically organized mathematics. Continuing the progression, we suggest that the computational media are in the process of creating a new, virtual culture based on the externalization of highly general algorithmic processing that will in turn lead to profoundly new means of embodying, enriching and organizing all aspects of human experience.

The Role of Mathematics in Making the Computational Medium, Hence Virtual Culture, Possible: Part 1—The Development of Human-Driven Symbolic Computation

Donald's analyses of each prior evolutionary transformation suggests that we should look for the roots of the development of the posited fifth stage of cognition in changes in the way we represent or model our experience of the world within the prior stage. That is, we should look at the cognitive processes that made computational media possible. Their development depends on two factors: (1) the ability to create explicit rules of transformation on well-formed systems of symbols independent of particular fields of reference, and (2) external physical systems capable of autonomously applying those rules. The second of these, while not independent of the first, is relatively easy to account for—the history of computational devices leading to the miniature integrated circuits of today. It is not our focus. (Note that we ignored the nature of the different physical media in the development of writing, but they surely played a significant role. In particular, the cuneiform script and its predecessors mainly used objects pressed into wet clay rather than a stylus; and later, more alphabetic writing gradually moved towards a stylus writing on papyrus, rolls of which provided convenient and efficient storage of large amounts of text.) Instead, we will look, in a dangerously brief way, at the first factor, which, just as was the case with the prior stage, had its foundations in mathematics.

As described earlier, the first, and certainly the most well-explored, systems of notation were designed, or evolved, to represent concrete, physical quantities, especially what we would today call discrete quantities. Importantly, the various number systems supported, to varying degrees and with varying degrees of explicitness, rules for operating on them, especially for addition and subtraction (Kline, 1972). We will skip over the rich history of notations for numbers (see Cajori, 1929) and jump to the base-ten placeholder system of numerals and the algorithms build upon it. Just as was the case millennia earlier, the needs of commerce drove the development and adoption of algorithms that we largely still use today as documented by Swetz (1987). For our purposes, the essential feature of such a notation system is that it was designed to support, with the participation of an appropriately trained human, a particular but broadly useful form of reasoning—not merely the static representation of information. It is an *action* notation system (Kaput, 1989).

A prodigious advance in the development of mathematics was the creation of another, more general and therefore more powerful set of algorithms for representing and manipulating quantitative relationships: namely, algebra and the rules for manipulating algebraic symbols to solve equations, transform character strings into one or another canonical form, and so on (Bochner, 1966). As is well known, this system gradually evolved from a "rhetorical" shorthand to one that used genuine mathematical variables with Vieta (Klein, 1968), and then to an action system in the hands of the those who needed it in the pursuit of equation solving and, more intensely, in the development of and exploitation of calculus (Kline, 1972).

In both the numeric and the algebraic systems it is essential that one can perform operations on the symbols without regard to what they might refer. In Bruner's terms (Bruner, 1973), the symbols are being treated as "opaque." That is, they act as objects with their own identity and rules of transformation, which is different from a use based on what the symbols stand for, which Bruner refers to as "transparent" (Bruner, 1973). Inevitably in practice a mix is used—as is especially the case in the computational chemistry and microbiology example mentioned above—the rules for acting on the representations are developed in relation to what the symbols stand for, computations are carried out, and then their physical significance is investigated. All these systems extend the *processing* power of the biological mind rather than its *memory*, and all require a human partner.

The Role of Mathematics in Making the Computational Medium, Hence Virtual Culture, Possible: Part 2—The Emergence of Formality and Its Instantiation in External Devices

Euclid's geometry served for 2000 years as an idealized model of the geometry of the world, and its main function was as a model of mathematical reasoning, which, in turn, served as an idealized model of human reasoning. This changed in the last 200 years with the development of non-euclidean geometries. About a hundred years earlier, Descartes, through a clever use of geometry, freed the notion of number from dimensionality and made products of any two numbers possible without worrying about the physical dimension of the product (Klein, 1968; Kline, 1972). In addition, various algebraic maneuvers in equation solving led to the appearance of such novel "unreal" things as zero, negative numbers, roots of numbers, and even roots of negative numbers. Gradually, the notion of number was generalized and abstracted, the idea of a number *system* emerged, and by the latter 18th and early 19th century the idea of universal, and then abstract, algebra began to emerge. Over the space of a few centuries, mathematics was loosening its tethers to material reality. Paradoxically, at the same time, of course, mathematics was being used to create an entirely new set of extraordinarily powerful models of the material world. This divergence of purpose gradually led to the fissure separating mathematics from science, and was an instance of the knowledge specialization that has marked western science since the Renaissance.

But within this newly freed mathematics, the idea of a logically consistent system independent of any kind of reality took hold, and, indeed, a notion of mathematics as a formal system defined only by logically consistent actions on symbols was put forth by Hilbert and others around the turn of the century—the formalist view. While the logical foundations of the formalist view of mathematics as a whole were undermined by Goedel's work, the idea of formalism and of a formal system not only survived, but has become an essential feature of the mathematical landscape. The idea that one could define well-formed formulas and explicit rules for their transformation set the stage for the idea of a computer program, made explicit in somewhat different ways by Turing and von Neumann (Von Neumann, 1966, Turing, 1992). While the idea of universal (as opposed to numerical) computing machines and logic machines goes back to Leibniz and even earlier, the underlying intellectual infrastructure was not available to render it viable until well into the twentieth century. Of course pragmatic factors, both military and commercial, as always seems to be the case, drove the actual physical realization and early applications of computers. But now the computations could be designed by a human, but executed independently of a human! (It should perhaps be pointed out that Von Neumann conceived of computers that could design themselves, and, more recently in the 1970's, John Holland (1995) developed the idea of genetic algorithm, wherein the program modifies itself across iterations by way of random mutations of its operation strings, yielding a new level of processing autonomy.) The human could now interact with the model, even change it "on the fly," but its underlying computations could be executed autonomously of the biological mind rather than in direct partnership with the biological mind as was the case with the previously discussed action notation systems.

Moreover, the success of mathematics as a means of modeling aspects of experience—not merely the physical world—had validated not only the utility of many different mathematical systems (e.g., non-euclidean geometries), but the idea of an abstract, formal model itself, one with no *necessary* connections to anything else. Once computers were available within which to instantiate those systems, the freedom to construct and explore such systems led to an explosion in the use of computer models, especially simulation models, and deep changes in the nature of the scientific enterprise (Casti, 1996). Space limits discussion of the kinds of models now possible, but we must acknowledge that, particularly through the exploration of dynamical systems, an entirely new view of the world is emerging (Heim, 1993; Cohen & Stewart, 1994; Hall, 1994; Holland, 1995; Kauffman, 1995; Casti, 1996; Resnick, 1994).

Two other, related, innovations feed the process of creating a vitrual culture. One is the connectivity revolution, currently in the form of the World Wide Web and in local networks, but soon to take the form of more flexible "just-in-time connectivity." This allows the widespread sharing of data, analyses, and, most especially, models and simulations—including the collaborative manipulation of such models, and a rapid distribution of new insight and modifications. The second innovation involves the feeding back upon itself of the computation processes to form new visual means for the presentation of models and simulations and new ways to interact with them. In particular, it is now possible to design and build human-computer interaction systems that take advantage of the highly sophisticated physical and perceptual competence of human beings. Hence it is possible to create manipulable worlds with increasingly arbitrary "reality"—but without the constraint of physicality (Kaput, 1996), particularly with freedom from the time and size scales of the physical world. The nature of modeling has both changed and been democratized in the sense that one need not be a programmer or mathematician to use models and simulations profitably.

In the face of these changes, we are being forced to reexamine the ideas of mathematical abstraction, idealization, and even the psychological idea of abstraction (see Nemirovsky 1998; Noss and Hoyles (1996; and Wilensky (1991). Briefly, as these authors variously suggest, we may need to make room in our notion of mathematical understanding for a kind of "concrete abstraction" that builds mathematical meaning "additively" as an active web of meaningful associations rather than "subtractively" by deletion of elements and features.

Comparisons to Prior Stage-Transitions

The hominds and their episodic mind were *of* their world. They did not model it in any explicit way and changes were extremely slow because they depended on physical evolution. The mimetic mind, millions of years later, began the process of building autonomy, a separation from their world that was both the basis of symbolic reference *and* the beginnings of self-initiated practice with the means of modeling actions and experiences, and communication. The possibility now existed for feedback cycles within which the individual could intervene. With spoken language and the mythic culture, ever more comprehensive narrative stories about the world became possible, and with them appeared new forms of experience and meaning, new ability to effect change in others and in the physical world, and new forms of knowledge. Change became even more rapid as feedback cycles tightened and more knowledge could be shared more widely. The move to writing broke the limits of the biological mind, provided external resources for mental activity, both memory and processing. Even the process of thinking, at least the stylized oral aspects of it, could be externalized and made available to be shared and improved by others and even across generations, enabling even more rapid cumulativity and reshaping of knowledge than that begun by the Greeks.

At the same time, the process of demythologizing and secularization of human experience into the theoretic culture continued and continues today. The "heavenly bodies" became celestial objects that move according to human-specifiable rules, the earth became just another celestial object, the human body became a subject of study and the heart an organ, humans were recognized to be yet another species, the mind became subject of study, the societies we live in became subjects of study, the idea of "life" has become yet another formalism, and even the process of knowledge building, even model building, became a subject of study.

The induction into the symbolic forms and the products of the use of those symbolic forms became an increasingly important part of individuals' development, requiring new institutions and methods—the idea of education. Importantly, education, while mediated by *written* material, maintained its goal of producing sophisticated *speakers* for more than 3000 years—should we be surprised that changes in mathematics education require generations, and that we seem to be educating people for the past? Within the past 300 years, change has accelerated. In particular, the focus of education shifted from narrative and the classics to the new products of the theoretic culture. As our means of understanding—rapid, shared modeling and simulation, for example—become incorporated into the *processes* of education, we can expect change to accelerate even more. The book will be supplemented by the simulation as the primary intellectual object[1] and the learning feedback loop will be both enriched and tightened. The reader is also invited to examine Shaffer and Kaput (in press) for more detailed discussions of the implications of these changes for mathematics education. In the plenary discussion I will offer some concrete illustrations.

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[1] This characterization was offered by David Shaffer (personal communication)

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REPRESENTATION AND EVOLUTION: A DISCUSSION OF DUVAL'S AND KAPUT'S PAPERS[†]

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Duval and Kaput present two very differently-oriented perspectives on the important issues of representation in mathematics education. Yet, without setting out to do so, each paper speaks directly to issues raised by the other. I shall structure my comments by first focusing on the two papers independently and then on the two together.

Comments on Duval

Duval opens his paper with a comment that I found refreshing because it is so true:

> Research in developmental psychology, new technologies, new requirements in assessment have supported [needed changes over the past 50 years]. But their impact has been more effective on mathematics curriculum and on means of teaching than on the explanations of the deep processes of understanding and learning in mathematics.

Such explanations require explanatory frameworks, systems of constructs from which a researcher can formulate descriptions and explanations of important phenomena. Duval focuses on issues he sees as foundational to our understanding what conditions are propitious for mathematical learning. In the process he touches upon a myriad of distinctions that attempt to clarify essential ideas underlying representation and visualization. Among these are

- We never deal with mathematical objects, but only with representations of them
- "Representation," as commonly used, is ambiguous – that there is a common confounding of issues in thinking of "internal" versus "external" representations
- Representational activity is fundamentally semiotic in nature, and that semiotic systems are never transparent and must be developed within themselves

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- Semiotic activity occurs within registers of representing – systems of semiosis.
- Mature mathematical processing is founded on coordinating processes across semiotic systems.
- Mathematical visualization is different from perceiving, "to look at [drawings] is not enough to see, that is to notice and understand what is really represented," and is based on "operative apprehension" – seeing a present figuration as being but one possible state of a system of transformations.

Out of this Duval identifies three conditions for learning mathematics:

> to compare similar representations within the same register in order to discriminate what are the relevant values within a mathematical understanding, to convert a representation from one register to another one, and to discriminate the specific way of working in order to understand the mathematical processes which are perform in this register.

And he finally concludes with a statement directed at mathematics education researchers:

> We are here in front of an important field of research. But it seems still often neglected because most didactival studies are mainly centred on one side of the mathematical activity, as though mathematcial processes were natural and cognitively transparent.

Duval's emphasis on "registers of representation" (words, symbolic expressions, graphs, diagrams) reminds me of Post, Behr, Lesh, and Harel's ideas regarding modes of representation in their Rational Number Project investigations (Behr, Khoury, Harel, Post, & Lesh, 1997; Behr & Post, 1980; Behr, Harel, Post, & Lesh, 1993; Lesh, Behr, & Post, 1987). But it is different, too. The RNP's attention was on external figurations and meanings they possessed, whereas Duval's notion of representation (semiotic system) is more attuned to the activity of the representer. But his point is well taken that we must give explicit attention in instructional design to students' coordinating representational processes across registers. I am unsure, though, what Duval has in mind that is different from what Kaput (1987a; 1987b; 1989; 1992) has described as translating among representation systems and working within a representation system. Kaput's definition of representation system is broad enough that it fits Duval's idea of a register, so it cannot be that Kaput talked about just (what might be called) symbol processing.

On the other hand, I wonder what, precisely, Duval means by a register, what he calls a system of representing. Is this an ad hoc construct, suggested to us by observing that there seem to be different but loosely equivalent ways of representing what appears to be a single idea? Or is it defined operationally by specifying cognitive operations that cohere into schemes that express themselves in equivalence classes of externalizations? That is, does Duval arrive at specific registers by identifying certain cognitive operations that express themselves in different settings in apparently different ways (thus, determining, from the individuals' perspectives, equivalent representations)? If so, the register is the scheme of operations. Otherwise, I don't know what a register is except that it is determined by social convention.

I also wondered whether Duval's appeal to semiotics was in the Sausserian or Peircian traditions. At times it is reminiscent of both – his reference to dyadic relationships being more Sausserian and his reference to triadic relationships being more Peircian. But it seems that Duval addressed a very different matter than either Saussere or Peirce. Saussere focused on semiotics without appeal to an external reality (whence dyadic relations between signifier and significant), whereas Peirce held a slot for an objective referent. But both Peirce and Saussere imagined an active interpreter who made a signifier into a sign. However, Duval agrees very much with Saussere and Peirce in the importance of talking about *people* developing and coordinating semiotic systems. As Chandler (1999) notes,

> This highlights the *process* of semiosis (which is very much a Peircean concept). The meaning of a sign is not contained within it, but arises in its interpretation. Whether a dyadic or triadic model is adopted, the role of the interpreter must be accounted for - either within the formal model of the sign, or as an essential part of the process of semiosis.

Comments on Kaput

Kaput places issues of representation into a larger perspective of evolutionary psychology. As I am unfamiliar with Donald's book, I shall take Kaput's fascinating account as being an acceptable presentation of it. He recaps Donald's (1991) theory that three major advances in human culture occurred in consonance with fundamental changes in human cognition.

Socialization emerged with the emergence of episodic memory, foundations of semiotic man emerged with the emergence of the capacity to use one item of experience to refer to another. Historical and persisting cultures emerged by way of humans' capability to experience events vicariously through telling and listening to stories. Theoretic culture emerged as a byproduct of humans' capacity to reason formally about their actual use of semiotic items – to attend to matters of form in their use of signs and symbols.

Kaput extrapolates from Donald's theory to suggest that human culture is at the dawn of yet another stage, a stage that is enabled by human's capacity to produce autonomous computations. This is the stage of virtual culture, brought about by informational interconnectivity on a massive scale.

While I am fascinated by Kaput's ideas, I wonder if he has changed Donald's thesis in subtle, fundamental ways. His presentation of evolution has, at times, a decidedly Lamarckian and teleological flavor.

Modern genetics differentiates between genotype and phenotype. As I understand it, a genotype has to do with the genetic structure inherited across generations, whereas phenotype is the set of characteristics exhibited by members sharing a common genotype. It is a tenet of modern genetics (I am told by my science education colleagues) that phenotype cannot influence genotype. Put simply, children of weight lifters will not inherit the fruits of their parents' efforts. They must exercise, too, in order that their bodies show the same characteristics as their parents' bodies. Now, the children of weight lifters may have a higher percentage of weight lifters among them than the general populace, but that is because they are around people who lift weights, not because of an inherited trait. Lamarckian biology, as I understand it, proposes that the phenotype can, in fact, influence the genotype.[1] This is not widely accepted, I am told, and is at best controversial.

Teleology is the idea that nature evolves in a way to reach a particular end. This, too, is rejected in modern genetics. That is, it is considered a mistake to make claims like "Frogs developed webbed feet so that they could swim," and like "Birds developed wings so that they could fly." Rather, more appropriate claims would be "Frogs that had webbed feet swam faster and with greater agility than frogs that didn't, and therefore had a higher survival rate in areas where large fish also populated the waters." They did not develop webbed feet in order to escape from fish. Instead, those who *had inherited that mutation* ended up with higher escape rates. Not all mutations make a difference. Some make a positive difference, some make a negative difference, and some make no difference in survival rates.

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[1] Piaget subscribed to this view based on research he conducted in his youth on the shells of fresh water mollusks. Smooth-shelled mollusks moved from placid waters to fast-flowing waters developed ripples in their shells. Offsprings of these mollusks, placed in placid waters, had rippled shells as did their offsprings.

One way to test for Lamarckianism in our understanding of culture is to imagine an infant transported from its native culture to an adoptive family in another. Infant Papuans brought to the United States to live with an upper middle-class family will probably exhibit all the characteristics of someone born to that culture (except perhaps for characteristics due to interactions with others that express others' attitudes toward children adopted from another culture).

Kaput's extension of Donald's theory seems to break with Donald's Darwinism. For example, Donald (as in a quotation presented by Kaput) made it clear that he did not think of language as an evolutionary breakthrough. Rather, he considered integrative thought as the evolutionary breakthrough. Language was an expression of this new genetic mutation in the face of pressures of persisting. Now, I say "persistence" instead of "survival" for a reason. Survival has *existence* at its core. To survive means continue to exist as a living entity. Persistence has *coherence* at its core. Persistence entails survival, but it also entails the pressures of abduction, reflection, and socialization (coordinating competing perspectives).

Kaput's extension does not point to any underlying change in the human genotype, and if there is such a change, he implies that it is because of current human activity – whence the Lamarckian flavor. Increased sunspot activity would obliterate the virtual culture overnight, but our cognitive potentialities would be unaffected. Kaput is probably correct that we are entering a new stage in human culture, but I do not see a deep connection with Donald's evolutionary psychology.

What Kaput and Duval Say to Each Other

Kaput pointed to the accelerating emergence of increasingly virtual worlds with which and through which humans interact. Duval emphasizes that thoughtful, didactic attention must be given to helping students employ any register of representation powerfully and flexibly, and that deep mathematics emerges from their coordinating across registers their specific register-centered activities. It would be interesting were Duval to analyze Kaput's virtual worlds for what he sees as registers of representations and didactical strategies to make them evident, and for Kaput to analyze Duval's didactics to see wherein it could be empowered by infusing it with perspectives of a virtual culture.

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THE ROLE OF VISUAL REPRESENTATIONS IN THE LEARNING OF MATHEMATICS

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Introduction

Vision is central to our biological and socio-cultural being. Thus, the biological aspect is described well in the following (Adams & Victor, 1993, p. 207): "The faculty of vision is our most important source of information about the world. The largest part of the cerebrum is involved in vision and in the visual control of movement, the perception and the elaboration of words, and the form and color of objects. The optic nerve contains over 1 million fibers, compared to 50,000 in the auditory nerve. The study of the visual system has greatly advanced our knowledge of the nervous system. Indeed, we know more about vision than about any other sensory system". As for the socio-cultural aspect, it is almost a commonplace to state that we live in a world where information is transmitted mostly in visual wrappings, and technologies support and encourage communication which is essentially visual. Although "people have been using images for the recording and communication of information since the cave-painting era ... the potential for "visual culture" to displace "print culture" is an idea with implications as profound as the shift from oral culture to print culture." (Kirrane, 1992, p.58).

Therefore, as biological and as socio-cultural beings, we are encouraged and aspire to "see" not only what comes "within sight", but also what we are unable to see. Thus, one way of characterizing visualization and its importance, both as a "noun" —the product, the visual image— and as a "verb" —the process, the activity— (Bishop, 1989, p. 7), is that "Visualization offers a method of seeing the unseen" (McCormick et al, 1987, p. 3). I take this sentence as the leitmotif of this presentation in order to re-examine first, its nature and its role and then, the innovations of research and curriculum development .

Seeing the unseen - a first round

Taken literally, the unseen refers to what we are unable to see because of the limitations of our visual hardware, e.g. because the object is to far or too small. We have developed technologies to overcome these limitations to make the unseen seeable. Consider, for example, the photographs taken by Pathfinder on Mars in 1997. Or, for example, a 4,000 times amplification of a white blood cell about to phagocytise a bacterium, or a 4,000 times amplification of a group of red blood cells. We may have heard descriptions of them prior to seeing the pictures, and our imagination may have created images for us to attach to those descriptions. But seeing the thing itself, with the aid of technology which overcomes the limitation of our sight, provides not only a fulfillment of our desire to "see" and the subsequent enjoyment, but it may also sharpen our understanding, or serve as a springboard for questions which we were not able to formulate before.

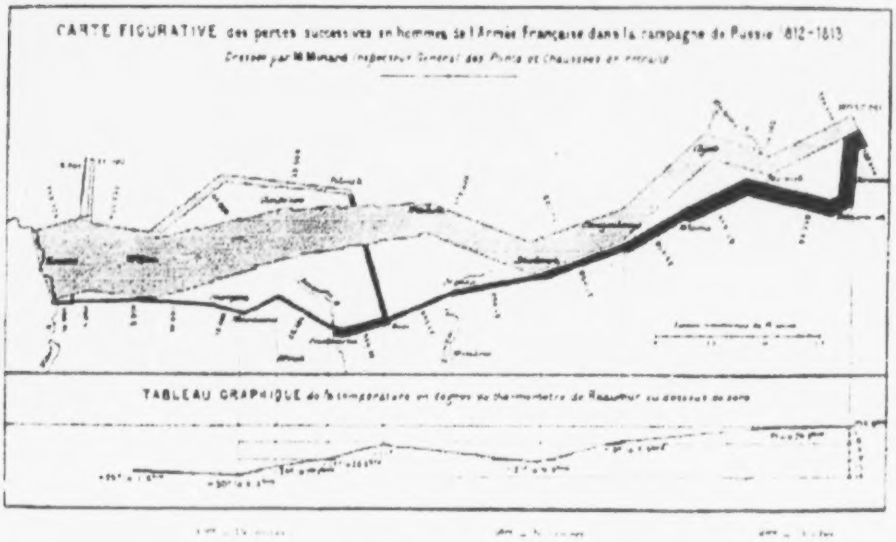
Seeing the unseen - in data

In a more figurative and deeper sense, seeing the unseen refers to a more "abstract" world, which no optical or electronic technology can "visualize" for us. Probably, we are in need of a "cognitive technology" (in the sense of Pea, 1987, p. 91) as "any medium that helps transcend the limitations of the mind ... in thinking, learning, and problem solving activities." Such "technologies" might develop visual means to better "see" mathematical concepts and ideas. Mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena (like planets or blood cells), relies heavily (possibly much more than mathematicians would be willing to admit) on visualization in its different forms and at different levels, far beyond the obviously visual field of geometry, and spatial visualization. In this presentation, I make an attempt to scan through these different forms, uses and roles of visualization in mathematics education. For this purpose, I first blend (and paraphrase) the definitions of Zimmermann & Cunningham (1991, p.3) and Hershkowitz et al. (1989, p.75) to propose that:

"Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings."

A first type of the "unseen" we find in mathematics (or allied disciplines, e.g. data handling or statistics) consists of data representations. The following example (See Figure 1) is a chart, considered a classic of data graphing, designed by Charles Joseph Minard (1781-1870), a French engineer.

Tufte (1983, p.40) considers this graph a "Narrative Graphic of Space and Time", and refers to Marey (1885, p.73) who notes that it defies the historian's pen by its brutal eloquence in portraying the devastating losses



suffered in Napoleon's 1812 Russian campaign. At the left, at the then Polish-Russian border, the chart shows the beginning of Napoleon's campaign with an army of 422,000 men (represented by the width of the "arm"), the campaign itself and the retreat (black "arm"), which is connected to a sub-chart showing dates and temperatures. The two-dimensional graph tells the "whole" story by displaying *six* variables: the army size, its exact (two-dimensional) location, direction, temperature and dates in a compact and global condensation of information. The visual display of information enables us to "see" the story, to envision some cause-effect relationships, and possibly to remember it vividly. This chart certainly is an illustration of the phrase "a diagram is worth a thousand (or "ten thousand") words," because of a) their two-dimensional and non-linear organization as opposed to the emphasis of the "printed word" on sequentiality and logical exposition (Larkin and Simon, 1987, p.68, Kirrane, 1992, p. 59); and b) their grouping together of clusters of information which can be apprehended at once, similarly to how we see in our daily lives, which helps in "reducing knowledge search" (Koedinger, 1992, p. 6) making the data "perceptually easy" (Larkin and Simon, 1987, p.98).

Anscombe (1973, p. 17) claims that "Graphs can have various purposes, such as: (i) to let us perceive and appreciate some broad features of the data, (ii) to let us look behind those broad features and see what else is there". And he presents the following example (See Figure 2) for which the data are described by the following identical parameters (See Figure 3)and which looks quite different when the data are plotted (See Figure 4)

In this case, the graphical display may support the unfolding of dormant characteristics of the data, because it does more than just depict. As Tufte

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I		II		III		IV	
X	Y	X	Y	X	Y	X	Y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

Figure 2

$N = 11$
mean of X's $= 9.0$
mean of Y's $= 7.5$
equation of regression line: $Y = 3 + 0.5X$
standard error of estimate of slope $= 0.118$
$t = 4.24$
sum of squares $X - \overline{X} = 110.0$
regression sum of squares $= 27.50$
residual sum of squares of $Y = 13.75$
correlation coefficient $= .82$
$r^2 = .67$

Figure 3

says, that in this case: "Graphics *reveal* data. Indeed graphics can be more precise and revealing than conventional statistical computations".

Seeing the unseen - in symbols and words

Visualization can accompany a symbolic development, since a visual image, by virtue of its concreteness, can be "an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p.101).

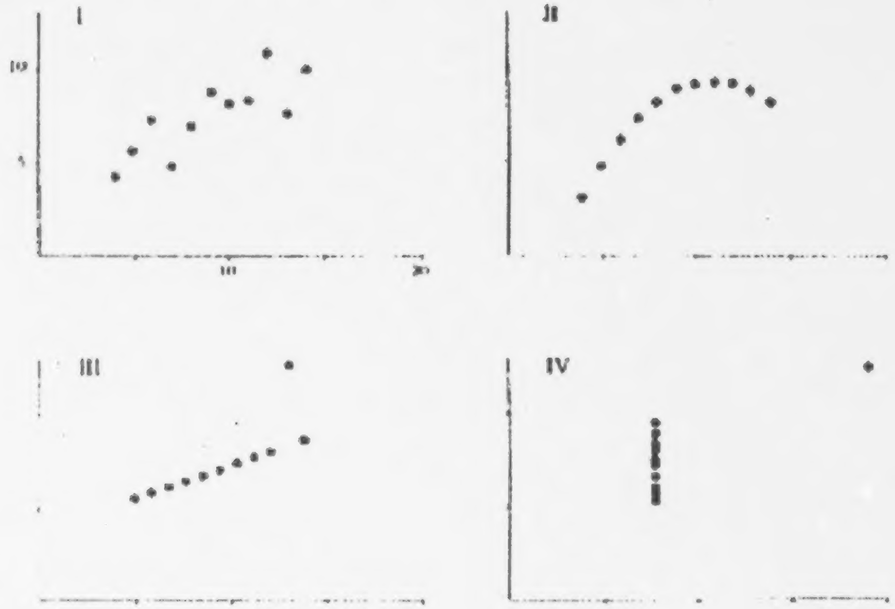


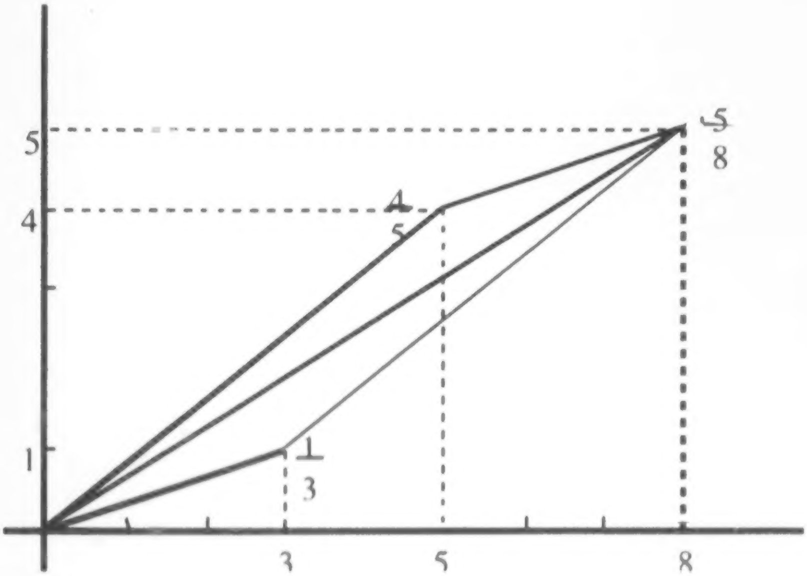
Figure 4

Consider, for example, the mediant property of positive fractions:

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Flegg, Hay and Moss (1985, p. 90) attribute this "rule of intermediate numbers" to the French mathematician Nicolas Chuquet, as it appears in his manuscript La Triparty en la Science des Nombres (1484). The symbolic proof of this property is quite simple, yet it may not be very illuminating to students. Georg Pick (1859-1943?) an Austrian-Czech mathematician wrote "The plane lattice ... has, since the time of Gauss, been used often for visualization and heuristic purposes. … [in this paper] an attempt is made to put the elements of number theory, from the very beginning, on a geometrical basis" (free translation from the original German in Pick, 1899).

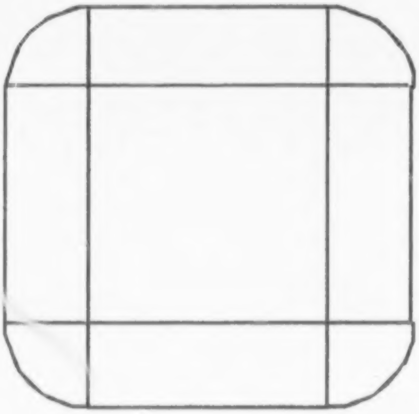
Following Pick, we represent the fraction $\frac{a}{b}$ (whether reduced or not) by the lattice point *(b,a)*. The reason for representing $\frac{a}{b}$ by *(b,a)* and not *(a,b)*, is for visual convenience, since the slope of the line from the origin O to *(b,a)* is precisely $\frac{a}{b}$, and hence fractions arranged in ascending order of magnitude are represented by lines in ascending order of slope. Visually, the steeper the line the larger the fraction. (Note also that equivalent fractions are represented by points on the same line through the origin. If the lattice point P represents a reduced fraction, then there are no lattice points between O and P on the line OP). Now, the visual version of the mediant of $\frac{1}{3}, \frac{4}{5}$, which is $\frac{1+4}{3+5}$, is represented by the diagonal of the parallelogram "defined" by $\frac{1}{3}$ and $\frac{4}{5}$.



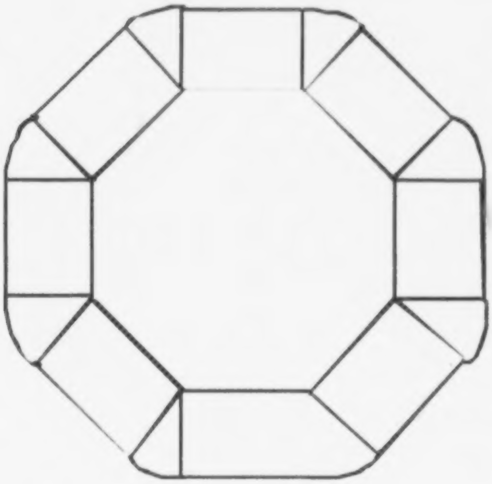
The same holds in general for $\frac{a}{b}, \frac{c}{d}$. I would claim that the "parallelogram" highlights the reason for the property, and may add meaning and conviction to the symbolic proof. In this example, we have not only represented the fraction $\frac{a}{b}$ visually by the point with coordinates (b,a) - or the line from the origin through (b,a) - but capitalized on the visualization to bring geometry to the aid of what seem to be purely symbolic/algebraic properties. Much mathematics can be done on this basis; see, for example Bruckheimer and Arcavi (1995).

Papert (1980, p. 144) brings the following problem. "Imagine a string around the circumference of the earth, which for this purpose we shall consider to be a perfectly smooth sphere, four thousand miles in radius. Someone makes a proposal to place a string on six-foot-high poles. Obviously this implies that the string will have to be longer." How much longer? Papert says that "Most people who have the discipline to think before calculating ... experience a compelling intuitive sense that "a lot" of extra string is needed." However, the straightforward algebraic representation yields $2\pi(R+h)-2\pi R$, where R is the radius of the Earth and h the height of the poles. Thus the result is $2\pi h$, less than 12 meters, which is amazingly little and independent of the radius of the Earth!

For many, this result is a big surprise, and a cause for reflection on the gap between what was expected and what was obtained. Papert wa- uncomfortable with the possible morale from this example, that our initia intuitions may be faulty, therefore they should not be trusted, and it is onl the symbolic argument that should count. His discomfort led him to propos a visual solution, which would serve to educate, or in his own words t "debug", our intuitions, so that the symbolic solution is not only correct but also natural and intuitively convincing. His non-formal and graphica solution starts with a simple case, a string around a "square Earth"

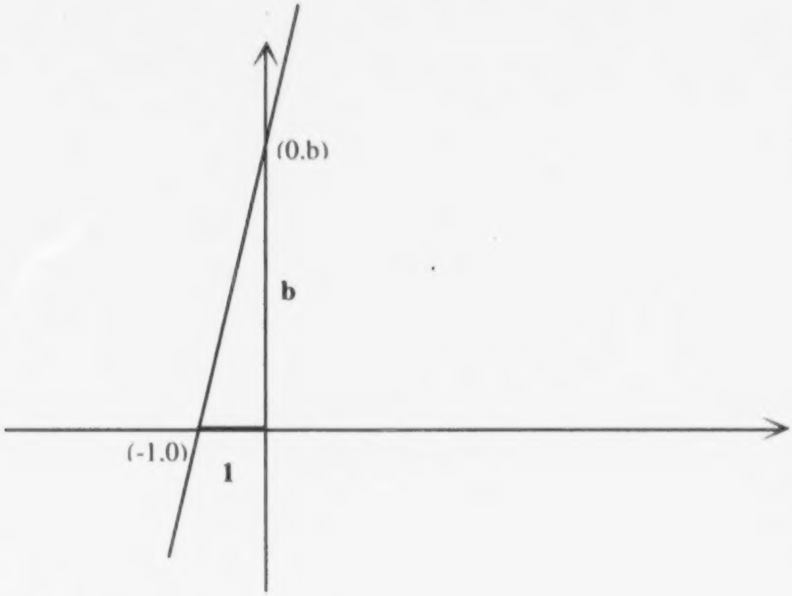


"The string on poles is assumed to be at distance h from the square. Along the edges the string is straight. As it goes around the corner it follows a circle of radius h.... The extra length is all at the corners... the four quarter circles make a whole circle... that is to say $2\pi h$." (p. 147). If we increase the sides of the square, the amount of extra string needed is still the extra four quarters of a circle. Then he proceeds to deform "continuously" the square towards the round earth. First by looking at the shape of an octagon.



The extra pieces of string "is all in the pie slices at the corners. If you put them together they form a circle of radius h. As in the case of the square, this circle is the same whether the octagon is small or big. What works for the square (4-*gon*) and for the octagon (8-*gon*) works for the 100-*gon* and for the 1000-*gon*." (p. 149). The formal symbolic result becomes now also visually (and thus intuitively) convincing. After such a solution, we may overhear ourselves saying "I see", double-entendre intended. Visualization here (and in many similar instances) serves to adjust our "wrong" intuitions and harmonize them with the opaque and "icy" correctness of the symbolic argument.

Another role of visualization in an otherwise "symbolic" context, is where the visual solution to a problem may enable us to "see", that is to engage with concepts and meanings which can be easily bypassed by the symbolic solution of the problem. Consider, for example, the following: "What is the common characteristic of the family of linear functions whose equation is $f(x)=ax+a$?" The symbolic solution would imply a simple syntactic transformation and its interpretation: $f(x)=ax+a= a(x+1)$ - regardless of the value of a, all the functions share the pair $(-1,0)$. Compare this to the following graphical solution, produced by a student. The first a is the slope, the second is the y-intercept. Since slope is "rise over run", and since the value of the slope is the same value as the y-intercept, to a rise with the value of the y-intercept must correspond a run of 1.



Sophisticated mathematicians may claim to "see" through symbolic forms, regardless of their complexity. For others, and certainly for mathematics students, visualization can have a powerful complementary role in the three aspects highlighted above: visualization as (a) support and illustration of essentially symbolic results (and possibly providing a proof in its on right), (b) a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions, and (c) as a way to help us engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions.

Foreseeing the unseen - at the service of problem solving

Davis (1984, p. 35) describes a phenomenon which he calls *visually-moderated sequences* (VMS). VMS frequently occurs in our daily lives. Think of the "experience of trying to drive to a remote location visited once or twice years earlier. Typically, one could not, at the outset, tell anyone how to get there. What one hopes for is,…, a VMS…: see some key landmark … and hope that one will remember what to do at the point. Then one drives on, again hoping for a visual reminder that will cue the retrieval of the next string of remembered directions." In this case, visualization is a tool to extricate oneself from situations in which one may be uncertain about how to proceed. As such it is linked, in this case, not so much to concepts and ideas, but rather to procedures. One of the mathematical examples Davis (p. 34) brings is the following: "A student asked to factor $x^2 - 20x + 96$, might ponder for a moment, then write

$$\begin{array}{l} x^2 - 20x + 96 \\ (\quad\;)(\quad\;), \end{array}$$

then ponder, then write,

$$\begin{array}{l} x^2 - 20x + 96 \\ (x \quad\;)(x \quad\;), \end{array}$$

then ponder some more, then continue writing

$$\begin{array}{l} x^2 - 20x + 96 \\ (x - \quad)(x - \quad), \end{array}$$

and finally complete the task as

$$\begin{array}{l} x^2 - 20x + 96 \\ (x - 12)(x - 8)\text{"} \end{array}$$

The mechanism is more or less: "look, ponder, write, look, ponder, write, and so on." In other words, "a visual clue V_1 elicits a procedure P_1 whose execution produces a new visual cue V_2, which elicits a procedure P_2,... and so on."

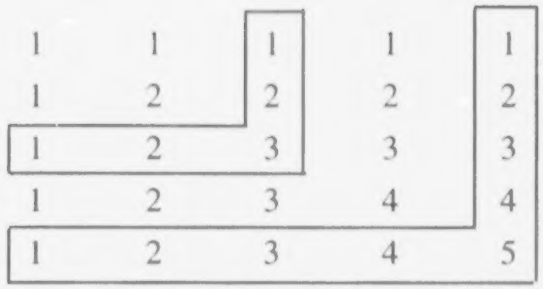
Visualization at the service of problem solving, may play a central role to inspire a whole solution, beyond the merely procedural. Consider, for example, the following problem (Barbeau, 1997, p. 18).

Let n be a positive integer and let an nxn square of numbers be formed for which the element in the ith row and the jth column ($1 \leq i, j, \leq n$) is the smaller of i and j. For $n=5$, the array would be:

Show that the sum of all the numbers in the array is $1^2 + 2^2 + 3^2 + \ldots + n^2$.

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

A solution.



"Algebraically, we see that the sum of the numbers in the kth gnomon consisting of the numbers not exceeding k in the kth row and the kth column is

$(1 + 2 + \ldots + (k-1))\, 2 + k = k^2$. The result follows." (Barbeau, p. 20)

This solution has some elements of visualization in it: it identifies the gnomons as "substructures" of the whole in which a clear pattern can be established. However, an alternative solution presented by the author is even more interesting visually.

> "We can visualize the result by imagining an nxn checkerboard. Begin by placing a checker on each square (n^2 checkers); place an additional checker on every square not in the first row or the first column ($(n-1)^2$ checkers); then place another checker on every square not in the first two rows or the first two columns ($(n-2)^2$ checkers). Continue on in this way to obtain an allocation of $n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2$ checkers; the number of checkers placed on the square in the ith row and the jth column is the smaller of i and j."

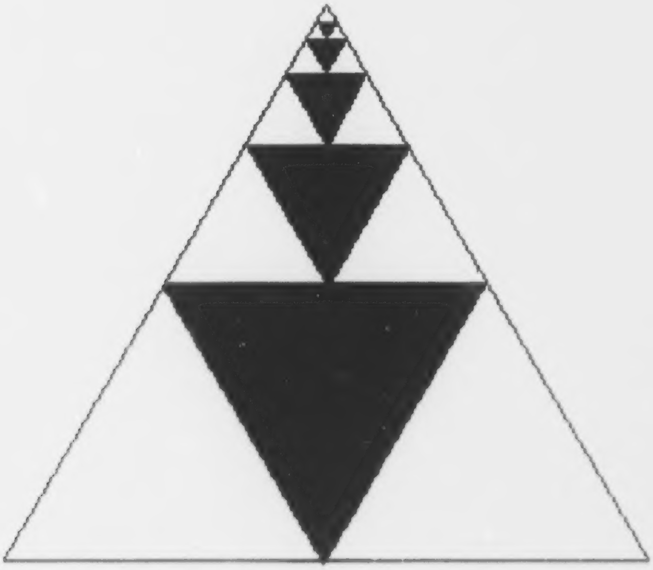
In the examples in previous subsections, visualization consisted of making use of a visual representation of the problem statement. I claim that, in this example, visualization consists of more than just a translation, the solver imagined a strongly visual "story"(not implied by the problem statement), he imposed it on the problem, and derived from it the solution.

Probably the inspiration for this visual story, was the author's previous experience and knowledge, which helped him envision the value of the number matrix as height, or in other words he probably saw a 2-D compression of a 3-D data representation. In any case, one's visual repertoire can fruitfully be put at the service of problem solving and inspire creative solutions.

Seen the unseen - more than just believing it?
Perhaps also proving it?

"Mathematicians have been aware of the value of diagrams and other visual tools both for teaching and as heuristics for mathematical discovery. ... But despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics. In particular, we are all taught to look askance at proofs that make crucial use of diagrams, graphs, or other nonlinguistic forms of representation, and we pass on this disdain to students." However, "visual forms of representation can be important ... as legitimate elements of mathematical proofs." (Barwise and Etchemendy, 1991, p. 9). We have already illustrated this in the example of the mediant property of fractions. As another example, consider the following beautiful proof taken from the section bearing the very suggestive name of "proofs without words" (Mabry, 1999, p. 63).

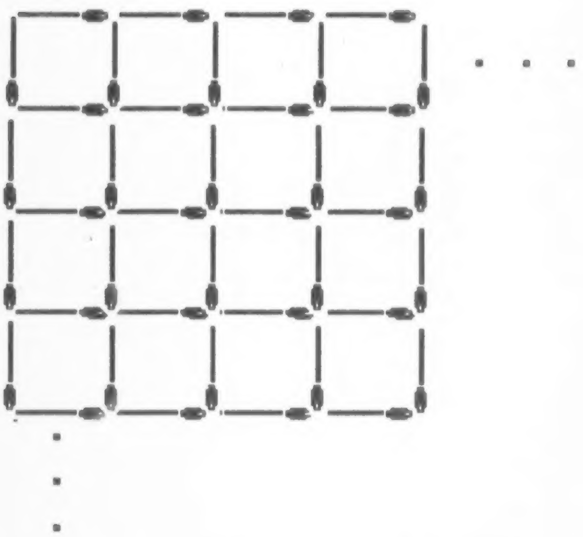
$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$



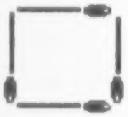
It can be argued that the above is neither (a) "without words" nor (b) "a proof". Because (a) although verbal inferences are not explicit, when we see it, we are most likely to decode the picture by means of words (either aloud or mentally); and (b) Hilbert's standard for a proof to be considered as such is whether it is arithmetizable, otherwise it would be considered non-existent (Hadamard, 1954, p. 103). As to the first reservation, we may counterargue that visualization as a process is not intended to exclude verbalization (or symbols, or anything else), quite the contrary, it may well complement it. As to the second reservation, there is a "clearly identifiable if still unconventional movement ... growing in the mathematics community, whose aim is to make visual reasoning an acceptable practice of mathematics, alongside and in combination with algebraic reasoning. According to this movement, visual reasoning is not meant only to support the discovery of new results and of ways of proving them, but should be developed into a fully acceptable and accepted manner of reasoning, including proving mathematical theorems." (Dreyfus, 1994, p.114)

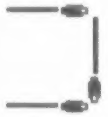
Consider, for example, the following.

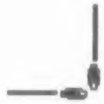
How many matches are needed to build the following nxn square?



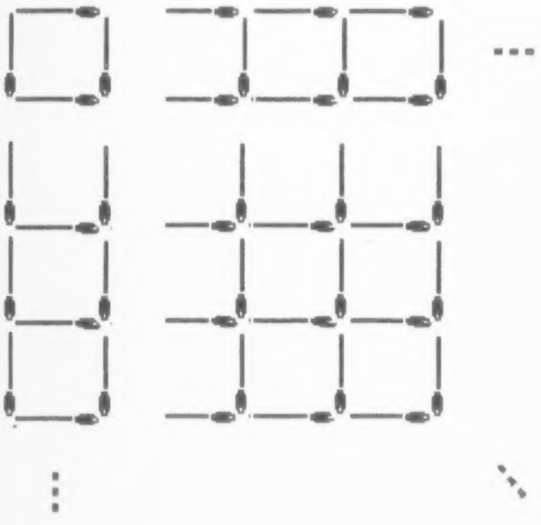
This problem was tried in several teacher courses in various countries and with several colleagues, and the many solutions proposed were collected and analyzed (Hershkowitz, Arcavi and Bruckheimer, submitted). The majority of the solution approaches were visual, yet they differ in their nature. There were those who decomposed the whole array of matches into what they saw as easily countable units. For example, a square, U's and L's,



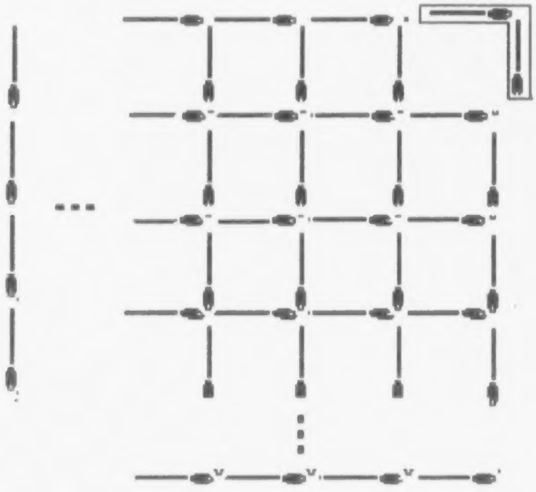




decompose the whole as follows

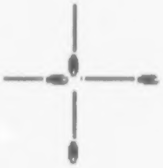


Others identified L's and single matches, and looked at the whole as

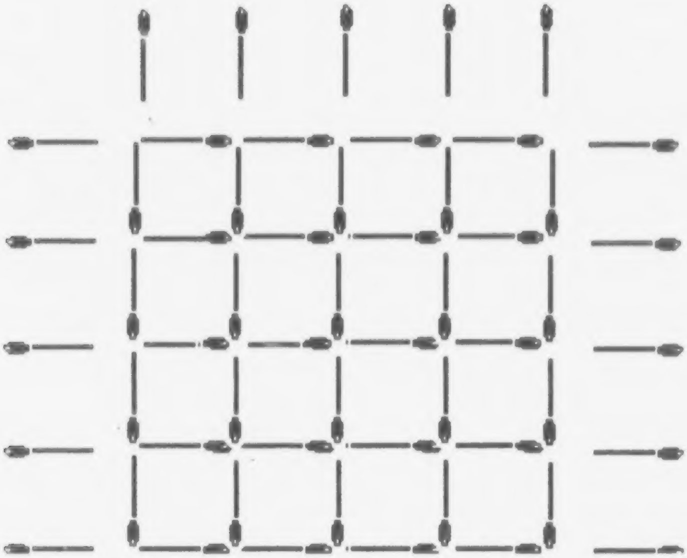


Some participants counted unit squares, and then proceeded to adjust for what was counted twice. Yet others identified the smallest possible unit, a single match and counted the *n* matches in a row (or a column), multiplied it by the *n+1* rows (or columns), and then multiplied by *2*. It would seem that the simpler the visually identified unit (one match), the more global and uniform the counting strategy became.

Some participants imagined units whose existence is only suggested: the "intersection" points,

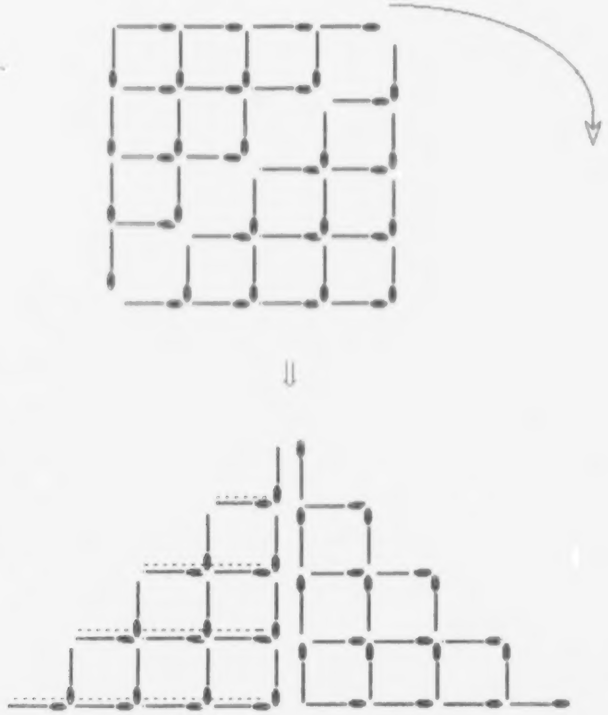


and then proceeded to make auxiliary constructions to make the count uniform

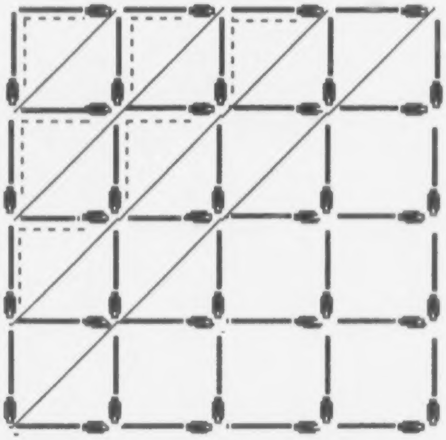


which then they adjusted for double counting.

Thus decomposition into what was perceived as easily countable units took different forms, but it was not the only visual strategy. Another visual strategy consisted of changing the whole gestalt into a new one, in which patterns are easier for the solver to identify. For example,



Change of gestalt took other forms as well: instead of "breaking and rearranging" the original whole as above, imposing "auxiliary constructions" whose role consists of providing visual "crutches", which in themselves are not counted, but which support and facilitate a certain counting strategy.



Surprisingly, visualization, for others, was sparked by their symbolic solution. Having obtained the final count in the form $2n(n+1)$, they applied a symbolic transformation to obtain $4x$. This transformation suggested the search for a visual pattern which would illustrate a counting strategy. This exemplifies how visual reasoning can also be guided, inspired and supported, by a symbolic expression, namely by "symbol sense" (Arcavi, 1994).

In sum, we found that visualization consisted of processes different in nature. However, all of them seem to corroborate Fischbein's claim that visualization "not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution." (Fischbein, 1987, p.101). We propose that visualization can be even more than that: it can be the analytical process itself which concludes with a general formal solution.

Do you and I see alike?

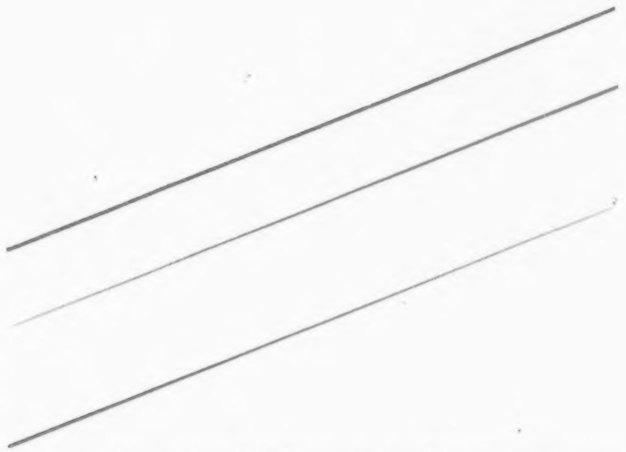
"We don't know what we see, we see what we know". I was told that this sentence is attributed to Goethe. Its last part: "We see what we know" applies to many situations in which students do not necessarily see what we as teachers or researchers do. For some this sentence may be a truism, already described in many research studies, nevertheless it is worth analyzing some examples.

Consider the following taken from Magidson (1989). While working with a graphing software, students were required to type in equations one at a time, and draw their graphs. The equations were $y=2x+1$, $y=3x+1$, $y=4x+1$, and students were asked what do they notice, in which way the lines graphed are similar and in which way they are different, and to predict (and test the prediction) about the graph of $y=5x+1$. The expectation was that the task would direct student attention (at least at the phenomenological level) to what an expert considers relevant: the influence of the number that multiplies the x, and that all lines go through $(0,1)$. Presmeg (1986, p. 44) stated that many times "an image or diagram may tie thought to irrelevant details", irrelevant to an expert that is. Some of the answers reported by Magidson certainly confirm this: there were students who "noticed" the way the software draws the lines as "starting" from the bottom of the screen. Others talked about the degree of jaggedness of the lines, which is an artifact of the software and depends on how slanted the lines are. And there were those who noticed that the larger the number, the more "upright" the line, but when asked to predict the graph of $y=5x+1$, their sketch clearly did not go through $(0,1)$.

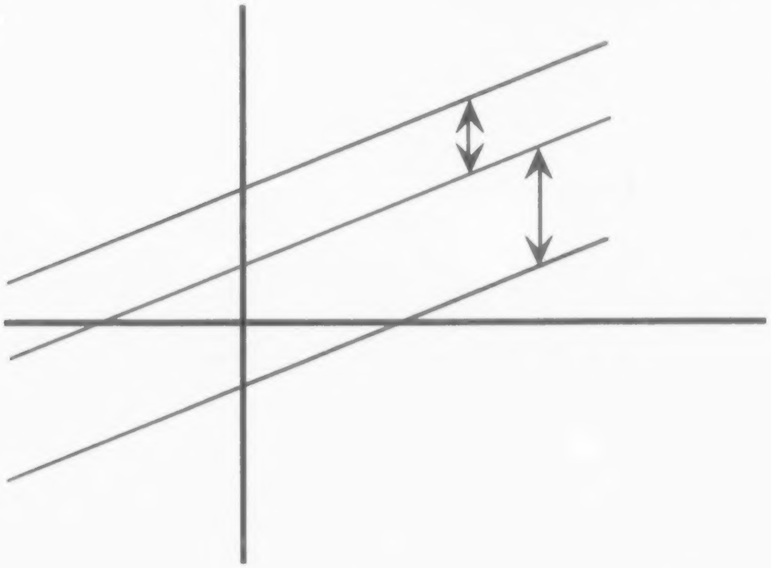
A similar phenomenon in a slightly different context is reported by Bell and Janvier (1981) where they describe "pictorial distractions": graphs are judged by visually salient clues, regardless of the underlying meanings.

Clearly, our perception is shaped by what we know, especially when we are looking at what Fischbein (as reported in Dreyfus, 1994, p. 108) refers to diagrams which are loaded with an "intervening conceptual structure". Some of the visual displays I have brought so far are either displays of objects (matches) or arrays of numbers which allow one to observe and manipulate patterns. Others were displays of data, for which a small number of ad-hoc conventions suffices to make sense of the graph. However, when we deal, for example, with Cartesian graphs of linear functions, what we look at has an underlying representation system of conceptual structures. Experts may often be surprised that students who are unfamiliar (or partially familiar) with the underlying concepts see "irrelevancies" which are automatically dismissed by the expert's vision, even to a point that they may remain unseen.

I would like to claim further: in situations like the one described by Magidson, what we see is not only determined by the amount of previous knowledge which directs our eyes, but in many cases it is also determined by the context within which the observation is made. In different contexts, the "same" visual objects may have different meanings even for experts. Consider for example, the following diagram.

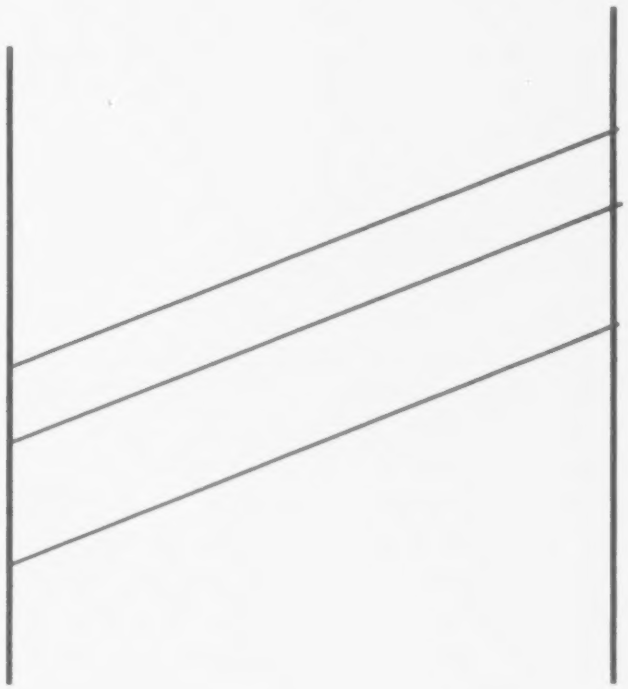


What we see are three parallel lines. If nothing more is said about the context, we would probably think about the Euclidean geometry associations of parallelism (equal distance, no intersection, etc.). Consider now the same parallel lines, with a superimposed Cartesian coordinate system. For a novice, this may be no more than two extra lines, for experts, this would probably trigger much more: the conceptual world of Cartesian representation of functions. The lines are now not only geometrical objects, they have become representations of linear functions, and hence suggest notions like, to each line corresponds an equation of the form $y=ax+b$ (or any other equivalent form), that these lines have equal slope (share the same a value), and the non-existence of a solution for any pair of equations. It may also re-direct the attention from the notion of distance between parallel lines (as the length of a segment perpendicular to both) towards the notion of the vertical displacement from one line to the other, namely



which is reflected by the difference in the b values.

If we now remove the superimposed Cartesian referents, and replace them by a system of parallel axes to represent linear functions, experts familiar with such a representation see very different things.



In this case, the three lines are the representation of just three particular ordered pairs of a single linear function. The inclination of the lines have to do with the slope (the "*a*" in $y=ax+b$) but for different reasons. The parallelism indicates that an interval of the domain is mapped onto an interval of equal length in the co-domain, namely that the slope is *1*. (Further details about the Parallel Axes Representation, and its visually salient characteristics, can be found in Arcavi & Nachmias, 1989,1990, 1993) Thus, many times our perceptions are conceptually driven, and seeing the unseen in this case is not just producing/interpreting a "display that reveals" or a tool with which we can think, as in many examples above. Seen the unseen, may refer to the development of a conceptual structure which enables us to see through the same visual display, things similar to those seen by an expert. Moreover, it also implies the competence to disentangle contexts in which similar objects can mean very different things, even to the same expert.

Visualization in mathematics education
- some unseens we are beginning to "see"

There seems to be wide agreement on the centrality of visualization in learning and doing mathematics. This centrality stems from the fact that visualization is no longer related to the merely illustrative only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving. Yet, there are still many issues concerning visualization in mathematics education which require careful attention.

Borrowing from Eisenberg and Dreyfus (1991), I classify the difficulties around visualization into three main categories: "cultural", cognitive and sociological.

The "cultural" difficulty refers to the beliefs and values held about what mathematics and doing mathematics would mean, what is legitimate or acceptable, and what is not. We have briefly referred to this issue while discussing the status of visual proofs. Controversy within the mathematics community, and statements such as "this is not mathematics" (Sfard, 1998, p. 454) by its most prominent representatives, are likely to permeate through to the classroom, via curriculum materials, teacher education etc. and shape their emphasis and spirit. This attitude, which Presmeg (1997, p. 310) calls "devaluation" of visualization, leaves little room for classroom practices to incorporate and value visualization as a an integral part of doing mathematics.

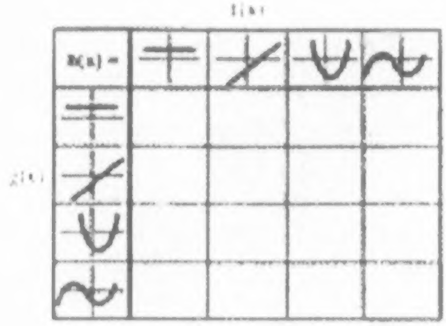
The cognitive difficulties include, among other things, the discussion whose simplistic version would read as follows: is "visual" easier or more difficult? When visualization acts upon conceptually rich images (or in Fischbein's words when there are intervening conceptual structures), the cognitive demand is certainly high. Besides, reasoning with concepts in visual settings may imply that there are not always procedurally "safe" routines on which to hang (as may be the case with more formal symbolic approaches). Consciously or unconsciously, such situations may be rejected by students (and possibly teachers as well) on the grounds of being too "slippery" or too "risky".

Another cognitive difficulty arises from the need to attain flexible and competent translation back and forth between visual and analytic representations of the same situation, which is at the core of understanding much of mathematics. Learning to understand and be competent in the handling of multiple representations can be a long-winded, context dependent, non-linear and even tortuous process for students (e.g. Schoenfeld, Smith and Arcavi, 1993).

The sociological difficulties, include what Eisenberg and Dreyfus (1991) consider as issues of teaching. Their analysis suggests that teaching implies a "didactical transposition" (Chevallard, 1985) which, briefly stated, means the transformation knowledge undergoes when it is adapted from its scientific, academic character to the knowledge as it is to be taught. It is claimed that this process, by its very nature, linearizes, compartmentalizes and possibly also algorithmetizes knowledge, stripping it (at least in the early stages) from many of its rich interconnections. As such, analytic representations, which are sequential in nature, seem to be more appropriate and efficient for teachers.

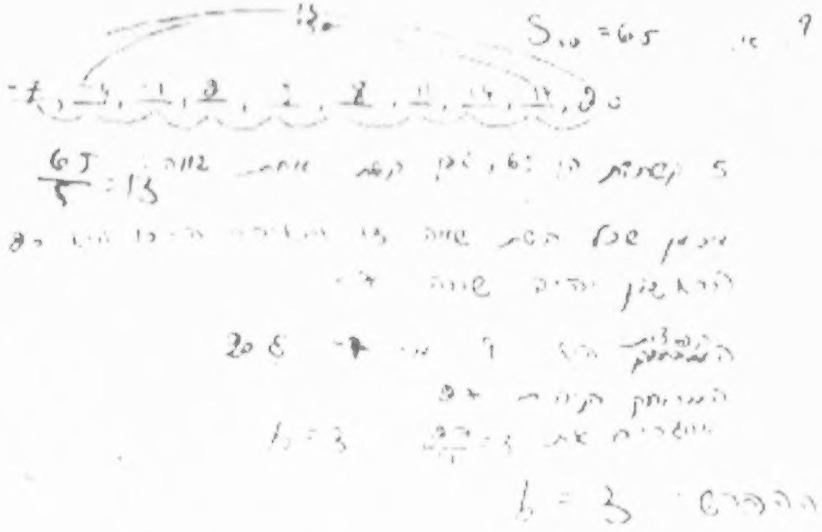
Another kind of difficulty under the heading "sociological" (or better socio-cultural), is the tendency of schools in general, and mathematics classrooms in particular, to contain students from various cultural backgrounds. Some students may come from visually rich cultures, and therefore for them visualization may counteract possible "deficits". In contrast, visualizers may be under-represented amongst high mathematical achievers (Presmeg, 1986, 1989).

Recent curriculum and research studies are taking into account some of the above difficulties and to address them, in order to propose and explore innovative approaches to understand and exploit the potential of visually oriented activities. Consider, for example, the following task from Yerushalmy (1993, p. 10),



in which the goal is to sketch the graphs of different types of rational functions of the form , obtained from the given graphs of f(x) and g(x), and analyze the behavior of the asymptotes (if any).

Arcavi, Hadas & Dreyfus (1994) describe a project for non-mathematically oriented high school students which stimulates sense-making, graphing, estimation, reasonableness of answers. The following, solution produced by a student learning with this approach we found surprising and elegant.


$S_{10}=65$
-7, -4, -1, 2, 5, 8, 11, 14, 17, 20
65/5 = 13
b = 3

Given were the tenth element of an arithmetic sequence ($a_{10}=20$) and the sum of the first *10* elements ($S_{10}=65$). The student found the first element and the constant difference mostly relying on a visual element: arcs, which he envisioned as depicting the sum of two symmetrically situated elements in the sequence, and thus having the same value. Five such arcs add up to *65*, thus one arc is *13*. Therefore the first element is *13-20=-7*. Then, the student looked at another visual element: the "jumps", and said that since there are *9* jumps (in a sequence of *10* elements starting at *-7* and ending at *20*), each jump must be *3*.

DiSessa et al. (1991) describe a classroom experiment in which young students are encouraged to create a representation for a motion situation, and after several class periods they ended up "inventing" Cartesian graphing. By being not just "consumers" of visual representations, but also their collective creators, communicators and critics, these students developed meta-representational expertise, establishing and using criteria concerning the quality and adequacy of representations. Thus visualization was for them not only to work with pre-established products, but also was in itself the object of analysis.

When a classroom is considered as a micro-cosmos, as a community of practice, learning is no longer viewed only as instruction and exercising, but also becomes a form of participation in a disciplinary practice. It is in this respect that Stevens and Hall (1998, p. 108) define "disciplined perception". Visualization by means of graphs, diagrams and models is a central theme which "develop and stabilize … in interaction between people and things". Ways of seeing emerge in a social practice as it evolves.

Nemirovsky and Noble (1997) describe a research study, in which, a student makes use of a physical device which served as a transitional tool used to support the development of her ability to "see" slope vs. distance graphs.

In sum, new curricular emphases and approaches, innovative classroom practices and the understandings we develop from them, re-value visualization and its nature placing it as a central issue in mathematics education. This should not be taken to mean that visualization, no matter how illuminating the results of research, will be a panacea for the problems of mathematics education. Some difficulties may be solved, others not. However, understanding it better should certainly enrich our grasping of aspects of people's sense making of mathematics, and thus serve the advancement of our field.

Paraphrasing a popular song, I would suggest that "visualization is a many splendored thing". However, borrowing the very last sentence from (the English poet) Thomas Gray's (1716-1771) poem entitled "On the death of a favourite cat, drowned in a tub of gold fishes" (whose story can be easily imagined), I would also add

> "Not all that tempts your wand'ring eyes
> and heedless hearts, is lawful prize;
> Nor all, that glisters, gold."

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DISCUSSANT'S COMMENTS: ON THE ROLE OF VISUAL REPRESENTATIONS IN THE LEARNING OF MATHEMATICS

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Making visual representations of things is a natural cognitive activity, which is valued for good reasons. For example, visual images can facilitate the recall of facts and events; they can also be crucial in the search for solution(s) to mathematical problems (Polya, 1945). Not only that, anyone who has studied mathematics can, in all likelihood, recall an instance or two where a visual clue made all the difference in his/her learning of a mathematical concept or procedure. In spite of these advantages, "visual representation remains a second-class citizen in both the theory and practice of mathematics" (Barwise and Etchemendy, 1991). Most mathematicians just distrust visual thinking, and look down on proofs that make heavy use of that form of thinking. Professor Arcavi seems intent on changing these old attitudes.

In his paper, Professor Arcavi makes a strong case in support of a small but growing group of mathematicians who want to "make visual reasoning an acceptable practice of mathematics, alongside, and in combination with algebraic reasoning."

To begin his enlightening exploration of visualization and its role in the learning of mathematics, Arcavi defines visualization as "the ability, the process and the product of creation, interpretation, and use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings." He then identifies three roles which visualization may play in the learning process. Visualization, he asserts, can serve as (a) support and illustration of essentially symbolic results (and possibly providing a proof in its own right), (b) a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions, and (c) a way to help us engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions" to problems. To defend these claims, Professor Arcavi uses engaging problems to show real situations where visualization actually plays these roles. I find these illustrations quite compelling; they make a case with which few, if any, in this assembly would disagree.

The professor then turns his attention to three types of difficulty, which the concept of representation faces in the mathematics community. The first type stems from cultural (mathematics culture, that is) beliefs and prejudices about the nature of mathematics and what it means to do mathematics. As indicated earlier, most mathematicians do not consider visual thinking as any form of mathematical thinking. Instead, they see it as too "slippery" and, therefore assign low status to research on mathematical representation.

I think that the widespread misperception of visualization may be due in part to the different definitions of the word. The Oxford English Dictionary, which shapes popular understanding of the meaning of English words, defines visualization as "the power or process of forming a mental picture or vision of something not actually present to the sight"; or "a picture thus formed. When one compares Professor Arcavi's definition with that of the Oxford English Dictionary, it is obvious that the two definitions refer to different phenomena. It is quite possible that the acceptance of the former, richer definition may lessen the level of opposition to representation within the mathematics community.

Another type of problem arises from the slow but challenging task of learning to make competent translations between visual and analytic representations of the same problem. Still a third type involves the issue of equity for students from visually rich cultures. I think this last issue is best dealt with by re-valuing visualization in mathematics.

Although Professor Arcavi makes a very strong case for using visual thinking in the learning and doing of mathematics, I am not sure that his reasons will change my minds. Feelings of opposition those boarders on prejudice seldom yield to reason. Thank you.

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CONCEPT AND REPRESENTATION IN THE RESEARCH ON PROBABILITY EDUCATION

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Abstract. This work refers to the way in which elements in the constitution of concepts in the individual are being considered in a research project on stochastics in education. Emphasis is given to the link between concepts and representation by means of semiotic registers. Results from research exhibits how the way of using semiotic registers in classroom sessions on probability can hinder further development of the concept at issue and may impose constraints on the students' interest in the study of random phenomena.

Introduction

As a result of an investigation carried out on students' understanding of fundamental ideas of probability at pre-university level (Ojeda, 1994), a research project on stochastics in the Mexican system of education has been conducted over the last five years. Three factors are at the core of the investigations: the epistemological, the psychological and the social factors. The first one concerns the conceptual development of stochastics, either from a phylogenetic or an ontogenetic point of view. The psychological factor considers reasoning and biases when individuals face uncertainty. Very briefly stated, we assume in our investigations that knowledge results from social interaction, and that knowledge is actively built in, that is, knowledge constitution demands the individual's active involvement. Therefore, the three factors we have been considering are closely intertwined, although some of our studies may rather stress one or two of them.

The epistemological stance in our research points to the difficulty of considering a conceptual system provided by the mathematical model vis-à-vis the individual, for him or her to understand the basic ideas it involves and to apply this model. That is, we are concerned with a conceptual system *produced* by mathematicians, the fundamentals of which is expected to be *re-produced* by the educational system. This problem, including the way in which it is decided what is "fundamental", has been discussed by Chevallard (1991).

As researchers in mathematics education, our task aims at the discovery and interpretation of facts and phenomena in mathematics education, in order to understand and to improve the teaching and the learning of mathematics. In this task, an educational system, student and/or teacher and means and conditions to communicate ideas, are to be considered in respect to a specific mathematical content. This mathematical model is a system of abstract entities and relations logically and deductively stated, from axiomatic grounds, as definitions, theorems, corollaries, propositions. The terms we use to refer to the mathematical content in relation to the individual, such as *notions, ideas, concepts*, are not deprived of ambiguity. Often they are indistinctly used. For instance, *concept* is defined in the dictionary as something *conceived* in the mind, as a thought, a notion; as an abstract or generic idea generalized from particular instances (Merriam-Webster's Dictionary). However, in order to take into account different levels of abstraction in the individual's intellectual activity, we refer to notion, then idea and finally concept.

The term concept has been borrowed from philosophy, in particular from the analytic school of philosophy, where concept is a logical entity. From a social perspective regarding knowledge and its constitution in the individual, Sfard (1996) quotes Foucault to characterise *concept* in discursive terms, as '...a virtual entity "constituted by all that was said in all the statements that named it, divided it up, described it, explained it, traced its developments, indicated its various correlations, judged it ..."' (p. 403). In addition to the conditions that our sensorial system imposes on the way we perceive our reality (Schmidt, 1996, pp. 386-388), and to our daily experience (in a social community), by means of education we have a view of the world structured from projecting over it (the world) our concepts. Once certain concepts are introduced in a determined way, we only can use them by following the profiles that reality adopts by projecting over reality those concepts (Mosterin, 1964, pp. 11-39); that is, we consider reality according to the conceptual schema we use for that consideration. A concept is a rule that may be applied to decide if a particular object falls into a certain class. Concept formation refers to the process by which one learns to sort one's specific experiences into general rules or classes; whereas conceptual thinking refers to one's subjective manipulations (that is, to treat, relate or operate with) those abstract classes.

Fundamental ideas of stochastics

As a general orientation for our project, we have regarded ideas of probability and statistics as fundamental in the sense in which Heitele has spelled them out in his proposition for the curriculum of stochastics (1975). That is, as those ideas that provide the individual with an explanatory model of the (random) situation with which she or he is concerned, with no changes in essence but in their linguistic presentation and sophistication at the different levels of his or her conceptual development. We could add to this characterisation the meaning of *model* as "a system of postulates, data, and inferences presented as [or that provide the individual with] a mathematical description of an entity or state of affairs", after The Merriam-Webster's Dictionary. More specifically, Heitele proposed as fundamental ideas for the curriculum of stochastics the following: norming our beliefs (in the mathematical sense of norm), sample space, addition of probabilities, independence and the product of probabilities, equiprobability and symmetry, combinatorics, random variable, the law of large numbers, sample, urn model and simulation. It is by posing in the teaching of probability rich situations from whose study several interrelations among these ideas could be laid out, that chance, and probability, can be put into focus (Ojeda, 1994; González, 1995; Alquicira, 1998).

However, this general guide is not deprived of the main drawback one has when facing chance, since a duality lies at the very meaning of probability. The fact that the meaning of probability has so much attracted the attention of philosophers and scientists of all times (e. g. Hacking, 1975; Krüger *et al.*, 1987) suggests that probability requires a different way of thinking from the one needed for other mathematical concepts. Lack of instruction in this subject matter would result in an incomplete background to face a wide range of world situations. Freudenthal pointed out that "the usual mistakes in this field differ greatly from mistakes in mathematical techniques. Those who never had the opportunity of making these mistakes, also did not either get the opportunity to *unlearn* them" (p. 587).

Yet this complexity does not imply that education in probability be beyond the scope of children before the age of preparatory or university level. On the contrary, delay of instruction in this discipline may result in the rooting of misconceptions (Fischbein, 1975). Even pre-school children and children aged 6-8 years old can be posed activities involving chance (Limón, 1995; Gurrola, 1998), although appropriate teacher training should be required (López, 1998).

Freudenthal expressed the importance of probability and statistics in the mathematics educational task (1973, p. 581) stating that "probability provides the best opportunity to show students how to mathematize, how to apply mathematics - not only the best, but perhaps even the next and last opportunity after elementary arithmetic ..." (p. 592), as this topic is the privileged field of mathematics applications. Still research in primary education has offered evidence that a correct performance with fractions does not imply a correct performance in probability, and that correct insight in probability may occur without a correct handling of fractions (Perrusquía, 1998).

Freudenthal recognized the difficulty involved in the concept of probability, and expressed this by quoting Poincaré (1896) at the beginning of his discussion:

> *Calcule des Probabilités. Première Leçon. I. L'on ne peut guère donner une définition satisfaisante de la Probabilité* (p. 1)
>
> *Le calcul des probabilités offre une contradiction dans les termes qui servent à le désigner, et, si je ne craignais de rappeler ici un mot trop souvent répeté, je dirais qu'il nous enseigne surtout une chose: c'est de savoir que nous ne savons rien. Fin.* (p. 274)

The epistemological triangle

There is a double conceptual dependence that comes to the fore whenever the meaning of probability is at issue, as Hacking has expressed it: "[o]n the one side it is statistical, concerning itself with stochastic laws of chance processes. On the other side, it is epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background." (1975, p. 12). From a formal point of view, Bernoulli's theorem (the weak version of the law of large numbers) accounts for what probability is. This mathematical result can be used to express the duality of the concept of probability, since a gradually established statistical regularity is explained by the limiting *a priori* probability *p* which, in its turn, is explained by the tendency of the relative frequencies shown "in the long run". Nevertheless, it is this duality (an empirical/*a priori* idea) at the core of probability which seems to us to provide the best example of the way in which Steinbring has schematised the constitution of mathematical concepts (1997): as gradually built in from an interplay between contexts of reference (objects), symbols (signs) and the concepts themselves from previous stages in their constitution. He refers to this basic scheme in the constitution of mathematical concepts as the "epistemological triangle" (1997). With one example, Figure 1 freezes this spiral-like process, which evolves whenever the individual overcomes epistemological difficulties that force him to refine, modify or even to change his or her previous conceptual scheme. Figure 1 shows the way in which we identify the elements in the triangle with regard to the analysis we carry out in our researches. The example concerns the law of large numbers as it is proposed that secondary school teachers introduce it in the classroom setting. That is, to carry out sequences of Bernoulli's trials (for instance, tosses of a coin) and to register the occurrence of the outcome in each trial (heads or tails) in a diagram (sign) in order to focus on the gradually attained stability of the relative frequency (concept) (SEP, 1993). In the following section we explain how to register the outcomes.

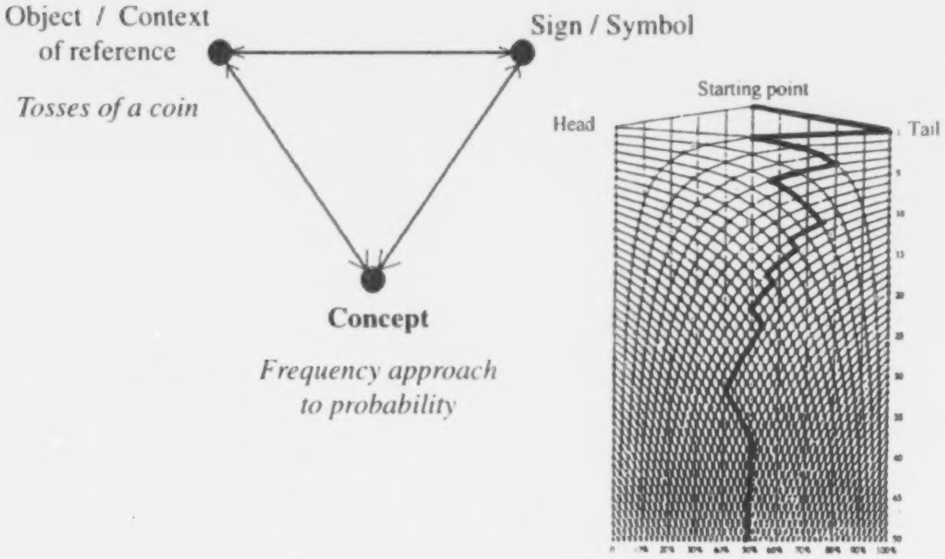

Figure 1. Example of the identification of the elements in the epistemological triangle.

The *object* toward which the individual's action is directed, or the *context* to which his or her activity *refers*, must be distinguished from the signs (symbols) used to denote the attribute or attributes at issue. Equally, in this scheme the object (or context of reference) must be distinguished from the concept. It is from the *reinterpretation* of the result obtained, or of the actions executed with regard to the object, -that is, to explain or to judge the result or the actions- that the constitution of the concept, or its evolution, takes place; concept formation builds on itself. This interpretation underlines the dynamic character of the process in the sense that it involves not only a sequence of actions to be reconsidered (reviewed) but their connection as well.

Three factors have to be pointed here. Firstly, that there is no restriction on the nature of the object, as its degree of abstraction can vary from physical world to conceptual objects, in particular, to the mathematical concepts themselves; that is, there is a transition from context-dependency to context-independent for the concept to evolve (Steinbring, 1998). Secondly, that it is the individual him or herself who is compelled to re-interpret his or her result and actions with regard to the object/context of reference in order for the concept to evolve. The individual's activity can refer to a mathematical concept, but the reinterpretation of the actions or of the result obtained regarding the object results either in its being refined, corrected or even discarded, depending on the degree of correspondence of previous experiences (with respect to that concept) and the reinterpretation made.

Thirdly, it seems necessary that notions be rooted by facing at first concrete physical situations from which a kind of ontological control could be established over the development of concepts, for the individual to make sense of his activity at every stage of abstraction. Notwithstanding this claim has been made on the grounds of theories of genetic epistemology (Piaget, 1951), which have greatly influenced curriculum design, it seems to be a postulate often neglected. In the case of probability, an example of overlooking this recommendation is the presentation of the classical definition for the *a priori* probability, as if this idea be innate, that is, as if it had emerged without any need of empirical acquaintance. Moreover, in the case of probability, it seems that contravening this order can result in difficulties in understanding the law of great numbers even at university level. For instance, even though the classical definition of probability follows from a logical reflection about the geometric properties of a physical random device (a die, coin, a pin, a spinner) considered as "ideal", taking it as the first step for teaching the law of large numbers instead of experiencing with the actual occurrences of possible outcomes from successions of independent Bernoulli's trials using one of such devices, can result in anchorage in the idea of equiprobability even when facing contrary evidence of relative frequency tendencies from long sequences of trials (De León, in progress with students of Social Sciences).

The epistemological triangle suggests that the evolving process of recurrence among object, sign and concept, results in the definition of a mathematical entity, which is more and more precise to the extent to which the individual has to reinterpret, to actively consider, his or her actions regarding the different aspects of the concept. However, this indication is not necessarily observed in probability education, and it is common instead to have the teaching of probability starting from definitions, even in open educational instances (e.g. see Vázquez for the case of Mexican TV secondary education, 1998).

It is worth stressing here the importance of the context of reference in the teaching of probability. Freudenthal pointed this out by stating that there should be awareness of the value of the isomorphism of problems (the same formal problem presented in different contexts) for the constitution of the mathematical entities. It is regarding the context of reference that psychological factors may give an account of drawbacks for the selection of the attributes from the context of reference concerning the concept. For instance, chronological order among events may be a drawback to understanding conditional probability (Ojeda, 1994; 1998).

We have referred to the epistemological triangle for the analysis of didactical activities on probability proposed for elementary and university mathematics education.

The role of semiotic registers of representation

In the epistemological triangle, the context of reference is as important as the signs we use to represent the attributes with which we are concerned and to have a physical support for establishing and keeping track of the logical relations among them.

Natural signs are constituted according to experience; they suggest to us the actions to be carried out. Hence, the interpretations that children make of the tasks they are asked about may not inform about their understanding of that task as was intended. Among other aspects, Gurrola (1998) studied the answers of children aged 6-7 years given to questions concerning the idea of chance, by proposing to them a seesaw trial to mix randomly equally sized marbles in two colours. The trial had a small divider in the middle as a reference for the arrangement of the marbles. The device was presented to a child, Almendra, at first showing all the marbles in one colour on one side, and all the marbles in the other colour on the other side. After several movements to and fro of the trial, it seemed that the girl did not have a notion of chance, which was in agreement with Piaget's results (1951). However, it turned out that she was focusing the task as a game to control the arrangement of the marbles since, pointing to the divider, she stated that *this is meant to keep them in their place*. Therefore, further questioning once the divider was removed revealed that she did have an idea of chance, what was confirmed with the explanation she gave using a drawing of the trajectories of the marbles that she did (see Figure 2).

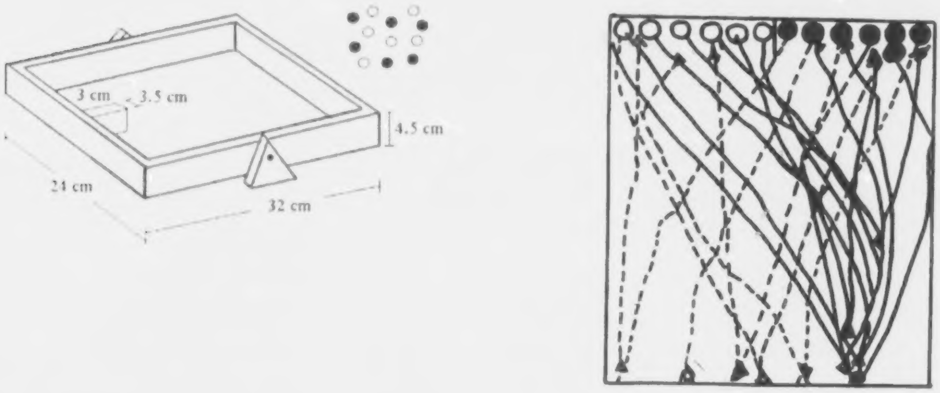

Figure 2. Figurative register produced from the random mixture experiment.

The signs we produce result from the need to overcome the limitations of natural signs (availability, being unwieldy) and to fix aspects of the context of reference, in order to focus on and to establish logical relations either among those attributes or with others. For secondary school pupils, labels for the relevant elements of a concrete physical situation (urns) had a mediator role between the situation and the diagrams that pupils produced to solve combinatorics problems referred to that situation (Elguea, 1998).

Written or printed forms are physical existences with a value of representative of meanings (of those attributes); they are produced under control and suggest the actions to be carried out in order to attain a target (to answer a question, to solve a doubt).

From the cognitive point of view, for the students to be introduced to the basic stages of the subject of probability, the use in teaching of means of organising the relevant information about a particular situation (context of reference) and to treat their data, results in prompting their mathematical activity (Ojeda, 1994). In more general terms, for the individual to develop and to communicate a mathematical activity, a system of signs, a semiotic register support, is necessary. A semiotic register, according to Duval (1996), constitutes a system of representation if it allows three cognitive fundamental activities: its production, an inside treatment, and an in-between treatment or conversion between different semiotic registers. The semiotic registers used in the mathematical activity are the algebraic, the graphical, the figurative and the natural language. Different aspects of a mathematical concept and of its levels of sophistication (formal structure) demand the staging of particular semiotic registers (Duval, 1996). For example, whereas the formal notation for Bernoulli's theorem (let ε, $\eta \in \Re$, ® $\varepsilon > 0$, ® $\eta > 0$, $\exists N \in \mathbf{N}$ ™ ® $n > N$, $P(|f_n - p| < \varepsilon) > 1 - \eta$) allows a more analytical and precise description of probability as a limit of relative frequencies, the diagram in Figure 3 shown some paragraphs below prefigures this result as an account of the frequency approach to probability for a particular random situation (a sequence of tosses of a coin).

Figurative registers provide a global structured organization of the relevant information for the students to have support to articulate their (mathematical) activity and to carry out a consecutive sequence of actions. For instance, at the same level of abstraction of conditional probability, the use of Venn diagrams with areas corresponding to the probabilities of the events represented, resulted in preparatory students understanding better the theorem of total probability, whereas tree diagrams favoured their understanding of Bayes' theorem (Barrera, 1994). Even though the work realized was complemented with the use of algebraic and numerical registers, the former figurative registers had a mediating role between the situation posed and the use of the mathematical results presented by means of formal notation, as the labels did in the case of secondary pupils.

However, the didactical value of the use of semiotic registers different from the algebraic register is not to be neglected with regard to the latter. More over, the semiotic registers cannot be used indiscriminately, without awareness of what they should be expected to suggest to the students, for their mathematical activity to be prompted. It is necessary to understand the way in which one can form and express with semiotic registers the specific mathematical result at issue vis-à-vis the cognizing individual. For instance, a tree diagram with proportionally varied scale so to allow the register of the outcomes from a relatively large number of trials of Bernoulli (50) is proposed in the guide for teachers (SEP, 1993) with the following instruction: *From the starting point, draw a line to the next point on the right if the outcome on the right occurs, and to the left if the outcome on the left occurs, and so on to the end* (p. 373). It consigns the number of trials on the right side and, at the bottom, the percentages corresponding to each branch, that is, the percentage of total occurrences of the outcome on the right, or path, in 50 trials. Figure 3 shows the resulting path using correctly the diagram for tosses of a coin, whereas Figure 4 was drawn by secondary school pupils during a classroom probability session, after the teacher's instructions (for each trial as quoted above), and according to his conducting of the class (Alquicira, 1998). The way in which this graph was drawn by the students prevented them from following the sequence in which the outcomes were occurring, as for each draw, the corresponding line started at the top of the graph; therefore, instead of one branch (path), there were as many branches as trials were carried out. Thus, no treatment of the register was possible. The diagram in Figure 4 does not allow one to thoroughly reconstruct the results of the sequence of trials; a precise reference to "number of occurrences" (as a random variable) is not possible there, as it can just be done by counting the "peaks" in Figure 3. Even more, an analytical way to obtain different paths ending in the same percentage, or in slightly different percentages, hence emphasising the idea of chance, cannot be worked out with Figure 4. The session in which this diagram was obtained was as follows. The teacher proposed that the class draws at random, with replacement, from an urn containing marbles in two colours, but in proportions unknown to the students:

> *We are going to carry out an experiment with the urn; you are going to mark on the graph* [he shows his own copy to the students] *with a line from the starting point, where the first point is in the*

> *first curve. Let's see; draw a line dividing at the middle* [he points at an imaginary vertical line going down from the starting point on the top of the graph].

Thus, since the beginning, the class accepted (as they drew the line) what they were supposed to discover on the grounds of facts, that is, the proportions of the two kinds of marbles in the urn.

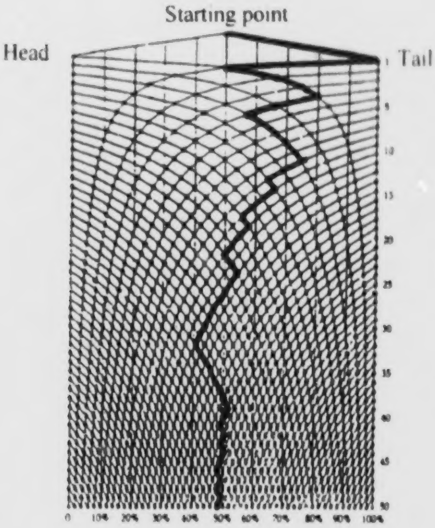

Figure 3. Tree diagram register sheet, produced correctly.

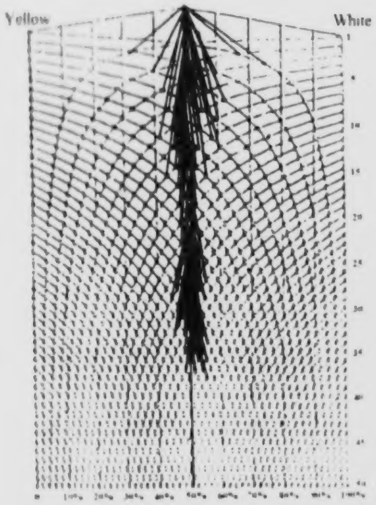

Figure 4. Pupil's graph after teacher's instructions.

> *On the left, if you agree, we are going to register the* [occurrences of] *yellow marbles and on the other* [the occurrences of] *the white marbles. ... Let's carry out the experiment as many times as possible, and you are going to repeat* [say] *by watching the graph if there are more of these or of these* [he shows a marble of each colour], *and then we'll try to agree on how many of each colour. The graph may say if there are more of these or of these. And then we'll measure how many in all.*

After some draws were recorded, the teacher stated:

> *More we draw, more it's going to approach the corresponding probability, isn't it?*

Having completed the sequence of 50 draws, the teacher said to the class:

> *This graph is pointing at 50%. Of course, that's probable, isn't it? It's not exact, it can't be exact ... We see there is a fluctuation, but it tends to 50%. The more repetitions we do, the more it's approaching 50%; that's a personal appreciation.*

At least three points may be considered here. Firstly, that the pupils were not given the opportunity to find in the tendency of the data a suggestion to relate the number of occurrences of the two possible outcomes in one trial with the *proportion* of the two kinds of marbles inside the urn, and to reflect about the conditions under which the phenomenon can be repeated; that is to say, to take the graph as a sign of the evidence obtained from the draws. Secondly, that transgressing the order by announcing "the" answer beforehand, deprived the activity of incentive for the students to make sense of the task proposed, to appropriate the question posed. Thirdly, this question was not answered at the end.

In terms of the epistemological triangle, the aim of the activity is misguided since the initial drawing of a line through the middle that the teacher proposed: using *his* knowledge of the proportions of the marbles in the urn (the context), he favoured the sign corner over the context of reference corner in the epistemological triangle. The class did not interpret the resulting 50% in terms of the context of reference (the composition in the urn). Finally, the teacher's remark at the end of the session regarding the progressive tendency towards 50% as to be taken as *personal appreciation* (opinion), withholds the activity from objectivity, whose search should be the aim when facing chance. As Hacking (1975) puts it, "[t]he old medieval probability was a matter of opinion ... [which] was probable if it was approved by ancient authority" (pp. 43-44).

Remarks

An account of the link among aspects of concepts (the elements, the relations they involve and their interpretations regarding different contexts), and their expression by means of semiotic registers can provide the research with grounds to carry out systematic analysis in order to complement the information concerning the understanding of those concepts.

The attention to registers produced by the individuals themselves involved in the research at issue is a way in which either their misinterpretations of the situation they are posed may be revealed or the information about their understanding of it may be complemented. However, research is also needed concerning the constitution and transition in the individual from natural signs to artificially produced signs (we refer here to written or printed signs).

Teacher training is needed on the use of semiotic registers in order to profit in probability education from the advantages that could be derived from didactical activities whose designing includes intentionally combining these resources on the basis of the aspects of the concepts at which they aimed. In the same vein, designers of textbooks and guides for teaching should be aware of the potential that the use of semiotic registers can supply to prefigure concepts of stochastics, and should provide instructions for that use accordingly.

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ON REPRESENTATIONS AND SITUATED TOOLS

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Introduction

Since ancient times, philosophers have dealt, in their attempt to explain the issues that arise from the theories of knowledge, with the idea of *representation*. To illustrate the importance of this notion, we can mention that the *Encyclopaedia Britannica* has 1741 references of the various ways in which the idea of representation is used in different disciplines. Representation, however, is a slippery concept. And because of its long past, using Seeger's words (Seeger, 1998, p.311), "it does not seem possible just to define what is understood as representation and make a fresh start".

In modern times, Kant's epistemology introduced an important distinction at the core of what is understood by representation. In his *Critique of Pure Reason* (1797), Kant asserts that whenever a person is in contact with the world, the impressions he/she receives are put through an organizing process by the innate cognitive structures. In much the same way as a liquid is given shape by the container that holds it, so are the sensory impressions shaped by the innate cognitive structures that process that information. Our knowledge of the world is therefore just an *interpretation,* as given by our intellect, of that external reality. In Kant's view, knowledge of reality is not an isomorphic copy of that reality. The idea of knowledge as a reflection in a mirror is abandoned. Knowledge is now considered a *representation*, a *map* of a territory, not the territory itself, as is described by Korzybsky (Le Moigne, 1995, p.69). Kant believed that all objects of sensation must be experienced within the limits of space and time. All objects therefore have a spatial-temporal location. Because space and time are the backdrop for all sensations, he called them *pure forms of sensibility.* This approach to the theory knowledge had a profound impact from its very inception. The cognizing subject was given a central role in the production of knowledge; this became the foundation for contemporary constructivist epistemology —which answers many of the issues raised against the Kantian school.

In Western culture, it is in the Renaissance that the individual comes to the foreground, and with it the conception of knowledge as a phenomenon centered in the individual subject. It would not be until much later, in the first decades of the twentieth century, when, as a consequence of various social and cultural movements of the nineteenth century, that new conceptions of the human being arose. These new conceptions rejected the inmutability of ideas such as "time" and "space"; they introduced a contingent element in the nature of knowledge, and with it, a more diverse interpretation of the term representation.

In order to understand a human being, it would no longer be enough to study his present: it became necessary to take into account the various genetic domains that constituted his history. Different theories then emerged, such as that of epistemological constructivism and the social-cultural approaches, all of which –from different cognitive perspectives–, gave a primary importance to the genetic domains.

We should comment that the existence of these two different approaches: one which emphasizes the role of the *subject* in cognitive activity, and the other which puts the emphasis on the *social-cultural dimension,* has been reflected in the educational field as an almost irreconcilable tension between cognition and culture. In our view, this is an unfortunate state for education; as Noss and Hoyles (1996, p.107) explain: "if teaching is to figure alongside learning, we have as much to gain from Vygostky as we do from Piaget". We need to be able to articulate these different points of view in a way which would allows us "to treat mathematical learning as both a process of active individual construction and a process of enculturation" (Cobb et al., 1997). This is not an easy task.

On Signs and Representations

The production of signs and representations is crucial in order for human beings to be able to assimilate what is external to them and communicate the results from those assimilations, to other human beings.

The use of signs and representations inside a culture gives them a conventional character and an agreed meaning. It is thus that a ring may be an *indication* that the person wearing it, is married. Depending on the nature of the relationship between the sign and the object represented, signs are differentiated, according to Pierce, into *icons, indexes, and symbols* (Deacon, 1997, pp.70-71). Because this denomination *is dependent on the relationship* of the sign with an object, no sign is intrinsically an icon, an index, or a symbol. "The differences between iconic, indexical and symbolic relationships derive from regarding things either with respect to their form, their correlations with other things, or their involvement in systems of conventional relationships" (p.71).

We would like to suggest then, that the systems of representations that we use —specially those we use in mathematics— have a cultural origin and therefore, so does the knowledge which is produced with their help. It is important to point out that this assertion does not compromise the objectivity of knowledge. It does force us, however, to reformulate the issue of objectivity in terms which differ from those inherited from the epistemological realism.

There is wide evidence that suggests that the sign systems play a fundamental role in the development and use of concepts. They act as *mediators of thought.*

Sign systems offer users the possibility to approach a problem in diverse ways (according to the sign system being used). In a learning situation, signs are part of the structuring elements in the interaction between the subject and the emerging concept. By changing a system of representation we could highlight different characteristics of a concept. Metaphorically speaking, we could say that each system of representation allows us to see a different facet of the object-concept being studied.

On Cognition and Historical Development

Theories of learning have to respect a fundamental principle: cognition is mediated by tools either material or symbolic (Wertsch, 1993). Technology, in all its forms, modifies, substantially, the process of knowledge production.

Learning involves the construction of representations. It is through the construction of representations of an observed phenomena, (or of a mathematical concept) that we make sense of the (mathematical) world. *Representations become mediational tools for understanding.*

It is possible to support these ideas with some historical examples. Once I say this I have to make clear that I am not suggesting to transfer this support from a genetic domain to another one.

The decimal representation of numbers is, perhaps, one of the best examples of how an adequate symbolic representation becomes an instrument with which to explore: The reasoning is (almost) impossible without this representational system.

The geometric continuum appeared, in Euclidean mathematics, as an abstraction of the physical continuum. Because of the characterization of continuity as neverending divisibility, it was possible to conclude that the continuum was not made of indivisibles. On the other hand, number was the prototype of discreteness; number was a collection of units (and the unit was not a number).

This scenario changed radically with the work (1585) of Simon Stevin (Waldegg, 1993). The Greek concept of number had developed as a result of an abstraction process applied to the material world. Stevin challenged the Greek viewpoint to accommodate the utilitarian matter of measurement in the real material world. In his work (Moreno, L. & Waldegg, G.), he identified number and magnitude, attributing numerical properties to continuous quantities and continuity to numbers. From this point on, *it is not possible to separate the concept and its symbolic representation*. For example, the infinite divisibility of number corresponds to the operation of division made possible by the decimal notation. The new concept of number was partially possible through a reflective abstraction. Number was understood as *"that through which the quantitative aspects of each thing are revealed"*. Therefore, arithmetic operations are sustained, at a first moment, on the actions that are carried out on quantities (Waldegg, 1996). Afterwards, the symbols used in the decimal notation are identified with real numbers. In a sense, this means that symbols becomes icons. In fact, cultural icons. This narrative also seems to illustrate Damerow (1988) viewpoint on cognitive structures. According to Damerow, the initial conceptual change "might be exogenous to the cognitive system and historically as well as culturally determined. Part of individual development would then consist of the effort to appropriate culturally developed cognitive structures" (Nicolopoulou, 1997, p.208).

Cognition and Context

When the epistemological realism is applied to the educational field, it is accompanied by a conception of cognition as that of a mechanism which extracts information from a stable and objective world. As has been pointed out, this view of realism deeply constrains the possibilities for the study of learning mechanisms as they happen in a local, socially constructed environment (i.e. in the classroom). It is from there that the theories of situated cognition have emerged. In the field of mathematics, however, situated cognition has presented us with a formidable challenge (Noss & Hoyles, 1996, p. 36). Mathematics does not accept propositions anchored to fixed referentials, which are dependent on the accidents of the context.

Street mathematics, as it has been called, take advantage of the meaning of the context in which problems arise. In school, the manipulations of syntax make it possible to carry out operations of the type "the price of an apple multiplied by the number of apples". We cannot help at this point but to remind the reader of the state of the mathematics of magnitudes before the emergence of Algebra (see Galileo's Dialogues concerning Two New Sciences).

Nunes et al., (1993) point out, that there is evidence that the pragmatic schemes of street mathematics can be generalized: users resort to the symbolic system provided by money in order to carry out activities of situated transference. Thus, a person's situated knowledge can be used as support for expressing more general relationships as well as for inducing from there a reflection on the activity. We enter here into the complex world of abstraction. We have given this illustration because it gives some insight into the solution to the problem of how the representational systems being used can be used to cross the gap between a situated exploration and the need to systematize.

On cognition and computing tools

Our goal in this section is to *articulate a reflection* on the ways in which computational tools mediate the construction of mathematical concepts.

Computing environments provides a window for studying the evolving conceptions of students and teachers, as they use the tools provided by that environment.

We have noticed that –according to the students– *mathematics* refers mainly to a set of symbolic expressions. Knowledge of this mathematics means to be able to use algorithms to transform a symbolic expression into another. Now the presence of graphing tools tends to shift the attention from symbolic expressions to graphical representations. At this point, it is important to highlight the importance of articulating the different systems of representations. These are tools for understanding and mediating the way in which knowledge is constructed.

Our efforts to articulate a reflection on computational tools lead us to consider the phenomenology we can observe on the screens of calculators and computers. The screen is a space controlled from the keyboard but that control is very much one of action at distance. The desire to be able to interact with the screen objects provide a motivation for struggling with the complexities of a computing environment (Pimm, 1995, p.36). On this respect, Balacheff and Kaput (1996) have talked about a "new mathematical realism" due to the new experiences while working within a computational environment. We suggest that this new realism is due, mainly, to the nature of computational representations. Computational representations are *executable representations*. Consider, for instance, the environment provided by Cabri-Geometre. Therein one can transform (by dragging) a triangle into another while trying to invalidate a property (for instance, that the bisectors always intersect in an interior point of the triangle). The impossibility of invalidating the property leads to a proposition we call a "theorem". The Cabri environment is well suited to close the gap between the notion of drawing and the notion of geometrical object.

There is another attribute of executable representations on which we want to cast light. It is the fact that they serve to *externalize* certain cognitive functions which formerly were executed by people. That is the case, for instance, with the graphing of functions. Now the student has the opportunity to transform the graph into an object of knowledge. This is similar to what the Greeks did with writing. They used the writing system not only as an external memory but as a device to produce texts on which to reflect. As Donaldson (1993, p. 342) has said, the Greek's critical innovation consisted of "*externalizing the process* of oral commentary on events".

The interaction between diverse executable representations facilitates the construction of *situated abstractions* (Noss&Hoyles, 1996, pp. 105-107; Sacristan, 1997). Situated abstractions refers to the understanding and encapsulation of processes within the context in which they have been explored. Let us explain: At a first moment students can make some observations situated within the computing environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (by dragging) a property of a geometrical figure and they cannot. That property becomes a theorem expressed through the tools facilitated by the environment. As a result we have a *situated proof* (Sacristan, A. op.cit. ; Sacristan, A. Noss, R. & Moreno, L, 1999, in preparation)

We retain the use of the adjective "situated" just to call attention to the role of the environment and the tools employed. A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization.

T. Nunes et al. (1993) have observed that Brasilian children used the money system to transfer their arithmetic strategies from one context to another. The money system, paper and coins, worked as a universal referent. Similarly, observing our students while working in a computing environment, we noticed they could articulate the results of their explorations in a way that could be taken beyond the environment in which they were found. The students purposely exploited the tools provided by the computing environment to explore mathematical relationships and to "prove" theorems (in the sense of situated proofs). Let us illustrate this point with the case of continuous non-differentiable functions. While working with the algebraic calculator TI-92, the students observe the dynamic drawing process of the polynomial approximations to Weiestrass' function, and begin to understand the randomness "hidden" in such a function. They realize this characteristic of the function under consideration, thanks to the dynamics provided by the executable representation of the Weierstrass' function.

Drawing by hand and using a computing device to draw, are different cognitive activities. Of course, the nature of the mediational tools applied in each case, support this assertion. We suggest that the executable nature of the computer's representations "orients" the reflections of the students to structural considerations: An apt tool for exploring the present example is the zoombox. Through its use, students may discover the degree of complexity of the function, perhaps only from a visual point of view. The tool is used, here, as a microscope, and not as a magnifying glass. We think this difference is important: What one can see by using a magnifying glass belong in the same structural domain as what one can see with naked eyes. The microscope, on the other hand, allows one to enter a new structural domain. It is not just a matter of scale, but of obtaining a new object of knowledge. This way, students can generate and articulate relationships that are general to the computational environment in which they are working.

Note. The field work, supporting our assertions, is part of the project "La incorporación de nuevas tecnologías a la cultura escolar" (referred to below) and of several joint works with Ana Sacristan (see references).

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DISCUSSION OF THE PAPERS BY MORENO AND OJEDA

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Both of these papers provide us with some important, although fairly familiar insights. Moreno considers representations from a very general point of view and Ojeda explores concepts in the context of probability. The two reports also share a certain lack of clarity or at least incompleteness regarding their central concerns. Ojeda tries to explain what she means by concept by giving us a number of characterizations that can be quite different but without either choosing or synthesizing.

Moreno, on the other hand tells us that "Representation…is a slippery concept" and also that it is not just a copy of an external reality, but he does not appear to even try to capture this mercurial notion.

What I find most interesting in Moreno's discussion is the notion of "situated abstraction" for which he credits Noss and Hoyles. The insight here, which also appears in Ojeda's paper, is that if we include as part of the "environment" mathematical and mental realities as well as physical ones, then yes, even the most abstract mathematical concepts are "situated" that is, inextricably linked to their contexts. This moves the argument about context independence/dependence to the question of whether an abstract concept, such a mathematical group can be considered as part of a "real" context. Moreno helps even further by pointing that (you should excuse the espression) situated well between physical and mental reality is the reality of computational environments provided by computer software. I would go so far as to say that these insights, if carried to their natual conclusion render irrelevant the whole notion of situated cognition and the controversies surrounding it .

It would be nice if such insights could be applied to enhance our understanding of representation but as I indicated above, by not really getting into what he means by representation, Moreno foregoes that task. He does, on the other hand take on the issue of cognition versus culture. Here, unfortunately, he perpretrates some common misunderstandings. Vygotsky does *not* advocate social *interaction* as a source of mathematical knowledge so much as the *transmission* of meaning from a parent or teacher or *advanced* fellow student to the novice. And the idea that mathematical learning is "both a process of individual construction and a process of enculturation", for which Moreno credits Cobb, was strongly expressed decades earlier by Piaget who also includes maturation as a third component of the process. It is certainly true that (as Moreno quotes Hoyle and Noss as saying) "we have as much to gain from Vgotsky as we do from Piaget". But I do not think this is the case so much for the sources of mathematical knowledge as it is for the mechanisms through which it can develop such as the zone of proximal development, the use of language, reflective abstraction and the intra-, inter-, trans- triad.

A point of major importance is that although the present paper by Moreno is an essay in which a number of statements are made without supporting evidence, we are told not only that the evidence exists but is soon to appear in works being prepared in collaboration with A. Sacristan.

We can look forward to the appearance of this material which can move Moreno's ideas from opinions to the results of scientific research.

I was very pleased but a little dismayed with the paper by Ojeda. On the one hand, I think it is really important for the field of mathematics education to confront some of the very difficult (for students and for mathematicians) topics, to try to understand what it might mean for an individual to understand such a topic and to think about what kinds of instruction will or will not help our students achieve such understanding. Ojeda's paper is a serious step in this direction.

On the other hand, as I indicated above, I had considerable difficulty in understanding what the author means by *concept*. I don't think this is a nit picking point about wording because both the title and the text indicate that Ojeda's concern is with concepts in the area of probability. She is concerned with epistemological, psychological and social factors involved in learning a concept and tells us that in order to consider levels of abstraction, she will "refer to notion, then idea and finally concept." It seems like an important point, but she tells us nothing about her meaning for first two of these three term and several different, perhaps conflicting, things about what a concept is. We get the Webster dictionary meaning, the concept-as-category meaning, the point that "probability requires a different way of thinking from the one needed for other mathematical concepts" and the formal mathematical definition. I have tried, without success to see what Ojeda might mean by concept, for example in the case of the concept of *random*, which has always given me a great deal of difficulty. I was not helped in my quest by anything in this article.

Ojeda announces her position that "knowledge results from social interaction, and that knowledge is actively built in", presumably by the individual as a result of this social interaction. I would argue, with Piaget (as I indicated above) that social interaction is a major factor, but equally so is an individual's interaction with the rest of her or his environment and individual maturation of mental and physical resources. On the other hand, I applaud the strong stand the article in favor of basing the development of understanding on direct human experiences. Although not a very new idea, this is extremely important, especially for the more complex mathematical concepts such as probability. Ojeda's structure of an epistemological triangle involving an object or context of reference, a formal definition, and a sign or symbol is a useful tool. I also enjoyed reading her description of an instructional experience in which the teacher takes the game away from the students by giving all of the answers to questions without allowing the students any space to develop their own ideas. As much as we all perhaps agree with Ojeda's point here, I think we cannot be reminded too often of how difficult it is, in practice, for a teacher to refrain from saying too much too soon.

LANGUAGE USE IN EMBODIED ACTION AND INTERACTION IN KNOWING FRACTIONS

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If one takes an embodied view of language use in mathematics knowing then one needs to consider not only the mathematical sign and its referent, but also the person using the language in mathematical knowing actions. But this person's knowing coemerges with a cospecified environment and in particular, with others in that environment. The individual can be viewed as being involved in a number of simultaneous conversations which he or she shapes through action and language use and in which her or his language use is occasioned. This essay is based on the observation and interpretation of use of mathematical language of children, primarily eight years old, as they engage in situations involving the use of fractional numbers. This interpretation includes the consideration of various levels of language use, particularly informal metaphoric and metonymic uses of fractional number language, of the interplay between language use and the children's mathematical construct use, of the roles of interaction in and with environmental features as well as the relationship of language use and mathematical understanding as a process.

Bertrand Russell, in his *Introduction to mathematical philosophy* (1924) suggests that a natural number like three can be characterized as the set of all sets which can be put into one to one correspondence with any constructed trio. It might be expected then that when a person, particularly a child uses the word "three", he or she will be pointing to some exemplary trio. The characteristic of "threeness" is necessarily abstract and relational in character and in making reference to a situation using a sign for three, the child at least unconsciously might be making a one to one correspondence. Of course, we know that young children say, "I am three" without necessarily invoking such a correspondence, although one most often observes such a statement accompanied by the raising of a trio of fingers which is at least a cultural, if not a mathematical correspondence. Even this last rather simple action language example we observe some of the ideas discussed by Sierpinska (1998) in her sociohistorical analysis of mathematical language and communication: children use [mathematical] language to communicate with others [and othernesses], and the child's lines of thought and language use interact.

The discussion above suggests that in using symbols for three in action, a child will not be simply pairing the number word/symbol with a referent object in some way, but will be enacting a relationship involving correspondences, or successors for example, intertwining language use with her or his thought. If this is the case for natural numbers like three, it is even more evident for fractional numbers such as three-fourths. Even in the case of one half where it is evident that children have some form of this concept and can use the word half referring to an amount or an active process ("I'm three and a half", "My half is bigger", "I will cut the cake in half") at an early age, using one-half as a word is not simply a process of matching a word to an indicated object., For consider the half piece in Figure 1 below. If one centres that piece on the unit piece vertically many persons, even adults, will notice that the piece no longer "looks like" one half. Thus in using fractional number language, a child will be intertwining that language use with thought/actions which are relational in character and involve what have been observed as constructive mechanisms in children's fractional numbers (Behr et al., 1992; Kieren, 1976; Confrey, 1998; Steffe, 1999) such as partitioning, splitting, or unit reconfiguring and which involve the use of at least proto-ratio notions.

Embodied use of fractional language

The best way to understand the embodied use of fractional language by children is to consider some language samples drawn from the artifacts of such actions. The first set of such examples are given in Figure 2 below. They have been drawn from the work of eight year old children from three different classroom studies of fractional knowing in a suburban school in Edmonton, Canada. These classes involved children with a wide variety of histories of performance in mathematics. Children were involved in using a variety of materials in lessons which were aimed at engaging them with the use of the various fraction sub-constructs - e.g. quotients, operators, measures - (Kieren, 1976). The "fractional pieces" relating to the students' language samples are shown in Figure 1.

These particular fractional language samples were drawn from work done in a lesson which was early in a series of lessons and projects which involved what Simmt (1999) calls variable entry prompts using the first kit in Figure 1 above. They are called prompts rather than problems because it is the students and their lived histories that determine what are taken to be the problems in the setting for each of them and not the set task itself. Their variable entry nature reflects the proscriptive rather than prescriptive nature of the prompts and is evident in the variety of "good enough" responses based on very varied student backgrounds. These prompts and the subsequent student actions and language use were at the heart of lessons which Schonfield (1998) would characterize as emergent - that is the na-

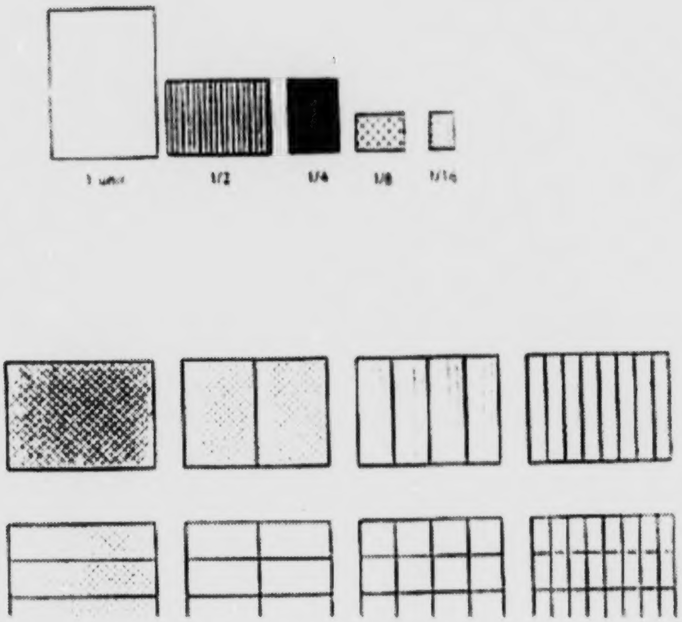

Figure 1. Pieces from Two Fraction Kits

ture and direction of the lesson coemerged from the actions and language and interaction of the students and the subsequent actions and interactions of the teacher; the lesson "direction" was not pre-determined by the teacher, even though the prompt was based on the teacher's view of prior student actions, thinking and language use. In a way the curriculum in this lesson was "wrapped around" the previous lived curriculum of the students and the teacher. In this setting, even though the teacher used standard symbols and words in her communications, the students were free to think about fractions and use fractional language in their own way as long as they could share and explain their work to others. For research purposes, the regular teacher, a research teacher, an observing researcher and a research assistant video taping were present. The language samples below are drawn from student written work or from captured board work, while the interpretation is supported in part by various other "data" sources including tape viewing, transcripts, and research team meeting notes developed daily

Interpretations: How are these eight year olds using fractional number language? In a sense, these diverse examples answer that question on their own. They need no further interpretation. This is even more true if one considers that these samples represent only a part of the observed diversity. Nonetheless, it is useful to address the "title" question of this essay

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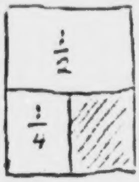
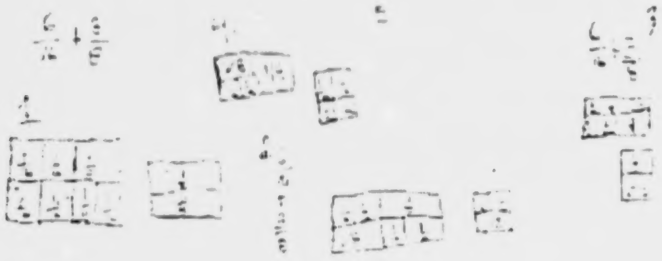
Here are some things which children wrote or drew about three fourths (3/4).
-Three fourths is less than one whole.
-1/2 + 1/4 equals 3/4.



Your turn: On this sheet or on the back write down or draw pictures of 5 things about three fourths.

Here are some of the responses of the children in the class:



Pat's idea of 3/4

Richy's idea of 3/4

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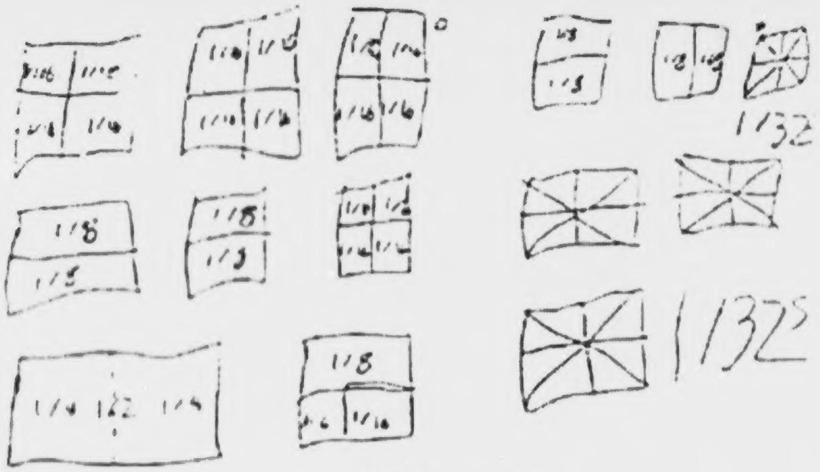
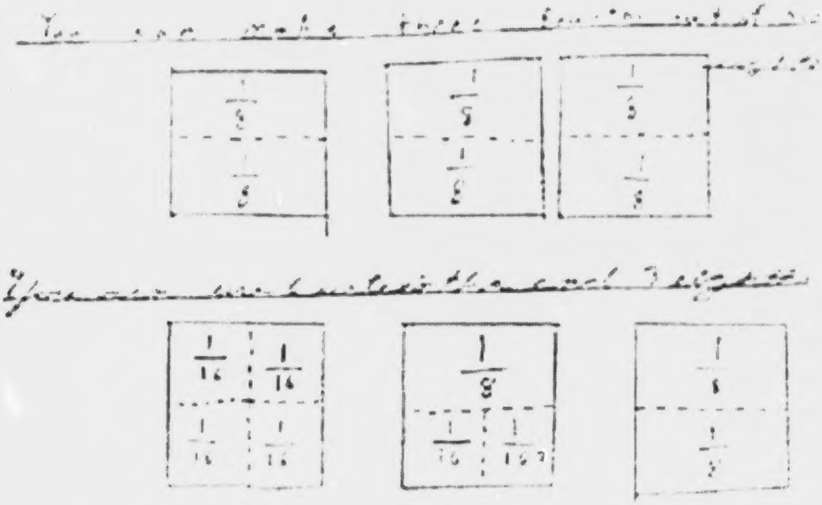
Figure 2a: Student language use relating to three fourths


1/16
1/8
1/32
1/32s

Wally's idea of 3:4



Mark's idea of 3:4

Tad's idea of 3:4

Figure 2b. Student language use relating to three

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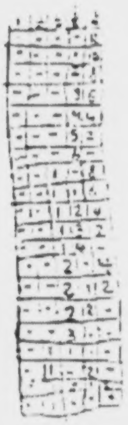
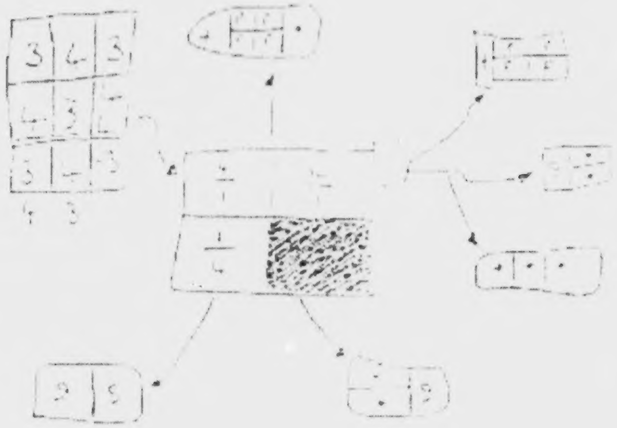


Sandy's idea of 3 4



Brent's idea of 3 4

Figure 2c. Student language use relating to three fourths

and ask what might it mean to say this language was embodied. For me, the idea of embodied knowing is derived from the work of Maturana and Varela (1987). They see knowing not as representing a pre-given world or even as problem solving per se; instead they see personal embodied knowing as the bringing forth of a world of significance (in our case, involving fractional number mathematics) with others in a sphere of behavioural pos-

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sibilities. This knowing in action is fully determined by the structure or lived history of the individual; but it is co-specified by the environment which occasions the action (including others in it like the teacher and physical features such as fraction kits and prompts). Looked at another way, the student is observed to act with the energy rich material in the classroom (tasks, materials, comments, and actions by teacher and peers) but in that action is observed to transform that rich input for her or his own use (after vonFoerster,1981). Looking at any of the samples in Figure 2, we can in one way or another observe these actions and transformations. But what of the use of language in all of this? Maturana claims that human beings exist in a world of language and engage in languaging (and perhaps especially in mathematical settings). They can, of course, engage simply in linguistic action or in the consensual coordination of actions. But it is only when human beings recursively coordinate those linguistic actions, or reflect on such actions by making distinctions in them and re-presentations of those distinctions for themselves that humans engage in what Maturana calls languaging. And it is in making such distinctions that humans make things things. Here we observe eight year olds making fractional numbers things for themselves in various ways. They are using fractional number symbols of various sorts to bring forth a world of significance involving fractions. Of course, they are "completing" a task set by the teacher. But had you been there, you would have seen in every one of these classes that the students persisted in working on the task long past its "completion" of saying five things about three-fourths. You would have, for example, seen Jodi and Rachel lying on the floor under their desks writing madly and challenging the class with "We have nine things!" and later "We now have found 23 things [about three fourths]". As might be expected, such claims as well as the work displayed around the classroom both provoked and occasioned other students to continued mathematical action.

A world of significance including fractional numbers Even the limited display in Figure 2 provides a view of the fractional number nature of this "world" and the language used in it. If one imagines that the classroom through the language artifacts of each students' own work as well as the extensive displays and explanations of others work, then one can see how students are offering fractional ideas to others including the teacher and how students might be occasioned to act in this world as well based on reacting to and transforming (for themselves) such artifacts, offerings and explanations. Thus one is aware of how this communication and potential interaction can be observed to expand the cognitive domain of possibilities for the children and even the adults in this setting. Consider for the moment the whole set of work in Figure 2 as well as work from this class in later figures; one notices the use of a variety of mathematical language - signs for fractions, plus and equal signs. How is this language used? For example, the plus sign is used to indicate the pulling together of fractional pieces to form three-fourths and does not stand for the addition of fractions in the usual sense. But even in these actions, the students are exemplifying the kind of combining and reconfiguring mechanisms envisioned in the work of Behr et al (1992). Thus, an observing researcher or teacher might ask how students might use or transform these actions into more abstract and more formal uses of this language. As these students both create and observe the creation of extensive sets of combinations of fractional units which yield three-fourths, they are generating a background of lived experience for at least the protoconcept of fractional quantities in an equivalence class - one based on a "is as much as" relation. Notice the work of Tad in Figure 2 and especially his sentence 3/4 = 3/4. He, Sandy, and a girl in the class named Jodi (whose other extensive work is not displayed here) were the only students to initially use such sentences. While using a sentence like "1/2 + 1/8 + 2/16 = 3/4" was OK, Jodi for example wasn't sure about using "3/4 = 3/4" even though she said that she knew it was true. Perhaps this query was an indicator of Jodi's awareness of a new use for "=" not as a signal for the result of a computation which would be a very typical reaction of an elementary school child, but as a sign indicating an equivalence relation.

What might researchers or teachers make of the kind of action and language use indicated in Figure 2? They might look to see in what ways the language used pointed to or was intertwined with the kinds of thinking that children were doing. For example, they might ask in what ways does the language used relate to the constructive mechanisms (such as combining or reconfiguring fractional quantities) it might be inferred that students were using? Or they might see the collected artifacts as a source for future knowing actions and lessons. For example, the teacher might use this collection of fractional sentences to occasion children's thinking about equivalence. Researchers or teachers might also ask in what ways the language used indicates similarities and differences in the thinking of students and how actions, utterances or language artifacts such those in Figure 2 illustrate different "levels" of sophistication in language use.

Northrop Frye (e.g. Frye, 1981) has pointed to four levels of historical use of language: hieroglyphic use or language used only in the presence of the object or action; hieratic use which is metamorphic or story-telling in nature in which the language is put for the thought/action with objects; metonymic use in which language use is independent of but analogical to actions. In such a metonymic use the language used can be observed in a "part for all" relationship with the total action on objects from which it might have originally derived its meaning. Finally in a demotic or analytic use language is independent of any other actions. Without doing too much violence to Frye's ideas, we can use this "taxonomy" to consider the sophistication of fractional language use by children. If we simply stick to the limited samples offered in Figure 2, we can observe several of these "levels" even among the work of these young students. For example, we might note that Pat simply reported on the various reconfigurations of the same set of kit pieces. One could imagine that for him 6/16 + 3/8 was not put for a representation about such action but was directly associated with it and not independent of it at all. We notice in Pat's work, as well as the work of many other students the close tie between fractional language use and the unit fractions associated with the pieces. It might be thought that this somehow indicates the primitive nature of unit fractions for these children, but as we will see in later examples, such thinking and language use is not independent of the setting or of the inferred mechanisms students use in thinking about and acting on fractions. One might think that the work of Richy or Mark or Wally reflects the same kind of hieroglyphic fractional language use as Pat's; but in Wally's case, we see that he is rather excited by invoking the fractional unit of 1/32. Here we see this language use pointing to the Piagetian constructive phenomenon of a child both constructing a new fractional unit from a given one (thirty- seconds from fourths) and using fourths shown as thirty-seconds (or 1/32 s as Wally puts it) to construct another quantity. Thus, it is likely that Wally is using his labeled figures at least to report on or tell stories about his actions on objects. But as the last two examples point to (and subsequent observation of Wally confirm) the labeled drawings have a generative character as well. Notice the adjectival use of language in Mark's samples, especially in the second example where the 6 and 3 (numerators) are adjectives for the sixteenths and eighths. This adjectival use of "numerator" words was observed as typical of many students' early work on tasks with the fraction kit as was the adjectival use of fraction symbols such as 5 1/8 which the children used for 5/8.

Tad's sentences (only a few of which are given) point to a more sophisticated use of fractional language. It is in this work as well as that of a few other students in each of the three experimental classes that we get a sense of a child trying to find [a set of] fractional unit combinations which are quantitatively equivalent to three-fourths. Such language use might be an example of an analogical, part for whole, metonymic use. This is much more clearly seen in the case of Sandy, the last example in the set.

But before turning to a more extensive interpretation of Sandy's work, let us briefly look at Brent's work. Brent could be observed in class to be trying to work like his partners and other children in class and in some way to coordinate his actions with his perception of their actions. He would move pieces around and write down symbols and make drawings that he interpreted to be like those of other children. But an observing teacher, researcher, or even fellow eight year old would find his language use to reflect that fact that Brent's actions were inappropriate to the task; in the language of proscriptive logic, while the other students' actions and intertwined language use were "good enough", Brent's was not. While Brent used numerals and symbols which resembled those of his peers neither his actions nor the semantics or syntax of his language use indicated that he was acting in a fractional number world. Neither Brent's actions nor his symbol use pointed to fractional ideas. Even when his symbols took a fractional form, Brent seemed unaware of the order of the numerals or the meaning of that order. Perhaps this action and language use points out that Brent's structure, his lived history of mathematical activity, simply did not allow him to take appropriate action even in this relatively simple fractional task and environment. For Brent, the language and action of fractions may have been out of his world.

Sandy's actions and related language use indeed show him bringing forth and living in a world of significance including fractional numbers. He used his table generatively to generate all of the possible combinations of a set of fractions which yield three-fourths. To understand his language use better, one needs to understand its history (Kieren and Simmt, 1999). On the previous day, Sandy and his classmates were trying to generate combinations of "half fractions" which were quantitatively equivalent to five-fourths. Sandy claiming that he was "trying to find all of them", had made up a chart with similar headings to those of his in Figure 2. He was observed standing at the board, looking up in the air, and occasionally making an entry in his table, now written on the board, reflecting his dreamed up combination that worked. During this time, he was interrupted by a teacher and other students who wanted him to explain what he was doing. Frustrated by this interruption, he started to work secretly on a new five-fourths table which he did not finish, but in which instead of using the table to record thought up combinations he used it to generate an orderly sequence of such combinations. Sandy's work in Figure 2 then represents his reluctant response to the teacher's request to "make a table like you did yesterday". But now when he presented it to her after only a minute or two, he confidently said "There, that's all of them!". Sandy's use of his table now had become formalized and was an analytic means of dealing with fractional number combinations which was independent of any actual combining of quantities in any physical sense or even of imagining such combinations one by one. In his work then unlike any of the other work in Figure 2, we see fractional language used not with action, not to report action, nor even as an analogy to action, but as an independent thought action in itself. Such language use might be considered an example of demotic or analytic language use.

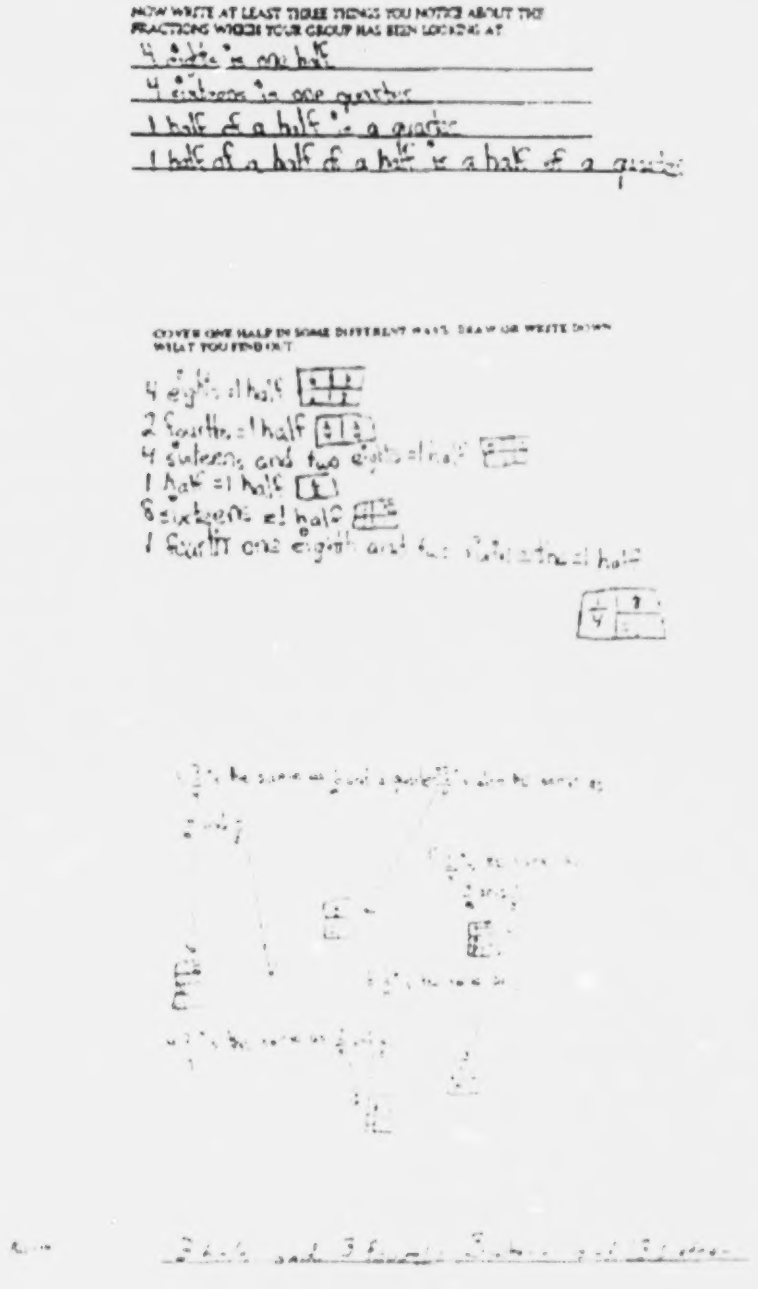
Several things should be noted about this "leveled" way of looking at fractional language use. These levels are not to be construed as stages of language use such that once a child reaches a certain stage, then her or his language use will continue to reflect that achievement. Neither is language use abstract and context free. Whatever the use of language, it occurs in and coemerges with the context in which it occurs and in the conversations with others within it. Both of these issues will be revisited in examples which follow. Finally, while it is beyond the scope of this essay to elaborate on this, language use in action is one of the useful indicators in the observation of a person's growing and changing mathematical understanding viewed as a process (e.g., Pirie and Kieren, 1994)

Changing use of fractional language

The fractional language samples in Figure 2 above represent a variety of language uses by different students in the face of a particular prompt. To get further insight into the nature of fractional language in embodied action, it is interesting to look at such language use by one child across tasks (and across time). The samples in Figure 3 below come from the work of Kara considered by her teacher to be a "typical Grade Three" student.

The first sample offered in Figure 3 arose as Kara and four partners worked on the following task: Each student in the group was to fold 4 unit sheets, one into halves, one into fourths, one into eighths and one into sixteenths. They were then asked to shade in and identify a fractional part of each. They then put all of their "fractions" into a common set and worked together to find relationships among twenty elements in that set. Before turning to Kara's writing, it is interesting to note that well over ninety percent of the students in the Grade 3 classes studied were able to predict the number of parts given the number of folds - at least from one to the next. All seemed very aware of the multiplicative and even exponential nature of this typical splitting task confirming the observations that Jere Confrey has made on this over the years.

There are many interesting features of Kara's use of fractional language. One striking feature of this is the use of "words" instead of fractional symbols. This occurred even though the teacher and some other students used fractions to describe folded and shaded amounts. A second



NOW WRITE AT LEAST THREE THINGS YOU NOTICE ABOUT THE FRACTIONS WHICH YOUR GROUP HAS BEEN LOOKING AT

4 eighths is one half

4 sixteens is one quarter

1 half of a half is a quarter

1 half of a half of a half is a half of a quarter

COVER ONE HALF IN SOME DIFFERENT WAYS. DRAW OR WRITE DOWN WHAT YOU FIND OUT

4 eighths = 1 half

2 fourths = 1 half

4 sixteens and two eighths = 1 half

1 half = 1 half

8 sixteens = 1 half

1 fourth one eighth and two sixteens = the 1 half

Figure 3. Kara's use of fractional language

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feature that Kara shared with nearly all of her class mates is evidenced in her use of "eights" and "sixteens". At first, the teacher was convinced that the children simply were not hearing the "th" at the end of the fraction words. But since this use was almost universal in this class and persisted even when the teacher emphasized the " th", the language use might be attributed to what the student perceived in the situation. As noted in the last two items in the first sample of Kara's work, fractions and fractioning in this setting were related to a process of [to an observer] iterative folding. The resulting parts occurred not as discrete items, but as connected sets. Thus, three halvings resulted in an "eights" set.

The second set of Kara's fraction language samples was drawn from very early in her experience with "half fraction kit" (Figure 1). Notice that her previous language use persisted, but changed as well. For Kara, it was only gradually that the fractional pieces (and related numbers) became units for her which could be combined and reconfigured and used in their own right as suggested by Behr et al (1992).

The third sample in Figure 3 is drawn from Kara's work on the task related to Figure 2 above. Notice that while complex fractional symbols were used, they on describe or report on previously completed actions on the physical materials where the unit fractions are used as labels. One might say that Kara was using fractional language metaphorically or using it to report on prior action. Notice also that the fraction symbols are not combined with other mathematical signs, even though Kara appeared to be thinking in terms of equivalences- as much as. Her drawings show that her fractional combinations were compared with (and in her drawings placed on) a unit or one. Thus, it appears that Kara's fractional language use changed with the setting of its use (and of course was related to but not dictated by the fractional language uses of her peers and the teacher). Further, it appeared that this language uses was becoming more "mathematical" and perhaps sophisticated with time and experience.

The fourth and last sample of Kara's work in Figure 3 allows us to comment on the last conjecture, but before turning to it, consider the two language samples in Figure 4 below. Barney, the boy who generated those samples was considered by his teacher to be mathematically gifted. The first of his samples arose as he worked on a task which was a student favorite - the Missing Fraction Mysteries. In the particular case, the students were to describe a "hidden" fractional amount from the two clues that it was more than one fourth and less than three fourths. From observing the work of his friend Sam, Barney noted that one could find such an amount "subtractively" by deleting a fractional amount from one. But more than that, he realized that one could engage in fraction play, adding and subtracting amounts which finally "solved" the mystery. The first sample in Figure 4 was just one of a full page of various such mathematically playful solutions which were created in their own terms (a demotic/analytic use) independent of and unrelated to any "concrete" actions.

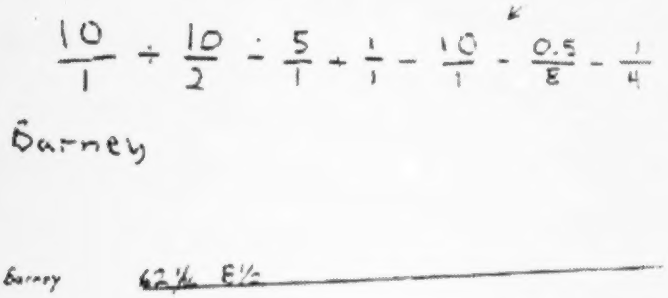

10/1 ÷ 10/2 ÷ 5/1 + 1/1 − 10/1 − 0.5/8 − 1/4
Barney
Barney
62 1/16 8 1/2

Figure 4. Barney's use of fraction language

It might be thought that Barney had very sophisticated mathematical and symbolic control over at least a subset of the fractional numbers which he could invoke in any setting involving fractions. But his second sample allows one to question this assumption. Notice that Barney had reverted to using an adjectival form of fractional language where 62 1/16 was used to describe the length of a long table measured on a "fractional measures scavenger hunt" using a constructed fraction tape which was a unit (one metre) folded into sixteenths. The second fraction, 8 1/2, is Barney's measure of the same table using a "half tape". Notice here that Barney's use of language is no longer abstract, but descriptive. Although later on another measuring task Barney noted that one could just measure with any tape and convert that measure to a measure based on any other fraction tape, at least early in his experience with fractions as measures Barney did not think in those terms. In Pirie and Kieren's terms (1994), Barney had folded back to a more local mode of understanding and perhaps his language use reflects this.

Turning back now to Kara's last fractional language sample in Figure 3, we see her reports of two measures of the same long table. Notice that although Kara had previously used fractional language in a more sophisticated manner with fraction kit pieces or fractions as discrete units, in this new "fraction as measure" setting her language looks much less sophisticated.

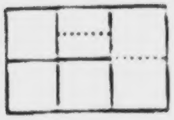
By looking at student fractional language use in various settings, a researcher or teacher needs to be aware that the students perceive those settings in their own terms and their fractional language use reflects their own structures, the nature of that setting and their lived experience in it. Such an embodied view of fractional language use suggests that while the taxonomy of levels is a useful observation tool, embodied fractional language use is not a monotonically staged phenomenon.

Fractional language use in interaction

The interpretations of the fractional language samples discussed so far have emphasized two things. Students use fractional language in a constructive manner - that is, their language use is related to and intertwined with their thinking and their re-presentations in fractional settings. Secondly, the context of the language use matters. The various samples in the figures above suggest that fractional thinking and language use, even if it is determined by the structure of the individual child, is coemergent with the setting in which it occurs. In this final interpretive section we turn to the communicative uses of fractional language. Figure 5 below contains two items which were drawn from one urban, multi-ethnic middle school class of 11 and 12 year olds with very varying histories of mathematics achievement. As a "warm-up" task during the introduction to fractional numbers, the teacher had asked the class to find the amount represented by 1/6 + 3/12 + 2/24 and to express this amount as a fraction. In doing this task students had available a fraction kit like the second one in Figure 1 above. Because of the very wide variety of "solutions" the teacher had many of the students sketch and explain their solutions. Van's work is typical of many of the solutions and explanations which involved the kit. Peter, a boy who considered himself to be a good student, apparently wanted to generate an "answer" which was different from any of the others. He went to the board and started the interchange which occurs in the second part of Figure 5 below.

Here we see Peter use fractional language in many ways. First he uses "eights" as a means of being different. While his quickly generated solution is correct, it was done mainly to "be clever". After being challenged Peter (3) uses fractional language as part of or intertwined with his thinking in which he can be observed to recombine units (trying to think of how eighths could be related to sixths, twelfths and twenty-fourths). Finally because he wants his thinking and fractional language use to be accepted in this conversation he shows how the language he used could be seen as analagous to the fractional action of inter-relating the eighths units and the sixths, twelfths and twenty-fourths which many of his peers had worked

Van: It makes $\frac{6}{12}$.



Justification:

Change $\frac{2}{24}$ into $\frac{1}{12}$, and change $\frac{1}{6}$ into $\frac{2}{12}$.

1. **Peter:** Use eights. Four eights. (Making this drawing)

2. **Teacher :** Wow! How did you ever think of that? (emphasis mine)

--- brief pause ---

3. **Peter:** Half of a twelfth plus one twenty-fourth plus one fourth of a sixth is an eighth – and that happens twice.
   Half of a twelfth plus half of a twelfth plus a fourth of a sixth is an eighth – and that happens twice
   Altogether there are four eighths.

4. **Teacher (Like nearly all of the class and the observer):** What?

5. **Peter:** See. I'll show you. (Makes this drawing)

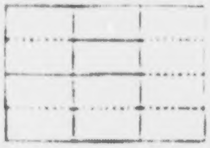


Figure 5. Fraction use in interaction

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with. In this action we observe Peter using fractional language in many ways to communicate things about himself and his thinking to others and to deliberately address himself to his peers in what he thinks might be their terms.

Concluding remarks

There are many ways to think about fractional language and its use. One could do this abstractly by thinking of the various ways in which fractional symbols might be related to situations to which they refer. Such an analysis would have one ask " how might fractions be used in setting x and why might this be so?" Or one might treat fractions in a textbook fashion where one would focus on how children might come to write, relate and symbolically combine fractions in ways which match some pre-given, prescribed outcomes. Such an analysis might prompt one to ask "how well do children perform with fractions and how might such performance be enhanced?" Rather than taking either of the approaches above, I have chosen to focus on the use of fractional language as an embodied phenomenon. In so doing I have tried to show how even young children use fractional language to think about, act in and bring forth a world which includes fractional mathematics. The interpretations offered here focused both on how the children used the symbols and on how this symbol use and related thinking could be observed to be occasioned by the environment and community in which they existed. Thus fractional language use was observed as subject to the internal structural dynamics of the individual and the social/interactional dynamics of the community in which he or she existed all-at-once. Further to the extent that the children were observed to be acting in mathematically legitimate ways with fractions such actions and language use were also embodied in the broader culture of the practices of mathematics.

What difference does such an interpretive approach make? First I think allows researchers and teachers to understand the necessary diversity in a class which is coming to know fractions and use them in action. Such knowing and related use of fractional language appears to occur at many levels of sophistication both across the children in a classroom but also within the knowing actions of any one child. But such knowing is clearly related to the context and setting in which it occurs. Based on such observations the teacher can wrap the curriculum around the thinking, actions and language use of the children while still acting to expand their potential domain and sophistication of fractional language use. Such an interpretation does allow one to think about the relationship of fractional symbol to referential context through the context sensitive embodiment of the fractional language user. In addition it prompts one to think about curriculum and teaching of fractional number ideas so as to take into account the ways in which students use fractional language intertwined with their thinking, which includes the constructive mechanisms of children's mathematics; and so as to promote fractional language use as part of the maturing use of mathematics in a person's life with others.

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EMBODIED ACTION AND LANGUAGE: ITS IMPLICATIONS FOR FRACTIONAL THINKING

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In his paper, "Language Use in Embodied Action and Interaction in Knowing Fractions," Thomas Kieren accomplishes two valuable goals. He extends our understanding of the implications of his previous work with Susan Pirie applying the theories of Maturana and Varela to mathematics learning and teaching, and he challenges some fundamental, and dearly held, assumptions about learning fractions, and learning in general. My comments here focus on these broader implications.

Embodied action and language are critical to Kieren's analysis of mathematical activity. For embodied knowing, Kieren describes "...knowing not as representing a pre-given world or even as problem solving per se; instead they [Maturana and Varela] see personal embodied knowing as the bringing forth of a world of significance with others in a sphere of behavioural possibilities." What a far cry from a behaviorist trajectory of "getting the students to routinely perform the required action as a result of encountering the appropriate stimulus." As has been the trademark of Kieren's immense contributions over the years, the phrase indicates how central and fundamental is the child's eye view of knowledge. In addition, the statement suggests, not only is a child's previous knowledge central, but the aspect of their history which establishes their perception and experience of significance.

In the area of language, Kieren also pushes the field forward. Using Maturana's idea of languaging, which occurs only when human beings recursively coordinate [those] linguistic actions, or reflect on such actions by making distinctions [in them] and representations of those distinctions...(p 7), Kieren folds embodied action and language together. Reminiscent of Vygotsky's dialectic between thought and language, Kieren's treatment seeks to emphasize that mathematics is neither idealism or realism, but a human expression of significance. He emphasizes that its the idea of the idea that makes the mathematics delightful and worthwhile.

This description recalls for me a radical idea expressed by David Thurston, a Fields medalist mathematician, whose theoretical work was challenged as not worthy of record without formal proof by Jaffe and Quinn. Thurston's response was to suggest a more essential question, "how do mathematicians advance the human understanding of mathematics?" He offered a recursive definition of mathematics as, "Mathematics includes the smallest system whereby: the natural numbers and plane and solid geometry are included; mathematics is that which mathematicians study; and mathematicians are those humans who advance human understanding of mathematics." (p. 7)

This focus on significance and the commitment to meaning based not in intrinsic beauty or elegance but in the undertaking of significant activity among a group of people is critical to the future development of mathematicians. We cannot derive our curricular projections solely from logical or developmental trajectories and ignore the fact that human beings engage in activities that they find significant and compelling. And, furthermore, we cannot ignore the fact that when they do, they produce remarkable accomplishments, and when they do not, their products resemble the completion of bureaucratic tasks.

The implications of Kieren's arguments extend well beyond his examples with fractions to raise fundamental questions about the secondary curricula. How can one weave together a curriculum structured to consider logical, development factors along with significance? And is significance only an emergent feature, a local experience, or are their global qualities that can drive not just the interview level or even classroom level design of curricula but the district level or larger? In a recent paper, I have argued for a reconstruction of the term, content, to discuss "generative domain knowledge." This construct involves issues of genesis, discourse, modeling and design, and tries to construct what mathematics would be like for not a "research" but a "resourceful mathematician." (Confrey, 1999).

Kieren's task and examples illustrate beautifully how one only has to vary a task slightly, to invite open and diverse response, and to gain the intellectual commitment of young children. The question I am raising is if one's longer term commitment is to supporting and maintaining such energy, how would one want to develop the concept of rational number? My own bet would be towards issues of rate of change or probability and statistics, in order to keep a broad definition of significance, that competes successfully with the other information rich, communications-embedded facets of children's lives. Though this may be too ambitious of an extension of the Kieren position, it seems to me such questions are embedded in his theoretical presentation.

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Working Groups

GEOMETRY AND TECHNOLOGY

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Introduction

The working group on Geometry and Technology will continue the discussion started in Raleigh, North Carolina at the PME-NA XX conference. The focus of that working group was on the integration of geometry and technology from the student and teacher perspective. I will first discuss our objectives, the student perspective and the teacher perspective.

The objectives for our working group were to:

- explore three areas of research: environment, student perspective and teacher perspective,
- investigate research questions,
- coordinate future research in this area,
- identify commonalities in the research findings, and
- identify critical questions that are not being addressed.

We began to look at the environment created and caused by microworlds. We explored three questions: how can microworlds be used to change the learning environment? What features of the dynamic geometric environment encourages the development of student understanding of geometry? And how do we make the link between formal symbolic and graphical representations (classes of objects)? Chronis Kynos (Greece) demonstrated a variation tools program to help us better understand how microworlds can affect the environment. In 1999, we will further explore the research that has been done in this area.

The second focus of the working group is students and their use of technology in the study of geometry. We attempted to answer the following question using a concept map - What factors need to be considered when investigating student … in dynamic geometric environments? Each group was given the opportunity to insert an area of study that relates to students. Some selected areas included student achievement, student understanding, student problem solving and student motivation. As the groups presented their concept map, connections were made to future research questions. In 1999, the working group will explore the research as it relates to students and technology in geometry.

A growing number of teachers have used the dynamic geometric software programs as the basis for geometric construction and for the exploration of geometric relationships and proofs. Our third day in Raleigh was spent investigating the role of the teacher in geometry and technology. The focus in Mexico will be on the teacher preparation and inservice programs to assist teachers with this new role.

REPRESENTATIONS AND MATHEMATICS VISUALIZATION

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The Group on Representations and Mathematics Visualization was constituted at the PME-NA XX meeting at North Carolina State University. There were five presentations followed by corresponding discussions that addressed important issues related to the Group theme. Fernando Hitt gave a general introduction on the learning of mathematics and the role of the representations and mathematics visualization. Manuel Santos' presentation focused on the use of technology as a means to explore mathematics qualities in proposed problems, James Kaput addressed issues on multiple linked representations and co-ordinated descriptions, Luis Radford challenged traditional conceptions of representations and proposed something related to rethinking representations. And finally, Norma Presmeg spoke on visualization and generalization in mathematics.

Some of the issues that appeared during the development of the sessions included:

- Theoretical orientations and the role of the semiotic systems of representations.
- The influence of technology-based multiple linked representation in the students' construction of a mathematical concept or in a mathematical activity.
- Representations as a social construct.
- Visualization and generalization.

The original goal of this Working Group was to provide a forum in which participants could express their work freely. We expect that presentations and discussion of ideas eventually lead to a group publication.

For the PME-NA XXI meeting in Mexico, we have invited some scholars to give short presentations and share their extensive work in this area. Abraham Arcavi (Israel), Raymond Duval (France), Adalira Saenz-Ludlow (USA) and Pat Thompson (USA), have confirmed their participation.

Stephen Pape suggest on this that *"it is appropriate to have senior people in the area presenting their work; however, if we have the papers prior to the meeting then the presentations could be much shorter. Perhaps instead of presenting the papers outright we could actually have people prepare responses to them which would stimulate a discussion of* *'controversies' in the field. These presentations migh focus on 'what do we know about representations' followed by group 'brainstorming' on 'where should we be going in this field. The culminating activities might enable us to summarize 'what we know' and 'what we want to begin to research in the future'. I think that we need to set goals for the working group and then see what activities might help us attain the goals."*

We are taking into account Pape's suggestions, and we encourage participants of the Working Group to make short presentations that could include new ideas or re-examination of those discussed previously, including critiques. Something important to do is to bring to the next meeting some copies of our work to give to the people that are interested on it. We should also generate a list of papers to be discussed during the session time. Thus, a particular topic or idea could be pursued and discussed within the group and culminate in a written paper.

*Acknowledgement: Supported, in part, by Grant 26408P-S from CONACyT, Mexico.

THE USE OF TECHNOLOGY AS A MEANS TO EXPLORE MATHEMATICS QUALITIES IN PROPOSED PROBLEMS

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Introduction. The use of technology has become important in the learning of mathematics; however, the way in which it has been utilized in the classroom has significantly changed over the last ten years; for example, Kaput (1992) stated that "as of 1988 the majority of educational software was still of the drill and practice genre" (p. 548). Thus, the use of technology was to make efficient a traditional conception of students' learning. However, recent mathematical curriculum reforms have suggested that students' mathematical learning goes beyond the application of rules or algorithms (NCTM, 1998). Goldenberg (1996) criticizes curriculum changes based on adding or deleting mathematics content and proposes what he calls "habits of mind" as key organizers of the mathematical curriculum. For example, an important way of thinking in the mathematical practice is the abstraction of invariance and the curriculum should provide opportunities for the students to develop this habit. An important question here is: How can computers help students develop a way of thinking in which they exhibit habits of the mathematical practice?

Background to the Study

One of the sources that teachers consult to prepare their classes is the set of calendar problems that appear in every issue of *The Mathematics Teacher*. There are different ways in which teachers might use the problems (homework problems, small group discussion problems, examples, etc.). Indeed, the problems themselves offer different opportunities for the students to test their knowledge in diverse areas that include arithmetic, algebra, geometry, and probability. In addition, a significant number of the problems can be approached or solved by using different methods of solution. Hence, the students can discuss mathematical qualities associated with each method and see the importance of going beyond reporting only the solution of a problem (Santos, 1996). The use of computers or calculators has been identified as an important component in the learning of mathematics (NCTM, 1998). As a consequence, it is necessary to keep exploring ways in which this technology can be helpful for students. This paper examines mathematical aspects that emerge from the discussion of an example taken from the "Calendar" in which there is interest in showing the potential use of such computer software as CABRI-GÉOMÉTRE II. The solution process itself involves the use of different forms of representation and it becomes a natural environment to formulate and pursue other related questions.

Methods, Procedure, and Conceptual Framework

Six active teachers and the researcher participated in a seminar during a semester. We met once a week for three hours. The seminar agenda included the selection of mathematical tasks or problems and presenting them during the seminar. Here, each participant showed what mathematical ideas were fundamental to the solution process of the problem that he or she had chosen for the discussion. After each individual presentation, the group as a whole also examined the problem and explored other ways of solution. Thus, participants engaged in activities which continually examined, compared, articulated, refined, and reached a consensus about the strengths and weaknesses of the problems. For each session, there was an assigned party, responsible for keeping notes and all discussion was recorded. The work done on one problem often took more than one session. The ideas presented in this paper came from the discussion that took place during two sessions of the seminar.

To outline the conceptual framework we paid attention to aspects of the solution process that include the type of mathematical resources needed to adequately represent the problem, the importance of focusing on a particular type of representation, and the strategies and criteria used to support the solution. In addition, we identified three different phases that appear consistently during the process of problem solving:

(a) Initial interpretation of the problem. Here the participants often exhibited inconsistent ways of thinking about the essentials of the problem and spent some time trying to make sense of the statement of the problem

(b) Intermediate interpretation in which the participants went beyond organizing and processing isolated pieces of information and focused on relationships, patterns, or trends in the given information, and

(c) Final interpretation in which the participants identified the fundamental of the problem and showed advantages or disadvantages of exploring different approaches. Here, they also posed and examined other related questions.

The Problem: Two congruent circles are to be cut from a 9" x 12" sheet of construction paper. What is the maximum possible radius, to the nearest hundredth of an inch, of these circles? What percent, to the nearest tenth, of the area of the sheet will be cut? (Calendar, November 1997, #22).

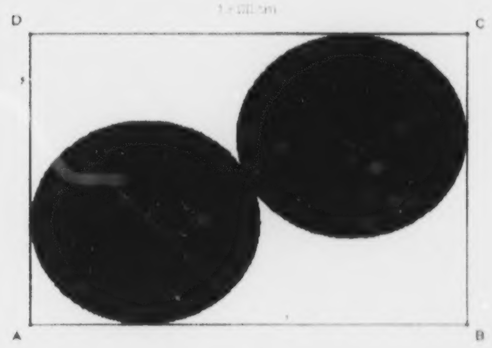

Figure 1

First Cycle: Understanding the problem. How do we represent the problem? What construction will allow us to find the maximum possible radius? These were two questions that appeared in the first phase of the solution process. Here, it was decided to explore key variables, focusing attention on a method of solution and the idea of examining particular cases helped to determine possible candidates. For example, four different ways of drawing the two circles seemed to be plausible to check:

(a) The centers of the circles being on the perpendicular bisector of the side AB.

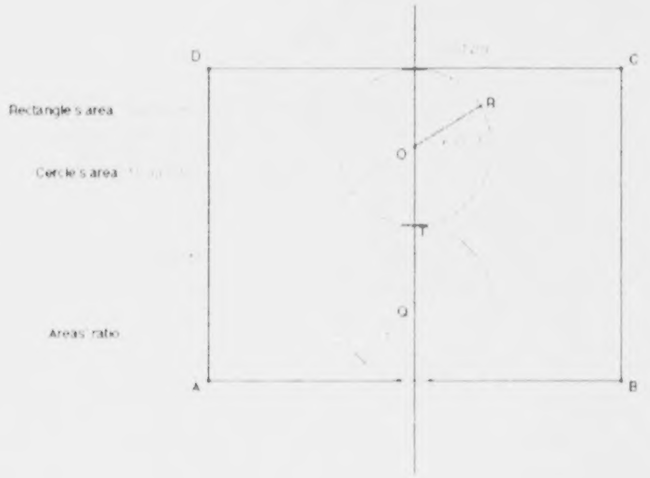

Figure 2a

(b) The centers of the circles being located on the perpendicular bisector of the side BC.

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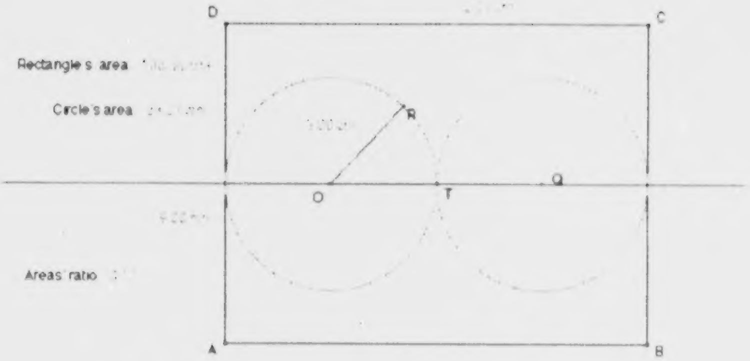
D
C
Rectangle's area
Circle's area
R
O
T
Q
Areas ratio
A
B

Figure 2b

(c) The centers of the circles will be located on the diagonal AC of the rectangle.

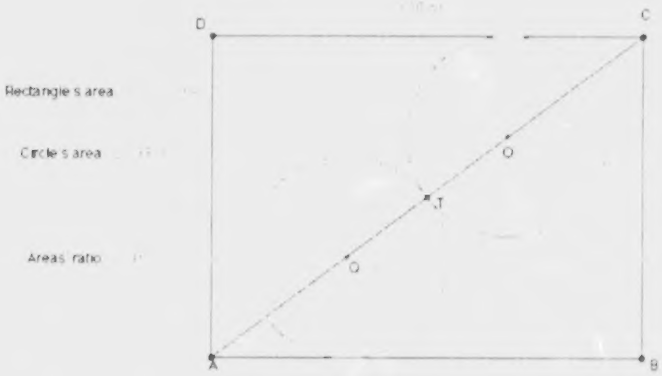
D
C
Rectangle's area
Circle's area
O
T
Q
Areas ratio
A
B

Figure 2c

(d) The centers of the circles located on the bisector of angles DAB and DCB respectively.

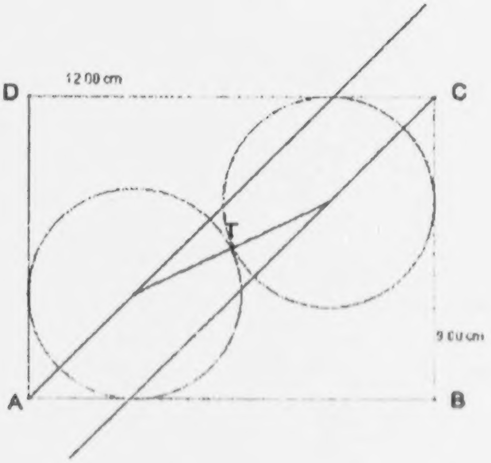

Figure 2d

From the figures, it was observed that the maximum radius for the circle (2b) and (2a) is 3 and 2.25 respectively. The center of the rectangle T seemed to be an important point in the figure. A conjecture that emerged from analyzing the above cases is that T functions as the tangent point of desired circles. It was also clear that the use and role of basic mathematical resources became crucial in representing and exploring the above cases. For example, how do I draw a rectangle? How do I draw an angle bisector? Or, How do I draw two pair of parallel segments? These are important questions that the teachers addressed during their approaches to the problem.

Second Cycle: Exploration of relationships via auxiliary constructions. While trying to represent the problem it was decided to draw line EF that passes through T and divides the given rectangle in two congruent figures. Based on the symmetry property of the rectangle, the construction of two congruent circles asked in the problem is reduced to finding the circle of the greatest area inscribed in one of the drawn figures. The ray EF meets the ray AD in G (extending rays EF and AD). Here the participants located triangle AEG and asked themselves: How can we inscribe a circle in this triangle? To answer this, they drew the angle bisectors in this triangle to determine the center of the circle (intersection of these bisectors). Then, they drew a perpendicular segment from the center to any side of the triangle AEG to find the radius of the circle. This circle does not necessarily pass through T. It was observed that the radius of the circle is less than the distance from the center of the circle to point T (Pythagorean Theorem). This is because OT is the hypotenuse of the right triangle OTM.

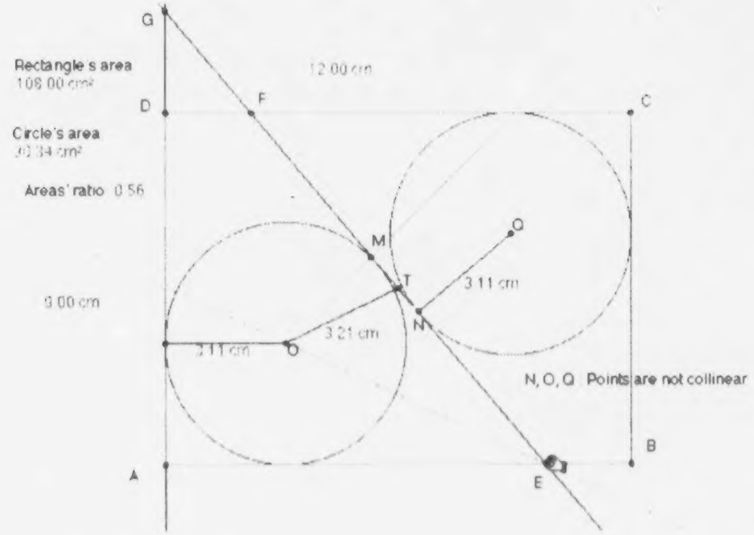
G
Rectangle's area
108.00 cm²
12.00 cm
D
F
C
Circle's area
30.34 cm²
Areas' ratio 0.56
Q
M
T
3.11 cm
9.00 cm
N
3.21 cm
3.11 cm
O
N, O, Q Points are not collinear
B
A
E

Figure 3a

Now by changing the inclination of the line EF (by moving E), it was observed that the radius of the inscribed circle will increase to reach the value OT. A similar construction on the other quadrilateral EBCF showed what the maximum value of the radius would be when both centers of the circles and T must be on the same line. Thus, the collinearity property became the key criterion to support the solution.

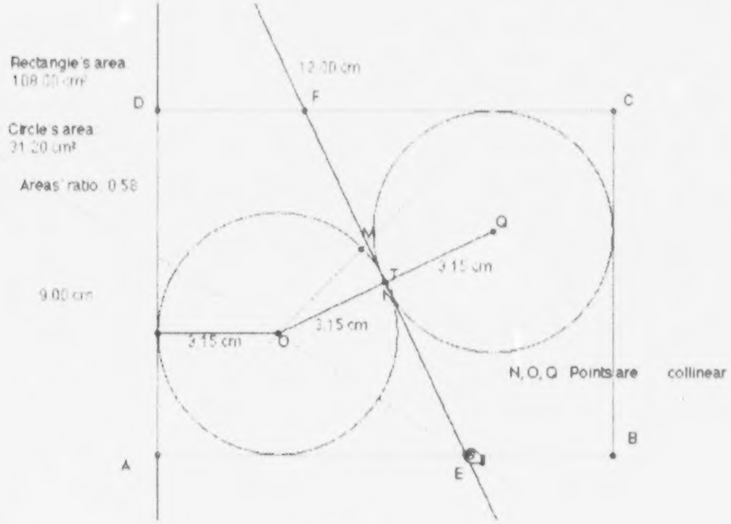
Rectangle's area
108.00 cm²
12.00 cm
D
F
C
Circle's area
31.20 cm²
Areas' ratio 0.58
Q
M
T
3.15 cm
9.00 cm
N
3.15 cm
3.15 cm
O
N, O, Q Points are collinear
B
A
E

Figure 3b

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Further explorations. How do we represent the problem, if one wishes to inscribe only the circle with the greatest area in the given rectangle? What is the maximum possible radius needed to inscribe three circles? Where should the centers be located? What happens if the dimensions of the rectangle change? These are some of the questions that could be explored with the help of the software. Figure 4a represents an exploration of the last question. In addition, the particular case in which the rectangle became a square showed that the angle bisectors are the same as the diagonals.

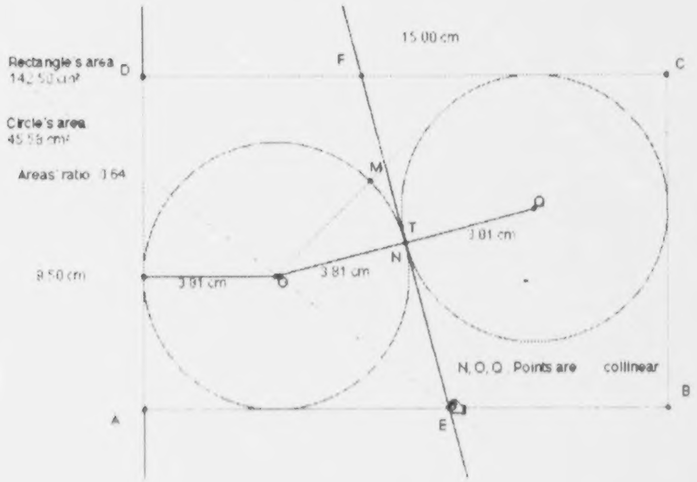

Figure 4a

Looking back. The method of solution described in the Calendar (p. 647) seems to take for granted that the reader will recognize that the centers of both circles are located on the angle bisectors respectively. In addition, it is also understood that the segment, which connects both centers, passes by the tangent point of the circles T (center of the rectangle). Here, the problem is simplified to find the expression (quadratic equation) whose solution gives the value of the radius. The solution achieved with the use of the software illustrates clearly how the same properties emerge naturally during the solution processes.

Concluding Remarks. The Cabri-Geometre software helped explore different approaches to the problems. In particular, the example presented here showed that the mathematical criteria employed to support their solutions displayed different features when compared with the traditional methods reported in the Journal. In addition, the process of working with

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the problems, via software, became a natural environment to introduce, pose, and pursue related questions. During the problem solving process key concepts and basic properties of the figures appeared as important resources to be utilized to approach and reach a solution. Indeed, seeing different dynamic representations of the problems helps visualize connections and realize other approaches to the problem. The paper also shows that even when the problem may not call for the use of computer directly, it was clear that the software became a vehicle to examine mathematical qualities of the problems.

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The Mathematics Teacher Calendar (1997, November), Vol.90, No.8.

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RETHINKING REPRESENTATIONS[1]

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The concept of representation is one of the more powerful psychological concepts used in the field of Mathematics Education in order to explain some important phenomena about children's way of thinking (see e.g., Janvier, 1987). This concept has a long-standing tradition in philosophy where it has been used with different connotations. For instance, as a faculty of the mind to *reproduce* the external world or as a kind of view that the individual *produces* from his/her own perspective. While the former has its roots in Cartesian philosophy and the idea of *re*-presentation as a mental mirror, the second one is related to the neo-Kantian idea of an individual who, in his/her acting in the world, produces an idea of it and its objects. Cognitivism proposed a distinction between external and internal representations whose link was theorized as a double-sense mapping from one kind of representation to the other and in which the external representations and mathematical notations were seen as aiding devices for the accomplishment of the cognitive mental processes (Meira, 1995). Since it is acknowledged that internal representations cannot be directly observed (De Corte et al., 1996, p. 502), experimentally, in the mapping perspective, external representations (e.g., students' use of mathematical notations, drawings) are seen as key elements to conjecture what is occurring in the head.

We argued elsewhere (Radford, 1998a) that the aforementioned classification of representations is clearly related to the traditional philosophical opposition between the individual and his/her milieu and the view according to which the milieu is the scene or space where individual thinking finds expression. However, with the progressive abandonment of the Cartesian cogitator and the solitary mind in contemporary approaches in psychology, the discourse about representations tends to include more and more the role played by others in the mental representations that an individual comes to form. It is beyond the scope of this paper to offer an overview of the many possibilities to theorize about these views. Therefore I shall limit my discussion to what a mental representation may be within the theoretical perspective I am advocating (Radford, 1998b).

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[1] This paper is part of a research program funded by the Social Sciences and Humanities Research Council of Canada, grant number 410-98-1287.

1. Representations as a social construct

First of all, a mental representation, I want to suggest, is not the direct product of the visual or sensual organs of the individual. The reason for this is that the individual is never in direct contact with his/her surrounding: this contact is mediated by the culture and the arsenal of concepts that the culture makes available to the individual. How Babylonian scribes conceived squares as different objects from us is an example. For us a side of the square is a segment. For them, it was *also* a side provided with a canonical projection, so that in fact the square had four rectangles added to it. If the side of the square has a length equal to "s", the width of the added rectangle will have "s" units and its length (the canonical projection) will have 1 unit (see Radford, 1996, in print). Their mental representation was different from ours. Of course, they did not necessarily use all four projections. It depended on the problem and the situation they were studying.

2. The context of mental representations

The previous point brings us to another feature we want to stress about mental representations, namely, their *contextual* nature. A mental representation is not independent of the context in which it is used. The "reality" represented in a mental representation is not the reflection of reality but the reflection of the reality as constructed by the individual in his/her interaction with others and participation in social practices. In this sense, in a child's drawing, it is not the object itself which is stressed but his/her *experience* with it. The experience, however, is not an individual act. To follow through with our example, a Babylonian scribe experienced a square differently from us. There is always a cultural aspect in experiencing objects. As it results from what we said in Section 1, any object is culturally embedded in a web of meanings which penetrate the ways we experience our world.

3. A paradox: The elusive concepts

The point stressed in Section 2 leads to some paradoxes. Indeed, the contextual and situated nature of mental representations can be seen as the impossibility for us to think of concepts in abstract terms. For it may be argued that if any mental representation is situated, and mental representations are our instruments of thought in order to think about mathematical concepts, then our thoughts will always remain situated and hence "concrete". When we produce a mental representation of a triangle, for instance, we do it for a specific triangle (it will be right or acute or equilateral or something else but it will be a *specific* triangle). A solution to this paradox is to consider mental representations as pointers (like signs) to abstract objects. But, in doing so, the problem is not solved. We merely succeeded in hiding it, for the question of the link between the concept and the mental representation still remains open. The concept appears as a sly fox always eluding the hunter's trap.

4. The concrete texture of mental representations

I suggested elsewhere (Radford, 1998a) that mental representations are genetically constructed on the basis of signs (e.g., words, mathematical notations, gestures). This position, of course, has been previously pleaded by Kaput (1991), Duval (1995) and others. What I have in mind is, however, something different. Since signs are not used at random but exist only within systems, within semiotic systems which provide the rules for their use and understanding (Radford, 1998b), in my view a mental (private) representation does not appear as a pointer to an abstract idea but as the contextual instantiation of social modes of knowing as expressed and contained in signs. As a result, the mental representation does not go beyond the signs but remains caught *in* them. 'Mental images' can be considered as materialized pieces of thought in the external sphere of the individual activity, colliding with the tasks at hand, and hence externally situated (Radford, 1998a, p. 291).

A practical pedagogical implication of this point of view may be illustrated through the case of functions. Functions play a central role in the new Ontario Curriculum of Mathematics. The learning of functions is seen as the capability of a student to move from one representation to another (tables, graphics, algebraic formulas, etc.). These "external" representations are seen as the expression of the *same* concept —that of function. In my view, each semiotic system (tables, graphics, etc.) leads to a particular concept (that, here, would be the one genetically constructed on the basis of the semiotic system in use). The mental image produced through sign use is not the same. It does not mean that all those conceptualizations are incommensurable among them. What this means is that a contextualization among representations will be needed to link them, and this requires a different pedagogical action.

In a longitudinal research program that I am conducting, a considerable effort is being made in order to understand the way students use representations and signs in the learning of algebra from the theoretical perspective of which I noted some aspects above. Instead of seeing external representations as merely the concrete manifestations and reflections of internal life, we are taking the former as the concrete aspect of the latter (see e.g., Radford in print-b).

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ON VISUALIZATION AND GENERALIZATION IN MATHEMATICS

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The renewed interest in visual forms of semiosis in the teaching and learning of mathematics highlights again the power and the pitfalls of these modes. Rather than describe again my research on visualization over the last two decades (Presmeg, 1980, 1985, 1986, 1991, 1992, 1994, 1995, 1997b), I shall make a few points about the nature of visualization, the nature of mathematical thinking, and some ways in which these may combine powerfully to use the strengths of the former in the service of the latter. After this introduction to reasons for the importance of finding ways to use imagery and diagrams in forms which support generalization in constructing mathematical ideas, I shall discuss several theoretical frameworks which have been useful in my research, or which show promise of being useful currently.

Krutetskii's (1976) mathematical interviews with highly capable students suggested to him not only that there is a large individual variation in preference for visualization in mathematics, but also that visual methods may both enable and constrain the mathematical problem solving of students. He suggested that students who prefer to use visual methods (his "geometric" type) manifest a certain imbalance in their thinking. He considered that, to a certain extent, it is a hindrance to be "riveted" to visual-pictorial schemes. However, he continued as follows.

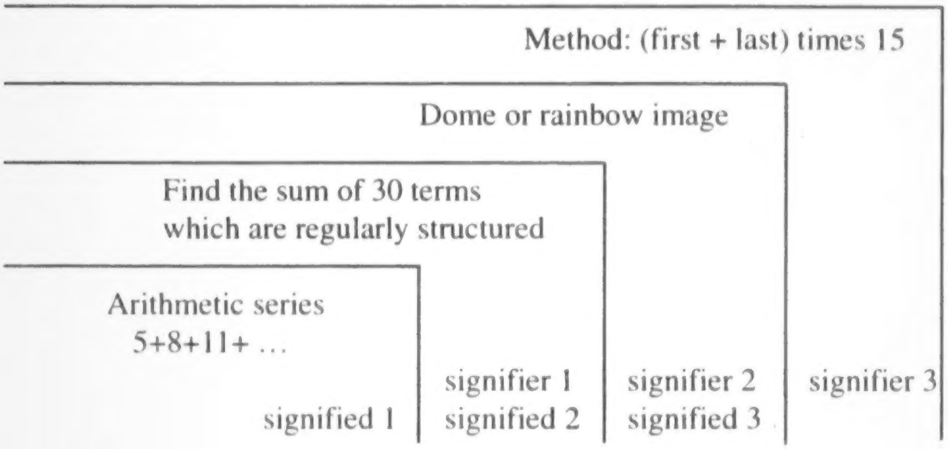
> "But only to a certain extent! The fact is that the graphic schemes used by these pupils [i.e., visualizers of high ability] are a unique synthesis of concrete and abstract. ... Botsmanova has shown convincingly that in a drawing - a graphic scheme - a schematization and a certain generalization of a visual image can occur. Supporting thought by, and even 'binding' it to, such a generalized visual image cannot prevent generalized thinking. In such a case this image is in a certain sense the 'bearer' of the sense and content of the abstract concept. The 'geometer' pupils feel the need to interpret a problem on a general plane, but for them this general plane is still supported by such images. In this way they differ from pupils of little ability - for whom visual images really bind thinking, push it onto a concrete plane, and hinder the interpretation of a problem in a general form" (pp. 325-326).

In my research too, all the difficulties experienced in using visual methods as the 54 visualizers in my first study (Presmeg, 1985) solved high school mathematical problems, related in some way or another to problems of generalization. Visual methods were effective when they were used in ways that supported generalization, as Krutetskii suggested. Then my question is, why is generalization an issue in the consideration of visualization in the teaching and learning of mathematics?

All questions regarding the teaching and learning of mathematics seem to refer back, directly or indirectly, to beliefs concerning the ontological question, "What is mathematics?" Definitions of mathematics with which I have found some resonance are those by Steen (1990), *the science of pattern and order*, by Thomas (1996), *the science of detachable relational insights*, and more strongly, by Ada Lovelace (Noss, 1997), *the systematization of relationships*. These three definitions share the aspect that some kind of generalization of patterns is an important characteristic of mathematics.But it is just this aspect which may cause problems in the use of some kinds of imagery or diagrams in learning mathematics. After all, a diagram by its nature depicts one concrete case or instance (see the delightful article "Let ABC be *any* triangle", Schwarz & Bruckheimer, 1988). Successful use of visual methods entails that some way be found of supporting generalization in their use. There are several ways in which this support of mathematical generalization may be realized, including use of more abstract forms of imagery such as pattern imagery, use of dynamic imagery, and use of imagery and diagrams in metaphoric ways (often idiosyncratic), in which the concrete image becomes the bearer of abstract information (Presmeg, 1992, 1997b). I see as extremely promising in this regard, the role of dynamic computer software and multiple modes of representation (Kaput, 1987).

A second and related question is, what theoretical frameworks are useful in investigating ways in which such generalizations may be supported by visual methods? My original research in this area was based in the framework constructed by Krutetskii (1976), but since then I have found it illuminating to examine data on visualization in the light of use of metaphors and metonymies, in the imaginative rationality of embodied cognition (Johnson, 1987; Lakoff, 1987). This metaphoric and metonymic use of visual representations of mathematical constructs resonates well with semiotic frameworks with which I am currently engaged (Presmeg, 1997a).

An example of an image used metaphorically in high school mathematics is the following, taken from data collected in an imagery project with high school students in Florida in Fall, 1991. Mark had been asked to find the sum of the first 30 terms of the sequence 5, 8, 11, ... After some dialog with the interviewer, he solved the problem correctly by adding the first and 30th terms, then the second and 29th terms, and so on, until he realized that there would be 15 of these sums. He obtained the answer by multiplication, and explained when questioned, "First I saw, like, a dome, where it's going like this, and it's continuing" (drawing concentric arcs) "until we got down to the two very middle terms. And I just deducted that if we have 30 terms, and we're taking two each time, then it has to be 15. So I multiplied by 15". Incidentally, this example also illustrates the efficacy of alternating use of visual imagery and logic. A different student in the same project saw Mark's "dome" as a "rainbow". In these cases, the dome or rainbow image was the vehicle of a metaphor in which the tenor was the process of finding the sum of a number of terms of an arithmetic sequence the 'Gaussian' way. For both of these students, the metaphoric image was an idiosyncratic way of representing a mathematical principle. An image often accompanies a metaphor, although the image alone is not the metaphor. Such images from the vehicle or source domain seem to give force and memorability to the comparison of domains, which constitutes the metaphor. In this manner, Mark's "dome" may come to encapsulate the mathematical principle with which it was associated metaphorically. In the following diagram the process is analyzed as a semiosic chaining.


Method: (first + last) times 15
Dome or rainbow image
Find the sum of 30 terms
which are regularly structured
Arithmetic series
5+8+11+ ...
signified 1
signifier 1
signified 2
signifier 2
signified 3
signifier 3

This example and many others highlight the myriad ways in which imagery may be used in the service of generalization in mathematics. I am finding semiotic models, both the triadic model of Peirce and Lacan's inversion of Saussure's diadic model (Whitson, 1997) to be illuminating frameworks in such analyses.

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SCHOOL ALGEBRA: THEORY AND PRACTICE

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A reflection on the relationship between theory and practice in the context of innovative approaches to the teaching and learning of algebra leads to take into account both the actual classroom practices and curriculum contents and the research results from new proposals.

In the present working group sessions we will focus our attention on the changes that have been produced in the classroom practices with the introduction of innovative techniques based or **not** in the use of information technology. We will predominately study the changes in the role of the teacher, the role of the students and the role of the environment in which the educational act is produced.

In particular, we will discuss the following issues:

- new forms of generalization and formalization;
- collaborative problem solving and the role of the computer;
- new approaches to analyze word problem solving processes;
- building-up algebraic syntax with graphic calculators;
- innovative organization of the classroom environment.

Algebra & Technology: New Semiotic Continuities & Referential Connectivity. (By James J. Kaput, University of Massachusetts at Dartmouth)

This informal discussion paper will deliberately take a somewhat different orientation to algebra than is usually the case, investigating how technology affects basic semiotic assumptions and habits, with a special focus on the algebra of functions – their definition, manipulation, and use as models.

Our historical applications of technology to help with both the learning and the doing of algebra have passed through several stages. The earliest involved facilitating manipulations of character strings, as was the case in the late 1960's with MACSYMA being used to do the complicated symbol manipulations required in General Relativity. In the 1970's people increasingly used computer technology to plot coordinate graphs of, and generate numerical data from, algebraic functions of one or more variables. In the 1980's these notations were increasingly linked to one another so that by the end of the decade one could make changes in one notation and these changes would be almost simultaneously reflected in any of the others. Two features were common to all the development up to this point. One was the central role played by character-string notations in both the definition and manipulation of the functions – whether they were closed-form or recursively defined functions. The second was the traditional relationships between the algebraically defined mathematical objects as models and the phenomena or situations that they were used to model or represent. Both of these features reflect a deep, but largely tacit view of formalisms as separate and distinct from informal notations or utterances and from the phenomena that they are taken to represent. In particular, algebraic statements are part of the universe of mathematical notations, with separate rules of reference, with syntax distinct from natural languages, and abstract independence from the media in which they happen to be instantiated. This view in turn is intimately integrated with a Platonist and representationalist philosophical orientation that takes:

- Mathematical objects as pre-existing, to act as pre-given reference objects for mathematical notations,
- Language as an inert representational instrument which does not help create mathematics but only enables us (if we are sufficiently skilled in its use) to see and do mathematics, and
- Mathematics as separate from the material and social worlds we inhabit.

All three of these positions are eroding in the face of an increasing flow of technology-enabled semiotic systems that offer:

1. Increased *semiotic continuity* between mathematical notations and our extra-mathematical methods of manipulating objects in our world, and
2. Increased *referential connectivity* between inscriptions taken to refer as models to phenomena or situations.

After offering a characterization of the kinds of 21st century mathematical activity that stimulate the *need* for new representational forms, the remainder my presentation paper will be devoted to illustrating and explicating these two assertions, developed in three sections.

1. Parametrization of Mathematical Objects and Relations, and the Increased Steepness of the 21st Century Learning Curve.
2. Semiotic Continuity with Ordinary Physical Actions.
3. Referential Connectivity between Inscriptions and the Phenomena or Situations They Are Taken to Model.

Collaborative problem solving by 13-year-old algebra students: didactive productivity, patterns of interaction, and the role of the computer. (By Carolyn Kieran, Département de mathématiques, Université du Québec à Montréal).

Background: The students featured in this research are in their first year of high school (13 years of age). They have just completed, along with their classmates, a seven-week introduction to algebra based on an object-oriented, functional approach that aimed at giving meaning to the symbols and transformations of algebra by means of graphical representations and operations with these representations—an introductory module prepared by Anna Sfard and me. Much of the seven-week sequence had involved pair-wise work with activity sheets, interspersed with classroom instruction by the teacher and whole-class discussions. The content that was emphasized was primarily situations involving linear functions, although some experience with quadratic and cubic functions was included. Several days of class work were spent at the computer, where students in groups of two were able to explore the role of the parameters in graphical and symbolic representations, and their relation to the problem situations.

Aims: The study to be presented to the Working Group is part of a larger research program focusing on alternate approaches to the introduction of algebra in environments that include a computer component. More specifically, pairs of students who had just completed the above 7-week introduction to algebra were presented with problems related to a family of functions that they had not experienced in class. The objective was to study the nature and productivity of students' collaborative discourse while they tried to adapt their recently-acquired algebraic and graphic thinking tools to the exploration of novel situations. Another primary focus was the extent to which students would choose to use the computer as a mediating tool in their problem-solving work, as well as the role of this usage in facilitating effective collaboration.

Methodology: The methodology that was used is one based on a technique by Hatano and Inagaki (1994). It involved joint problem-solving work, followed by individual report writing and individual work on problems analogous to those worked out jointly. First, students were selected who, in the opinion of the math teacher, had succeeded with both the mathematical content of the seven-week algebra introduction and the related computer work. Twelve were chosen for this study, all of whom had obtained at least 80% on the test given at the end of the seven-week introductory algebra sequence, and who had had some experience in working collaboratively with another student from the same pool. Thus, six pairs were formed (five male pairs and one female pair).

All pairs were given the same set of tasks to be solved. One pair of students was observed and videotaped at a time. After a couple of warm-up questions, the pair was given a single set of activity sheets. They were asked to collaborate in the solving of the given problems, taking as much time as they needed (the tasks were designed so as to take about 45 minutes to an hour to complete). A computer was beside their work table available for their use, should they so decide (the computer was set up with the same program that they had used in their math classes: *Math Connections: Algebra II*).

Upon completion of the joint work, each individual was asked to write a self-report in which she/he described some of the approaches used during the joint problem-solving work and some of the things they felt they had learned during this joint work. Immediately after that, each member of the pair was given a set of activity sheets containing tasks that were analogous to those worked on jointly.

The Working Group Presentation: The Working Group presentation will focus on the productivity of the mathematical discourse of six, quite mathematically strong, 13-year-olds working in pairs (in a computer-available environment) on a sequence of problems from a family of functions that they had never before experienced, and will relate that productivity to their patterns of interaction and to the role played by the computer in their conversations.

Verbal arithmetic-algebraic problems solution. (By Luis Puig and Fernando Cerdán, University of Valencia, Spain).

We will address the subject of transference of the algebraic operativity which has been recently learned to some other contexts, as would be the case of arithmetic - algebraic verbal problem statements.

Amongst the transfer processes of a given algebra operativity to problem contexts, where it could be used for its solution, are those which can identify the procedures for the solution, in which actually such operativity could be applied. These processes of simple recognition are only part of the complex transfer process (which includes, among others, the analytical reading processes of the statement, the production of a strategy and a representation system, as well). When reasoning a complex problem it is more than enough some kind of distraction for a child to focus on certain context, in which the recognition of what has already been learned and mastered at operational level could not be applied. The likelihood of experiencing these types of centering phenomena during the development of the procedure for solving the problem is not overlooked and if this is the case, all the procedure could be upset or still, the possibility of solving the problem could be hindered. The solution to these type of obstacles is a level of transfer of the operativity, in which the already mastered syntax elements could be drawn from the semantics of the context from which the problem is addressed (or solved).

Progress towards semantics. In spite of the confidence that is reflected by some students in being able to solve the new equations operationally when these appear in other contexts, to be able to speak of a true transfer of that operational capacity to the solution of problems, still to be considered are the processes that lead to understanding the statement and writing an equation. Among these processes are those of *representation* of the elements of the problem and this presupposes reading and analyzing the statement that distinguishes between what is given and what must be found and which allows the relevant information to be recovered while leaving aside whatever is not essential; and this might also precede (or sometimes follow) the *representation,* the production of a strategy to attack the problem. The consolidation of the first elements of algebraic syntax is based on their link with a non-algebraic semantics, in this case that of how problems are stated.

Returning to the matter of acquired operative transfer to broader contexts such as how to express problems, while the cases presented here cannot be considered to be an example of transfer in the fullest sense of the term, it may be referred to at another level of partial transfer, which consists in having thoroughly established the problem-solving operative actions to the extent that, on the one hand, the types of situations that require the application of such operational activity may be recognized (equations, in this case) and, on the other hand, the simple recognition of those situations serves as an impulse to release the corresponding operative actions (although teaching is required to implement them); this is a necessary, although insufficient, level of transfer in order to progress to the semantics (of problems).

When the performance of the students have already put into operation the new elements of syntax, there is evident progress in algebraic semantics (as far as its problem-solving use is concerned) which also implies progress in the use of syntax; the opposite is also true: progress in syntax implies progress in the semantics of algebra; this last-mentioned appears to be a fairly generalized opinion, since, in effect, a certain level of syntax is always considered to be a factor in helping to solve problems.

In the field of teaching, it would seem to be possible to close such cycles a relatively short period of time. This is not only desirable and necessary, but that it is indeed also possible; this opens the possibility of attempting, at the teaching level in the classroom, to arrive at a harmonious development of algebraic syntax elements and their corresponding semantic versions (whether as elements based on the context of the model(s) used to introduce them or, rather, as elements that are themselves used to model situations that stem from the statement of problems), so that they are combined to make proper and congruent use (as to these two aspects of syntax) of the algebraic language.

Algebra as a language in-use: First steps toward a model for using graphic calculators in the classroom.

(By Tenoch E. Cedillo A., Universidad Pedagogica Nacional, Mexico.)

My presentation is aimed at looking at a particular case that may provide us with elements for relevant discussion on how practice may inform about theory and vice versa.

Many years ago I came into an arts and crafts school workshop looking for the teacher and got amazed of how happy and productively children were working. Everybody in the room was busy and deeply interested in what they were doing. I could not find the teacher at a glance, it seemed he was no there. But the teacher was there silently going around the room observing children's work and giving some help when they ask for. Then I thought: that is precisely the way I'd like the mathematics classroom to be. A place where children come willingly to work, a place where children got actually engaged doing something while learning how to think mathematically.

Since then I've been trying to find out how to make the mathematics classroom look like that school workshop. The advent of calculators in the mathematics classroom has made me feel I am getting closer to that goal. Though the same may be said about computers, due to the aims of this paper the discussion will only focus on calculators. In the rest of this brief account a schematic view of a calculator-based study done in Mexico during the last few years will be addressed.

The powerful resources nowadays available from hand held calculators offer new ways for mathematical validation. Numerical-based strategies and visual approaches provided by graphic calculators currently challenge symbolic algebra as means of validation. These numerical and visual resources allow us to design teaching activities so that students work out mathematical challenges without previously having any formal approach to the formal mathematics involved. In other words, those resources allow us to put students in the situation of learning mathematics by using it in a similar way as we learn the mother tongue. Such an approach implies many other theoretical and practical issues, but there is no room to discuss all this here.

Research. The study was carried out in different stages and school settings. First (1994) within a highly framed classroom environment where the study was part of the regular mathematics course, with twenty-five 11-12 year olds and the researcher playing the role of the teacher. The results obtained from that stage were encouraging enough as to extend the study to different school environments.

Preliminary results from the last stage strongly contrast with the ones obtained from the earlier stages. There are serious difficulties having to do with teachers' adoption of the pedagogical approach. So far, optimistically, only 15% of those teachers have got an acceptable success in terms of their students' mathematical achievements and in recasting their previous teaching practice to the new approach.

Final remarks. The data available so far suggest that learning by exploring is a quite suitable approach for students if the teacher acts accordingly with the proposed teaching view. The case of teachers seems to be much more difficult, particularly due to their previous teaching experience, which needs to be radically modified on the light of the new approach. Nevertheless, the favorable results obtained with some of the teachers suggest that new distance means, such as video, e-mail and video conference should be tested in order to enrich the face to face fashion for teacher training.

General Discussion

Recently, much emphasis has been set in the use of computer environments, but less light (from a theoretical point of view) has been cast on how this have result in an innovative organization of the classroom. Computer use is a beneficial fact without doubt, been the traditional teaching practices so obsolete, but it seems that these practices changes can be implemented by other new approaches as recent research results indicates. Thus, it seems relevant to center our discussion in the innovative organization of the classroom environment, whatever is the way it is achieved.

THE ROLE OF ADVANCED MATHEMATICAL THINKING IN MATHEMATICS EDUCATION REFORM

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The Advanced Mathematical Thinking Working Group of PME-NA spent some time since the last meeting discussing what the term, advanced mathematical thinking, meant to members of the group. This paper will elaborate on some of that discussion and suggest some next steps.

In an effort to define advanced mathematical thinking, the working group decided that a reasonable approach was to start with examples. If we knew what constituted advanced mathematical thinking for members of the group, we could come to some common language and shared meaning for the purposes of the working group. To this end, there was a suggestion that the group use some organizing questions to classify examples of advanced mathematical thinking. These questions centered on the following areas:

- What are the characteristics of a person's mathematical work that make it advanced mathematical thinking?

- Is advanced mathematical thinking accessible to persons of all ages and educational levels? If so, is there something that is the same about advanced mathematical thinking across different ages and educational levels? Can one definition serve to define advanced mathematical thinking no matter the age level of the person? Or, does advanced mathematical thinking mean different things at different age or educational levels?
- Is the ability to provide mathematical justification a necessary attribute for thinking to be classified as advanced mathematical thinking? Is a necessary attribute of advanced mathematical thinking that is contains an answer to the "Why" question? (e.g., Why did you do that? Why does it work that way?)
- Are reflection and abstraction necessary components of advanced mathematical thinking? Must this reflection and abstraction occur voluntarily?
- Does advanced mathematical thinking require managing and representing complexity? Is managing and representing complexity a sufficient condition?

One perspective on advanced mathematical thinking centers discussion on the facility with mathematics that certain cognitive facilities give the thinker. From this perspective, generally speaking, advanced mathematical thinking is a term referring collectively to ways of thinking that enable an individual to learn, produce, appreciate, and apply mathematics. From this perspective, advanced mathematical thinking does NOT imply that a thinking ability that can only be acquired after non-advanced mathematical thinking has been mastered.

Underlying this perspective is the premise that the learning mathematics at all levels, elementary, secondary, and post secondary, requires the advanced mathematical thinking. An example of advanced mathematical thinking that is indispensable in all levels of mathematics is *multiple concept interpretation*. That a concept can be understood in different ways, should be understood in different ways, and it is advantageous to change ways of understanding of a concept while attempting to solve a problem, is a way of thinking commonly absent from most students' repertoires of reasoning. This deficiency can be particularly apparent in the teaching of linear algebra. The learner must understand, for example, that problems about systems of linear equations are equivalent to problems about matrices, which, in turn, are equivalent to problems about linear transformations. Students who are not equipped with these ways of thinking are bound to encounter difficulties.

The seeds for this type of advanced mathematical thinking are planted early in students' mathematical careers. Arithmetic can and should be the place where students begin to acquire this indispensable way of thinking. For example, students should be taught the advantage of each one of the following interpretations of fractions in solving related problems:

Counting: 3/4 is 3 things, each of which is 1/4.

Partition: 3/4 is the result of dividing (equally) 3 things into 4.

Measurement: 3/4 is the relative portion of 4 that goes into 3.

Object: 3/4 is a point on the number line.

Part-Whole: 3/4 is the selection of 3 out of 4.

Ratio: 3/4 is a representation of the set consisting of the fractions, 6/8, 9/12, 12/16, ...

Other examples of advanced mathematical thinking, whose seeds lie in students' early mathematical work, include: reasoning deductively; reasoning inductively for the purpose of exploring a conjecture; algorithmic reasoning; transformational reasoning (e.g., composition and decomposition of numbers in solving arithmetic problems); and entification (the mental process by processes and actions are conceived as conceptual entities). As we examine the role of advanced mathematical thinking in mathematics education reform, it is incumbent on us to account for the early development of these abilities.

In choosing from among the perspectives on advanced mathematical thinking, some working group members observed that much of the current work in mathematics education focuses on elementary and sometimes secondary mathematics, and that there is less research, in general, being conducted about students' thinking about more advanced mathematics. It was noted that there is some research on mathematical thinking in the context of calculus, less on mathematical thinking in the context of linear algebra, and even less research on mathematical thinking in the context of more advanced mathematics. These working group members felt a need for a forum involving the analysis of thinking about advanced mathematics, and one suggested a working definition of advanced mathematical thinking that concentrates on thinking about advanced mathematics, say calculus and beyond (or as Buzz Lightyear (a popular animated film figure) would have it—infinity and beyond).

Given a tendency to use advanced mathematics as a site for advanced mathematical thinking, a number of working group members returned to definitions previously offered by earlier mathematics educators. For example, Dreyfus (1991) suggested that advanced mathematical thinking be defined by the level of the material. That is, he offered the position that advanced mathematical thinking is equivalent to thinking about advanced mathematics. Tall (1991) stated that "the move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on these definitions." (p. 20). A major difference in these perspectives is that one focuses on the level of the mathematics and the other focuses on actions undertaken in doing mathematics.

A third perspective on advanced mathematical learning derives from Robert and Schwarzenberger (1991) who attempted to define advanced mathematical learning as follows:

> The search for single features which are specific to the learning of advanced mathematics proves to be inconclusive. Many proposed features are seen, on closer examination, to display strong continuity with the learning of mathematics at younger ages. Nevertheless, it seems that, when all these features are taken together, there is a quantitative change: more concepts, less time, the need for greater powers of reflection, greater abstraction, fewer meaningful problems, more emphasis on proof, greater need for versatile learning, greater need for personal control over learning. The confusion caused by new definitions coincides with the need for more abstract deductive thought. Taken together these quantitative changes engender a qualitative change which characterizes the transition to advanced mathematical thinking. (p. 133)

Some of these characteristics may be artifacts of the way advanced mathematics is sometimes perceived to be taught (more concepts, less time, fewer meaningful problems). The other features (the need for greater powers of reflection, greater abstraction, more emphasis on proof, greater need for versatile learning, greater need for personal control over learning) seem to "trigger" a need for advanced mathematical thinking. So, from this perspective, a characteristic of advanced mathematical thinking is the kind of precise, yet flexible, logical thinking that is necessary in the face of abstract ideas which are difficult (or impossible) to model in "real world" terms. A second common feature of this perception of advanced mathematical thinking is the idea of managing complexity. Third, from this perspective, advanced mathematical thinking is the type of thinking one has to do when one is thinking about the skeleton, the bare essence, of a mathematical idea without any "flesh" or "real world quality" attached to it.

Finally, one member of the working group offered a quote from Herscovics (1996) as food for thought about advanced mathematical thinking. He felt the quote contained some reasonable characterizations of advanced mathematical thinking as an activity in which one engages, although he thought that the situations and purposes needed to be part of what advanced mathematical thinking is all about.

> The third phase, that of abstraction, involves two phases, the first one consisting of the separation of the concept from the procedure, and a second phase characterized by the generalization of the concept, or by some conservation reflecting the invariance of the mathematical object, or by the reversibility and possible composition of the mathematical transformations. The fourth level, formalization, was evidenced by either the use of mathematical symbolism, or by the logical justification of operations, or by the discovery of axioms (p. 359)

Herscovics later continued, writing that

> ... formalization refers to its usual interpretations, those of axiomatization and formal mathematical proof, which at its elementary level, could be viewed as discovering axioms and finding logical mathematical justifications respectively. But two additional meanings are assigned to formalization, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree. (p. 361)

The working group on advanced mathematical thinking thus ends its work in the proximity of where it started, but more informed about the issues surrounding their topic. The pursuit of the meaning of advanced mathematical thinking and its impact on mathematics education reform has led to the identification of a list of distinguishing characteristics and to the recognition that the seeds of later advanced mathematical thinking are planted early in students' mathematical careers. Next steps include the identification of examples of advanced mathematical thinking at the college level, and clarification of the development of advanced mathematical thinking from early mathematical seeds.

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GENDER AND MATHEMATICS: EXPLORING THE ABSENCES IN RESEARCH

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Focusing on issues of gender and mathematics, the Gender Working Group has as its continuing goals the *synthesis* of knowledge generated through a broad spectrum of research traditions and the *identification* of directions and questions for emergent research. Because we intend the 1999 Gender Working Group to be a continuation of our prior work, a description of the 1998 sessions is a starting point for our discussion here. Taking the meaning of the term "working group" to heart, we planned on having all participants work for the three sessions we were allotted. We originally intended to focus the three days of our 1998 work session as follows: (1) Why do this work? and What do we Know?(2) What are the compelling topics/ideas/methods for further study? (3) How do we further this work?

On Day One we had introductions of all participants and a discussion of why we study gender and mathematics. Following short presentations summarizing the five papers written (and distributed) as starting points for our 1998, whole group discussion centered on the question of what we really know thus far.

As we (the organizers) reviewed the many contributions of participants to prepare for the next day's session, our first observation was that we had talked a lot about math and a lot about gender but very little about the connection between them. A matrix which used "the sex/gender system" to define columns and "the doing of mathematics" to define rows seemed both to fit the data and to suggest a way of addressing the missing connections. We decided to present these parameters of a matrix to the group and lead the group in filling in the cells with knowledge linking the two dimensions. Cells which remained empty would indicate an absence in the research, a need or direction for future study. What happened on Day Two told us that this organization was too simplistic. Instead, the matrix structure stimulated a rich discussion of its insufficiency for the problem.

Thus, we began Day Three wanting to know what kind of organization actually does fit the thinking being created in the group. After a brief review of the previous days' happenings each participant received multiple slips of white and yellow paper. They were asked to jot down research findings, ideas and observations about gender and mathematics which they found important over the past two sessions on white slips, and brief relevant questions on yellow.

Large sheets of poster paper taped together to make a surface were placed on the floor; all participants encircled this, some sitting on the floor and others on chairs. We began sorting through the slips of paper, reading them aloud, discussing and deciding as a group where to place them on what would become our affinity diagram. After all 64 white and 63 yellow paper slips were on the surface, we identified clusters that had emerged and then discussed linkages between clusters. The web that resulted represented the multidimensionalities of the concepts surrounding the study of gender and mathematics. Naming the clusters and other work toward elaborating the web would be a continuing project of the working group during the next nine months.

We finished Day Three with a discussion about the future of our group. Participants planned to maintain contact. Following these sessions, e-mail communication among group members was established; discussion of issues such as Gender and Standards 2000 was begun and continues. These discussions helped to frame the starting points for the 1999 Working Group sessions.

The organizers have worked to synthesize the outcomes of the 1998 sessions; this synthesis is another factor defining the starting points and goals for the PME-NA XXI sessions. Figure 1 presents tentative names of the clusters (nodes) identified by the group and the organization of these nodes in a web of relation between and among the factors uncovered and discussed in 1998. The web is elaborated further on the working group website (http://www.coe.ohio-state.edu/gendermath). The dynamic hypertext representation allows all group members to add comments and evidence including citations, links to other websites, summaries of unpublished research, and other materials.

Establishing Direction for PME XXI

We identify four clusters of statements and questions about gender and mathematics (Women, Plural; Other Factors/Conditions; and the two oppositional social clusters) are not linked with other nodes (clusters) of the web. This absence of connection indicates serious gaps in our understanding of the issues. Moreover, group members' analyses of individual clusters (see website) reveal that there are internal issues and problems which need to be addressed. In response to these absences, group members were invited to submit short papers addressing gaps for as starting points for work at PME XXI. Five papers will serve this function; their authors and (tentative) titles are:

> Dawn Leigh Anderson, University of Georgia
> *What We Know and What We Need to Know about the Voices of Women Mathematicians and What They Convey about Myths Surrounding Mathematics and The Mathematical Community*
>
> Peter Appelbaum, William Patterson University
> *What We Know and What We Need to Know About SmartGrrls and Glass Walls*
>
> Linda Condron, The Ohio State University at Newark
> *Mothers and Daughters: What We Need to Know about Positive Influences on Girls Attending A Math Summer Camp*
>
> Suzanne Damarin, The Ohio State University
> *Gender and Mathematics on the Eve of Y2K: What We Know and What We Need to Know about Technologies of Difference and Differences in Technology*
>
> Diana Erchick, The Ohio State University at Newark
> *What we Know and What We Need to Know about Women's Mathematical Voices: Talking about Specific Content of Mathematics*

These papers and a fuller report on the work of the group at PME XX will be available at http://www.coe.ohio-state.edu/gendermath and at the conference and will frame the 1999 sessions of the Gender Working Group.

Examining the Absences in the Web: Foci of the papers

Consider, for example, our node "Women's Agency and Choice." We asked about decisions women and girls make around mathematics. Are those decisions informed? How much do we "make" girls do mathematics? Why do some women and girls opt out? Why do some *not*? As such questions are raised, Dawn Leigh Anderson and Laura Condron work with women who are successful in mathematics. Anderson interviews women mathematicians using their life stories as models for young girls. Not only do we learn about why some women study mathematics, but we offer to young girls stories to help inform them of the purpose of mathematics, the options, and the consequences of study. In the past, Condron has interviewed women in math-using fields and reported on this research; with her current work she extends her work to include adolescent girls attending a summer camp. Many of the girls are supported in their attendance by mothers and grandmothers who are mathematicians or women in math-using fields. For

Condron, not only does the question of how women came to learning mathematics for their professional lives become important but also how those women influence the young girls in their personal lives.

Another opportunity to fill a gap within a cluster is finding stronger evidence of the role of mathematics content in developing voice and women's ways of knowing mathematics. The observations and questions that clustered in the node labelled "Women's Ways of Knowing Math" were focused on mathematics and what it is. Might there indeed be a "female mathematics"? What about the dichotomy of mathematics as exploration and as a deductively built finished product? What do women do when they do mathematics and how can the work of other researchers help us understand those processes? Diana Erchick has worked with *Women's Ways of Knowing* (Belenky Clinchy Goldberger and Tarule, 1986) and mathematics life histories to study the processes women employ as they develop mathematical voice. However, she has found that talk of specific mathematics content has been difficult to generate in the reflective data collection in her work. She now shares work where she includes in her mathematical life history work interviews around specific mathematics activities. The talk is from women who are teaching small groups of young girls and who answer questions addressing their responses to the girls and to their own epistemological interactions with the mathematics.

Work in gender and mathematics, like that of Anderson, Condron and Erchick, informs not only the topical nodes within which they seem to be embedded. Each has the potential to inform other nodes as well, such as "Intuition and Doing Math," "Feminist Informed Mathematics Education," "Standpoint Questions," and "Gendered Success?". Thus, these papers have the potential to contribute partial links between and among web nodes not currently linked directly.

Suzanne Damarin's paper speaks to topics and issues related to the nodes which are currently not linked with the rest of the web; in particular, she is concerned with technology and diversity in multiple relation to the study of gender and mathematics. As she argues elsewhere, in our times of rapidly shifting populations and rapidly advancing technologies, these topics need to be ... and can be ... addressed together. There are extensive literatures on gender and technology, technology and mathematics, mathematics and diversity (including ethnomathematics), diversity and pedagogy, and diversity among women which have not been integrated fully with the literature of gender and mathematics; such integration would be a fruitful endeavor. Drawing on these literatures, Damarin suggests ways in which the assumptions, questions, and implications they suggest link with other nodes on the web. Arguing that the absence of links between and among technology, diversity and the other nodes reflects a major faultline in research on gender and mathematics, she relates this absence to pervasive ideologies such as the ideas that America is a classless society, (contrariwise) that concerns about race and education should be subsumed by concerns about socio-economic status and education, and that (without qualification) technology is progress.

In responding to needs identified in our web, Peter Applebaum raises questions about how to address developing projects for fast track girls. He talks of "SmartGrrls and Glass Walls," how the glass walls are recognized and how projects are designed to remove the walls or support girls in pushing past or through them. His work connects to other nodes as well, responding in part to questions raised in "Feminist Informed Mathematics Education" about teaching, learning and assessment. Applebaum speaks of a "cultural commitment to 'good pedagogy'" and the role of Foucauldian notions of diffused power in work to support "smartgrrls." What he asks is not as much about how our web is constructed but about how it and the study of gender is situated within the field of education. Consequently, he and the rest of us continue to ask why some of us take on gender as a struggle while others do not.

Organization of Sessions

At PME XXI, the Gender Working Group will follow the same general guidelines as in 1998. That is, sessions will be devoted to discussion, with the organizers responsible for synthesizing and analyzing each day's work and, based on this between session work, determining the starting point and framework for the next session. The first session will be framed by the summary of work at and since PME XX and by the five papers. In the third session, we will try to firm up some of the questions developed in sessions one and two and to form interest subgroups around some of them. We will discuss potential funding sources. Also, we will discuss with the group any goals toward which we want to work throughout the coming year. In this context, we will return to discussion of our original idea to work toward a monograph.

In Closing

As a network, the gender working group continues to push for more questions and more answers. We continue to assess our progress and our needs, working toward action and publication (both in journals and electronically) and not willing to settle on merely knowing the state of affairs with respect to gender and mathematics. Our PME-NA XXI sessions are places where we can work toward specific research agendas and a set of questions which lend themselves to empirical and cultural studies research. We do not limit ourselves to the typical quantitative and qualitative studies of mathematics education, but also include historical, philosophical, textual analysis, media criticism, and other kinds of scholarly and academic studies. A major goal of our group is to define questions, to partially order them by importance and/or accessibility, and to identify potentially fruitful ways of studying them.

RATIONAL NUMBERS > REPRESENTATIONAL FLUENCY > 21st CENTURY CONCEPTUAL TOOLS: "GOING BEYOND CONSTRUCTIVISM"

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How will the Working Group Operate at PME/NA-Mexico?

At PME/NA-Mexico, our sessions will consist of a series of brief 5-minute "elevator speaches" (so called because they're similar to what a researcher might be able to say about "the essence of his or her work" if asked about it on an elevator) followed by discussions related to the following issues.

- Finding ways to *provide early democratic access to powerful conceptual tools* (constructs or capabilities) that enable all students to achieve extraordinary results.
- Finding ways to *assess deeper and higher-order understandings* that are likely to provide the foundations for success beyond schools in the twenty-first century.
- Finding ways to *recognize and reward a broader range of mathematical/scientific abilities* that are needed for success in problem-solving/decision-making in real life situations.
- Finding ways to *identify and encourage a broader range of students* whose outstanding abilities and achievements often haven't been apparent in settings involving traditional tests, textbooks & teaching.
- Finding ways to *provide new ways for students (and workers) to document their achievements and abilities* in ways that "open doors" for entry into desirable schools, professions, and jobs.

Invited "elevator speakers" will include a number of researchers from fields outside of mathematics education – including not only some science educators, but also some people who have been collaborating with us from future-oriented fields in which mathematics is used. These latter fiends range from aeronautical engineering, to business management, to civil engineering, to agricultural engineering, to other fields in which it's common knowledge that:

- When mathematics is used beyond school, relevant models conceptual tools often must be based on more than algebra from the time of Descartes, geometry from the time of Euclid, calculus from the time of Newton, and shopkeeper arithmetic from an industrial. Yet, the basic ways of thinking often involve new (graphic, iterative, dynamic) ways of thinking about basic rational number concepts.
- Problem-solving and decision-making often involves teams of specialists, working together for more than a few minutes, and using powerful tools to produce sharable, transportable, and re-useable conceptual tools – which involve much more than simple answers to questions of the type emphasized in traditional textbooks and tests.
- In the preceding situations, new stages of problem solving are emphasized (such as those that involve partitioning complex problems into modular pieces, and planning, communicating, monitoring, and assessing inter-mediate results) as well as new levels and types of understandings and abilities (such as those that involve representational fluency, or social or interpersonal abilities that go beyond traditional conceptions of content-related expertise).
- Past conceptions of mathematics, science, reading, writing, and communication are far too narrow, shallow, and restricted to be used as a basis for identifying students whose mathematical abilities should be recognized and encouraged. Students who emerge as being especially productive and capable in the preceding situations often are not those with records of high scores on standardized tests.

As time permits, other contributors will be welcome to lead brief "elevator speaches" that emphasize compelling examples. (Contact one of the organizers for more information.)

What central concerns will the Working Group Address at PME/NA-Mexico?

In the history of mathematics education , models for explaining the development of children's conceptual systems gradually evolved beyond being based on hardware (machine metaphors), to being based on software (computer metaphors), and toward currently emerging interests in wetware (organic metaphors). Gradually, we've come to recognize that the development of knowledge is less like the construction of a machine or a computer program, than it is like the evolution of a community of living, adapting, and evolving biological systems. But, even many self-taut constructivists continue to be little more that thinly disguised assemblyists who rely on ways of thinking that are fundamentally mechanistic in nature. Similarly, there has been a recurring tendency to treat means-to-ends into ends-in-themselves – teaching discovery for it's own sake, discourse for it's own sake, construction for it's own sake.

Each decade or so, schools alternate between focusing on *behavioral objectives* (BOs: low-level facts and skills) and *process objectives* (POs: content-independent problem solving processes) – with occasional attention perhaps being given to *affective objectives* (AOs: motivation, feelings, and values). But, throughout this history, the kinds of objectives that continually are neglected are *cognitive objectives* (COs: models and conceptual systems for constructing, describing, and explaining complex systems).

One of the foremost goals of the Rational Number Working Group has been to clarify the nature of these cognitive objectives of instruction – in a content domain that is both the capstone of elementary mathematics and the cornerstone of advanced mathematics. In general, we believe, researchers have done less than an admirable job clarifying (in terms that are meaningful to parents, policy makers, and other non-school people) what it means to develop the cognitive structures that underlie the most powerful and useful ideas in elementary mathematics.

What background theoretical perspectives are relevant?

Research on rational numbers was one of the first in mathematics education that focused on the fact that the meaning of relevant constructs tends to be distributed across a variety of representational systems – which may involve spoken language, written symbols, concrete models, diagrams or pictures, or experience-based metaphors. Whereas, research on whole number arithmetic, or research on early algebra, characterized development as a ladder-like linear sequence of stages; rational number research tended to emphasize the fact that development occurs along a variety of dimensions such as concrete-abstract, simple-complex, intuition-formalization, situated-decontextualized, internal-external (where the right side of these pairs should not necessarily be regarded as being "good" or as representing higher levels of development). In fact, students' early understandings of rational number concepts tends to quite unstable; and, it tends to be far more piecemeal, fragmented, and tied to specific experiences than researchers in other topic areas have suggested. That is, a student who appears to be a "stage N thinker" in one context frequently shifts to other (more primitive, or more advanced) levels of thinking in other contexts – or when other closely related ideas are needed (ratios, rates, fractions, quotients, etc.). In fact, if average ability middle school children confront special types of problem-solving episodes that we refer to as *model-eliciting activities*, their solution attempts often to go through a series of modeling cycles in which the conceptual systems that are used appear to move through several of the type of stages that have been employed in ladder-like descriptions of conceptual development. For example, for a problem that requires some form of proportional reasoning, students first interpretations often involve additive forms of thinking, whereas later interpretations tend to involve multiplicative forms or thinking.

Whereas research in most other topic areas has tended to focus almost exclusively on children's thinking about traditional kinds of word problems – where the aspect that's most problematic tends to be the fact that students must try to make meaning out of symbolically described situations –, rational number research has given equal attention to problems where almost exactly the opposite types of processes are problematic. That is, students must try to make a symbolic description of a meaningful situation.

When the products of problem solving that students are challenged to produce include descriptions, explanations, or constructions that must provide useful conceptual tools for accomplishing specified tasks, research on model-eliciting activities has shown that solutions tend to be characterized by a series of description-development cycles in which trial descriptions (explanations, constructions) are repeatedly expressed in forms that are tested and refined or revised. Consequently, even though students' first interpretations of problems often reflect quite low-level thinking, their final interpretations often involve thinking that is far more sophisticated than anybody dared try to teach them – based on predictions from past performances in situations involving traditional textbooks and tests. Furthermore, because such problems emphasize a much broader range of abilities than those emphasized on traditional word problems, a broader range of students often emerge as being extraordinarily productive.

Similar observations are commonplace in university programs in future-oriented fields that range form aerospace engineering, to business management, to agricultural engineering – where leaders in these fields often consider it to be obvious that the most important goals of instruction consist of helping students develop powerful models and conceptual systems for making (and making sense of) complex systems. This is why, in both instruction and assessment, such departments often emphasize the use of "case studies" which are: (a) simulations of real life problem-solving or decision-making situations, and (b) situations in which the goal is to develop a conceptual tool that is sharable and reusable (and that, consequently, focus on higher-order understandings and generalizations).

THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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Issues

Over the last several years, we have given serious attention to how students learn to reason probabilistically; that is, how learners build mathematical models, and how these models interrelate with each other and with data. A special focus has been on how students build and work with information. Some of this work has been reported and discussed at Singapore (ICOTS-5, June 21-26, 1998), at PME-NA 20 (North Carolina State University, Raleigh, North Carolina, October 31–November 3, 1998), and at the third Robert B. Davis (RBD) Working Conference (Snowbird, Utah, May 22-26, 1999). Discussion, investigation and collaboration continues at various sites around the world.

At PME-NA 20 (North Carolina State University, Raleigh, North Carolina, October 31–November 3, 1998) our working group (see Maher, Speiser, Friel & Konold, 1998) began to build a joint agenda for research and discussion in this sprit. Related work from the Rutgers-Kenilworth longitudinal study, now in its eleventh year is extensive (Kiczek & Maher, 1998; Maher & Martino, 1997; Maher & Martino, 1996; Maher, Davis, & Alston, 1991; Maher & Speiser, 1997; Martino, 1992; Martino & Maher, 1999; Muter, 1999; Muter & Maher, 1998). The working group at PME-NA 20 proposed the following more detailed agenda as an initial focus:

1. Reasoning from data (through modeling and making predictions);
2. Attention to how reasoning and thinking are enacted through communities of learners, teachers and researchers;
3. Studying the development of mathematical ideas as students reason with data;
4. Interpreting students' thinking through an analyses of their images, data representations, model building, and generalizations; and
5. Examining the role of the task, of the classroom environment, of the teacher/researcher, as well as student-teacher interactions and the flow of information and ideas among the learners.

The working group, in building this agenda, emphasized the importance of bringing together a community of researchers to engage in collaborative, cross-cultural research on the long-term development of probabilistic reasoning. We would like to extend the agenda, to address the interplay of combinatorial and probabilistic reasoning for constructing images and models in the course of task investigations. With these aims in mind, we invite participants to join us to collaborate and strengthen both our work and our community.

Theoretical Perspective

Our recent work, and the work of others, has emphasized the complexity and subtlety of probabilistic reasoning, even in very basic situations. Models can extend distortions, even as they help support the growth of understanding. Indeed, the variety of representations which learners find useful, and the complex relationships among the models learners build and the data which they seek to explicate provide rich opportunities for discussions focused centrally on sense and meaning. Given this complexity, the development of probabilistic thinking requires careful building over time, in which earlier inquiries are revisited, reconsidered, extended and reformulated. The tools available, the ways the tools are used, the ways in which ideas and information move among the learners, the teacher's questions, ideas and interventions also contribute (or fail to contribute) in important ways. Our view is that both research and teaching need to take the long-term building and the complexity into account.

Background

Related cross-cultural research on particular dice games, by researchers in several countries, using different methods of analysis across a range of settings and learner populations, was reported in joint sessions (Statistical Education at Elementary Level: The Emergence of Statistical Reasoning at the School Level) at the International Conference on the Teaching of Statistics (ICOTS-5, Singapore, June 21-26). The Singapore reports (Amit, 1998; Fainguelernt & Frant, 1998; Maher, 1998; Speiser & Walter, 1998; Vidakovic, Berenson & Brandsma, 1998) helped motivate the work at Raleigh.

Further discussions at the third RBD Working Conference (Snowbird, Utah, June 1999), addressed important aspects of the Working Group's agenda in the context of the growth of understanding, drawing case examples from the Rutgers-Kenilworth long-term study (Maher & Martino, 1992; Martino & Maher, 1999; Muter; 1999), from graduate courses in mathematics education at BYU and Rutgers, and parts of the BYU preservice mathematics teaching experiment (Speiser & Walter, 1998, 1999).

Examples

In the 1998 meeting at Raleigh, we considered data drawn from sixth-graders' work on two dice games (Maher, Speiser, Friel & Konold, 1998) which produced extremely rich discussion and analysis. Here, verbatim, are the task descriptions.

Game 1, a game for two players. Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets one point (and Player B gets 0). If the die lands on 5 or 6, Player B gets one point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. Is this game fair? Why or why not?

Game 2, another game for two players. Roll two dice. If the sum of the two is 2, 3, 4, 11 or 12, Player A gets one point (and Player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets one point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner. Is this game fair? Why or why not?

These tasks, and extensions, for example with tetrahedral dice, were developed for sixth-graders in the Rutgers-Kenilworth long-term study.

Here, for further exploration and discussion, we draw two more task examples from ongoing work at Kenilworth. The tasks were developed initially for eleventh-graders in the Rutgers-Kenilworth study. They share important similarities as well as differences and have triggered further rich discussions - first among the high-school students, then among the members of the research team, then among graduate students at Rutgers and BYU, and, further still, among a wider circle of researchers in mathematics education, at the 1999 RBD conference in Utah.

The World Series Problem. In a "world series" two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the "world series." Assuming that both teams are equally matched, what is the probability that a "world series" will be won:

a) in four games?
b) in five games?
c) in six games?
d) in seven games?

The Problem of Points. Pascal and Fermat are sitting in a cafe in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner take all." But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. The carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 francs be divided?

We invite our readers to build solutions for these problems, just as we have done, and to consider, as we have tried to do in detail, the specific thinking needed to explain with confidence why one is right. These problems, and how people think about them, in detail and depth, can help to anchor subsequent more theoretical investigations.

Discussion

How people in different settings work on problems such as those above, and then revisit and revise their understanding of them, can reveal much about how people think, more broadly, about data and prediction. How much *interpretive* work, for example, goes into the construction of a sample spaces in such a problem? What images, what representations, and what kinds of reasoning do different learner populations use? What kinds of argument and explanation?

One key issue here, suggested by our work with learners on the two problems above, might be the role (or roles) assumed by standard formalisms. In what ways, specifically, might formal thinking help or hinder learners' building understanding in a given case? In our work with high-school students in the Rutgers/Kenilworth study, who have come work quite well in realistic settings, formal arguments played different roles than for the graduate students and researchers in mathematics education who also undertook these problems. For some learners, direct construction using concrete models took precedence, while for others work with data (thought-experiments, connected to potential sample spaces) helped clarify, and sometimes deconstruct, more formal arguments.

As we think about cognition, we too can learn from special cases and their interplay with broader theory, as they illuminate each other. In doing so, we take care to note the special nature of each case as well as its relationship, often reciprocal, to models which we seek to build. Probability, indeed, is subtle and complex, and so is learners' thinking about it. Good and appropriate tasks produce rich data, which bear strongly on the questions we investigate.

These case investigations, as do others, raise key issues, which go far beyond the problem settings which initiated them, challenging widely held beliefs and offering fresh new perspectives on the growth of understanding. We, and others in the working group, will report what we are learning, share some of our current thinking, and invite others to join our conversations and investigations.

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USING SOCIO-CULTURAL THEORIES IN MATHEMATICS EDUCATION RESEARCH

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This Working Group began as a discussion group at PME-NA 1997 and first met as a Working Group met during PME-NA 1998. During the 1997 discussion group several researchers presented summaries of their research framed by socio-cultural theories. Out of these discussions grew an interest in pursuing these issues further and organizing a working group around this theme.

Focus and Aims of the Working Group

Socio-cultural theoretical perspectives have been used to frame research on learning and teaching mathematics (for some examples see Educational Studies in Mathematics, September 1995 Special Issue; Saxe, 1991). There are multiple interpretations of what socio-cultural perspectives say about learning and teaching and how these phenomena can be studied. The aim of this Working Group is to present and discuss different interpretations of socio-cultural perspectives and different applications of these perspectives to research questions in mathematics education.

The central goals of the Working Group are to:

1) Develop a shared sense of the contributions that socio-cultural theories can make to research in mathematics education.
2) Develop a plan for related projects using socio-cultural theories to explore questions about learning or teaching mathematics.

During the three sessions the participants will discuss research conducted using socio-cultural theoretical perspectives, analyze sample data using concepts from these perspectives, and discuss selected readings. These activities are intended to support participants in a) clarifying which specific versions, aspects, or concepts of socio-cultural theories are being invoked in different research, b) questioning key analytical concepts, and c) exploring which aspects and concepts can be useful for framing further research on learning and teaching mathematics.

The activities and discussion will address several ways to apply these perspectives to research design, data analysis, curriculum development, and teacher professional development. The anticipated follow-up activities for this Working Group include planning for a continuation of the Working Group at PME-NA 2000 and ultimately organizing a collaborative writing project on this topic.

Session 1

1) Introduction and overview of the Working Group.
2) Two brief (5-10 minutes each) presentations by panel members providing overviews of a project and examples of how researchers have used socio-cultural theory in their research. The purpose for these short presentations is to provide examples of how socio-cultural theories have been applied and show several different perspectives in a structured way.
3) Participants will analyze and discuss a segment of videotape data from a variety of socio-cultural perspectives, sharing their own experiences in data analysis as part of the discussion.

Session 2

1) Two brief (5-10 minutes each) presentations by panel members providing overviews of a project and examples of how researchers have used socio-cultural theory in their research.
2) Discussion in small groups of three selected readings. The readings will be mailed to participants who register for the Working Group and will also be available on the first day of the Working Group.

Session 3

1) Two brief (5-10 minutes each) presentations by panel members providing overviews of a project and examples of how researchers have used socio-cultural theory in their research.
2) Discussion in small groups focusing on the following questions: a) What aspects of socio-cultural theories have participants used in mathematics education research? b) In what areas that have not been linked to these perspectives might socio-cultural theories be useful? and d) How might this theoretical perspective inform participants' research projects?

Questions for Presenters:

1) How have socio-cultural theories informed your research project(s)?
2) What specific aspects (concepts, methods, etc.) from socio-cultural theories have you used in your research?
3) In what areas of your research were socio-cultural perspectives most useful (research design, data analysis, curriculum development, teacher professional development, etc.)?
4) How has your work extended or expanded socio-cultural concepts?
5) Which concept from socio-cultural theories do you find most puzzling? Most useful? Most misunderstood?

Readings for Day 2

1) Cazden, C. (1993). Vyogtsky, Hymes, and Bakhtin: From word to utterance to voice. In E; Forman, N. Minnick, and C.A. Stone (Eds.) Contexts for learning: Sociocultural dynamics in children's development. NY: Oxford University Press.
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Discussion Groups

SOCIAL AND CULTURAL CONTEXTS IN MATHEMATICS PEDAGOGY

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This discussion group is designed to expand the conversation in the mathematics education reform community to include consideration of the social and cultural context of mathematics instruction. Secada (1995) has described how in the Professional Standards for Teaching Mathematics (National Council of Teachers of Mathematics, 1991), little mention is made of what is required of teachers of diverse student populations both inside and outside the mathematics classroom. In this discussion group, the participants will provide topics for discussion that addresses this void. Each participant will discuss what teachers can do to address the cultural and social values and expectations of students in the classroom to make mathematics instruction more relevant and effective for all students. We focus in particular on what students, teachers, and parents bring to bear to improve mathematics instruction to develop student´s mathematics abilities.

The following topics will give an overview to set the stage for a lively discussion on the topic.

Bringing Diverse Lives into the Mathematics Classroom

In this discussion we will draw on our experiences in a 4-year project (*Children´s Math Worlds*) teaching mathematics to first-, second-, and third-grade Latino inner-city children in English-speaking classes and in Spanish-speaking classes. We will describe productive ways we have discovered to interweave elements of mathematics with children´s lives. These approaches have worked with Latino children and with children from other backgrounds who were in our urban classrooms. But our experiences in other kinds of classrooms, including middle-class and affluent upper-middle class schools suggest to us that these approaches are important for all children.

Because of the complex patterns of immigration to the U.S. from many different countries of the Spanish-speaking Americas, teachers' own life experiences may differ considerably from the life experiences of Latino inner-city children. Many children in a given class or school may emigrate from the same community (one typical pattern here in the Chicago area and elsewhere), or children may come from many different places. In any case, finding out about students' lives enables curriculum developers and teachers to develop and enact a curriculum that unfolds from children's real-life experiences, making mathematics learning and teaching meaningful, enlightening, involving, empowering, and creative.

Making Real Connections Between Mathematics and Students' Lives

In this discussion we will investigate the process by which real world problems become a meaningful context for the exploration and development of mathematics knowledge. A central idea in this work is that if we are to make real connections to students' lives, then problems should be motivated by the students' efforts to interpret their environment. There is an attempt to keep the curriculum close to the student (Dewey, 1956).

Accomplishing these goals requires more than simply developing contrived real world problems which often have only slightly more connection to students' lives than the abstract exercises in traditional mathematics textbooks. Not only must the topic be of interest to the students, but also the development of the topic must be based on students' understandings and interests.

Culturally Appropriate Mathematics Pedagogy

In this discussion we will discuss the role that teacher knowledge plays in mathematics learning.

Family Involved is Critical for Diverse Learners Success

This discussion will address the many ways that families have been involved in with schools and what strategies work for teachers in gaining more involvement.

The major goals of PME-NA are all addressed in our discussion group.

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PERFORMANCE ASSESSMENT OF K-12 PRESERVICE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE

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Both the Mathematical Association of America (MAA, 1991) and the National Council of Teachers of Mathematics (NCTM, 1991) have published sets of specific recommendations for the mathematical content preparation of preservice elementary, middle school, and high school teachers. The Interstate New Teacher Assessment and Support Consortium (INTASC) has published *Model Standards in Mathematics for Beginning Teacher Licensing & Development: A Resource for State Dialogue* (INTASC, 1998). In order to attain the vision described in these documents, there is a growing need for research and discussion about how to *assess* teachers' mathematical content knowledge.

Performance assessment of teachers' knowledge is a critical issue. For example, the National Council for Accreditation of Teacher Education (NCATE, 1998) is changing to a performance-based system of program evaluation, and the state of Indiana is changing to a performance-based system of teacher licensure. In response, an increasing number of teacher education programs will most likely require their students to develop assessment portfolios. These portfolios must include items that demonstrate the preservice teachers' mathematical content knowledge. Opportunities to develop these materials must be provided not only in mathematics methods courses but also in mathematics content courses for preservice K-12 teachers. This major paradigm shift in evaluating an individual's readiness to begin teaching raises new challenges for teacher educators as well as those less directly involved in teacher education, including mathematicians who teach content courses for preservice teachers.

The topic for this discussion group is performance assessment of preservice K-12 teachers' mathematical content knowledge. The discussion will focus on three main issues: 1) What research is in the existing literature on performance assessment of preservice K-12 teachers? 2) What suggestions for practice do the participants bring from their own experience? 3) What are some questions for research on performance assessment of preservice K-12 teachers' mathematical content knowledge?

The discussion will begin with a review of recent literature on performance assessment of preservice K-12 teachers. Next, discussion will focus on suggestions for practice and questions for research. Discussion questions may include the following: What sorts of performance mathematics activities have been successfully used? How can mathematics assessment portfolio items be developed? How are they evaluated? What constitutes a successful performance activity? What supports will be necessary for those less familiar with teacher education and its goals, including many mathematicians who will now be required to design and administer such assessment tasks? How might these assessment tasks enhance current mathematics instruction? Do we have to change what we have been doing in our mathematics content courses to better ensure that students will be able to meet new standards? How do we assess performance assessment? As questions arise, do they require answers from research?

Participants should bring examples of performance tasks used in their own classrooms.

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Advanced Mathematical Thinking

ON SCHEMA INTERACTION: A CALCULUS EXAMPLE

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The Action-Process-Object-Schema [APOS] theoretical perspective (Asiala, et al., 1996) was used to examine the way students solve a calculus graphing problem. The focus of this study was to determine students' work in terms of the schema used to solve the problem. Data consisted of extensive interviews with students who had completed at least two semesters of calculus. The complexity and the non-routine nature of the problem required students to rely on everything they had learned on graphing functions in calculus. In order to cogently describe the students' responses, we found it necessary to examine two important schema the students were using and their interaction. Additionally, a number of difficulties were demonstrated by students throughout and these problems are discussed in some detail.

Introduction and theoretical framework

In a complex problem situation it can be very difficult to analyse students comprehension and conceptions. This study intends to make a contribution in that direction and examines the way students deal with a dificult non routine calculus problem which consisted in graphing a function given its analytic properties (first and second derivatives, limits and continuity) on intervals of the domain.

Although a number of studies (Ferrini Mundi & Graham,1994; Asiala et al., 1997; Slavit, 1995; Thomson, 1994; Tall, 1996) have dealt with students' difficulties with calculus concepts and with problems related to different representations (Douady, 1985; Duval 1988, 1993; Rasslan and Vinner, 1995), little has been done on the extent to which students are able to integrate what they have learned when solving more complex situations. This paper is an attempt to do this. We present the problem and develop a theoretical framework that enabled us to analyse the interaction between two schema used by students in their attempts to solve it. In addition we discuss some specific properties of the graph that caused trouble to many students.

This study is based on APOS theoretical perpective (Asiala, et al., 1996); however it also extends APOS theory as it describes schema development in terms of the triad stages in more detail that has been done before (Clark, et al., 1997;Mathews, et al., 1998) and examines the interaction of two schema.

Schema development is a dynamic, ever changing process. Piaget and Garcia (1989) propose that knowledge grows according to specific mechanisms and that its evolution obeys a necessary order which consists of three stages called the triad. They hypothesize that these levels can be found in the analysis of any schema development. The nature of the stages is functional, not structural, and it is useful to describe general psychodynamical aspects of a given situation. In this paper this three stages are used to analyse the way students deal with the given problem.

As knowledge develops, people construct many coexisting schema, all of which are constantly changing and at varying levels of evolution. Therefore, there may be problem situations, as we would like to illustrate here, when a person finds it necessary to coordinate different schema in order to make sense of a particular situation. This coordination of schema can be useful when attempting to describe integration of knowledge.

Presentation of the problem and methodology

The question analyzed asked students to graph a function. The information for this graph was given as a set of conditions in interval form:

Sketch a graph of a function that satisfies the following conditions:

h *is continuous*
$h(0)=2$, $h'(-2)=h'(3)=0$ *and* $\lim h'(0)=\infty$, *as* $x\to 0$
$h'(0)>0$ *when* $-4<x<2$ *and when* $-2<x<3$
$h'(x)<0$ *when* $x<-4$ *and when* $x>3$
$h''(x)<0$ *when* $x<-4$, *when* $-4<x<-2$, *and when* $0<x<5$
$h''(x)>0$ *when* $-2<x<0$, *and when* $x>5$
$\lim h(x)=\infty$, *as* $x\to-\infty$ *and* $\lim h(x)=-2$ *as* $x\to\infty$

The 41 students who participated in this study were enrolled in engineering, mathematics, and science programs at a large Midwestern university and had completed at least two semesters of single variable calculus. After the class was over, the students completed a series of questions by intricate, audio-taped interviews, including the problem question. The students answered the question in detail, explaining their *thought processes, asking questions, and explaining the methods invoked in order to graph the function. In addition to the interviews, the students' sketches of the graph were also kept as part of the interview data. We designed a set of criteria that the students would need to exhibit in order to be categorized at a particular level of understanding. Each of the student interviews was independently analyzed by at least two researchers. These analyses were then compared for validity and no disagreements in the different analysis were found.*

The Two -Dimensional Calculus Graphing Schema

Our goal was to describe the level of schema development each student had attained and how this knowledge was applied to the problem. The mathematical components of the schema were defined, as they would be understood at this particular time for students at this level, in order to analyze the level of development.

What we called the calculus graphing schema is defined by a combination of the students' levels of development in understanding the derivative, limits and continuity, as well as their precalculus ideas. Through the analysis of the data we realized that there were two important components of the problem playing a significant role in students responses. The data showed that students were not only struggling with the conditions on the function, but also with the coordination of these conditions across the intervals of the domain. Therefore, we needed to form a model that involved the development of two different schema; one for the intervals and another for the properties. On one side, we saw students at differing levels in coordinating the properties of the graph as given by the conditions. On the other side, we saw students at differing levels in coordinating the graph properties across contiguous intervals. The development of the calculus graphing schema can thus be described by the interaction of these two schema and this lead us to define a two-dimensional triad: one of the dimensions is the Condition-Property Schema and the other is the Domain-Interval Schema. The data were described with pairwise levels of the triad of those two schema.

The Development Of The Property Schema

The triad levels for each of these schema were developed for this particular problem using the definitions of intra, inter, and trans levels as described below. The property schema involves two important aspects. One of them is understanding each analytic condition as it relates to a graphical property of the function and the other is coordinating these conditions within an interval. At the Intra Property level a student can interpret one isolated condition and relate it to a graphical property of the function. Students at this level typically utilized solely the first derivative condition. Often they were aware of other properties, but could not coordinate them in terms of the graph. If two properties overlapped, the student would describe the behavior of the graph using only one. In the case that the student would try to use more than one, the description would break down and the student would resort to using only one of the properties. At the Inter-Property level the student begins to coordinate two or more conditions simultaneously. This coordination, however, is not applied throughout all overlapping conditions. At the Trans-Property level the student can coordinate all of the analytic conditions to the graphical properties of the function on an interval. At this point, the student demonstrates a coherence of the schema. That is, it is clear that the student recognizes what behaviors of the graph of a function may be included and what may not.

The Development Of The Interval Schema

The important aspects related to the interval schema are understanding the interval notation, connecting contiguous intervals, and coordinating the overlap of the intervals. In the problem, each interval was assigned with certain conditions, and the integration of the intervals was essential to constructing the graph. The triad as applied to this particular schema is described below. At the Intra- Interval level the student works only on isolated intervals. The information is described interval by interval. The overlap of intervals or connecting contiguous intervals causes confusion. At the Inter- Interval level the student begins to coordinate two or more contiguous intervals simultaneously. This coordination, however, is not applied throughout all connected intervals. At the Trans-Interval level the student is able to describe the coordination of the intervals across the whole domain. He or she is able to overlap intervals and connect contiguous intervals. The student also demonstrates coherence for the schema by describing which manifestations in the graph are allowed by the overlap and connection of the intervals and which are not.

We consider this model to be appropriate for the description of the constructions needed in any kind of calculus graphing problem or situation. That is, if a student is given a function and asked to give the graph, it is necessary to find its properties, and determine for which intervals of the domain the properties hold. In addition, if one is examining the graph of a function, one needs to be able to determine which properties hold on different intervals of the domain. Although students are not always explicitly requested to indicate the intervals of the domain for each property, the data show that this can point to a weakness in a student's calculus graphing schema.

The Problem Points

There were three parts to the graphing problem that consistently caused students particular difficulties. These were the cusp at x = -4, the vertical tangent at x = 0, and it also became clear through the interviews that a significant number of students had a very limited understanding of the second derivative.

In particular in attempting to construct the cusp students demonstrated confusion with coordination of properties. Two difficulties were especially common. Students would want to either sketch the graph as curved and not as a cusp or would draw the cusp but it would conflict with their concept of a continuous function. The infinite limit on the first derivative function at x = 0 may be thought of in terms of the graphical interpretation of the limit process acting on h'(x) near x = 0. It requires students to have both a highly developed property and interval schema since it requires a large number of coordinations in both. Students' explanations of this limit condition fell into three distinct categories: Those who concluded that a vertical asymptote exists at x = 0, those who correctly described the existence of a vertical tangent line, but failed to incorporate this feature in the graph and those who did not know what effect this condition would have on the graph. One of the surprising results of the analysis of the interviews was the difficulties that students had in working with the second derivative conditions. They would ignore it, work from memorization solely or not be able to coordinate the first derivative and second derivative conditions across intervals .

Some Conclusions

The results obtained show that the triad is a useful tool in analyzing students' conceptions and difficulties and that it can be used in the study of more complex situations where possibly there are multiple schema interacting with each other. We were able to find students at all the two dimensional levels of the double triad except for the Trans- Property, Intra-Interval level, but we could not distinguish, from our data, any kind of path that could signal a preferred direction of evolution. The difficulties students have, to integrate different concepts to solve this problem, suggest that more emphasis has to be given in class to this kind of problem situations.

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REVISITING THE NOTION OF CONCEPT IMAGE/ CONCEPT DEFINITION

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Abstract. This paper discusses the results of a study of undergraduate mathematics majors' understandings of formal mathematical definitions in the context of real analysis and the implications these results have for the concept image/concept definition framework (Tall & Vinner, 1981; Vinner, 1983, 1991). During the study the researchers' theoretical framework included the notion of concept image/concept definition as described by Vinner and Tall. One of the results of the study indicated that students' understanding of the role of mathematical definitions in general could influence their understanding and use of the concepts represented by individual definitions. It seems that the nature of mathematical definition is a concept in its own right and that a student's understanding of this definition concept may influence the interplay of concept image/concept definition when the student is engaged in a mathematical task.

Introduction and Theoretical Framework

The purpose of this paper is to discuss the results of a study of undergraduate mathematics majors' understandings of the role of formal mathematical definitions in the context of real analysis (Edwards, 1997a) and the implications of that research on the notion of concept image/concept definition (Tall & Vinner, 1981; Vinner, 1983, 1991).

Formal mathematical definitions are of central importance in mathematics because they contribute to the basis of the theorems that provide the theoretical structure of mathematics. However, students often have difficulties understanding and using formal mathematical definitions (Moore, 1994; Tall, 1994; Vinner & Dreyfus, 1989). Part of the enculturation of college mathematics students into the field of mathematics involves their acceptance and understanding of the role of mathematical definitions – that the words of a formal definition embody the essence of the concept being defined and that the process of creating that definition is flexible and sometimes arbitrary (Tall, 1994).

A well-known framework for describing one's accumulated mathematical knowledge – concepts and their definitions – is that of *concept image/concept definition* (Tall & Vinner, 1981; Vinner, 1983, 1991). In this theory, *concept image* is "the total cognitive structure that is associated with the concept" (Tall & Vinner, 1983, p. 152). The *concept image* may include primitive notions as well as the understandings that have developed formally in mathematics courses. The *concept definition* is the body of words used to designate that concept. A mathematically acceptable definition may not be known to the learner, may be "separate" from the concept image or may be included, possibly incorrectly or incompletely, in the learner's concept image. The *evoked concept image* is the outward manifestation of a learner's concept image and can vary from situation to situation. This implies that careful and repeated contact (through in-depth, task-based interviews, for instance) is necessary to establish the best possible representation of a learner's concept image. This notion of concept image/concept definition contributed in an important way to the researcher's theoretical framework during the study reported in this paper (Edwards, 1997b).

Method

Eight college mathematics majors participated in four one-hour, in-depth, task-based interviews over a period of fifteen weeks during which they were enrolled in an introductory real analysis course. The intent of these interviews was to investigate each student's understanding and strategies in dealing with formal definitions, including definitions which the students had previously encountered, definitions which students were currently first encountering in their introduction to real analysis course, and definitions which students had not encountered before. Formal definitions for pointwise continuity, infinite decimal, absolutely continuous function, and connected set (both set-theoretic and path connected) were used during the interviews. Students were also asked in the first interview to explain their understanding of the concept of limit, although no formal definition of limit was given to them during that interview.

The purpose of the study was to determine students' use and understanding of formal mathematical definitions, and the researcher held no expectations that, during the interviews, students would come to understand definitions of which they had no previous knowledge. It was also not the purpose of the interviews to determine whether or not students "remembered" the formal definitions for concepts they were studying or had studied in the past. The researcher's concern was to investigate students' use and understanding of accurately worded definitions. During the interviews, when students were asked to discuss the formal definition of a concept, they had the printed definition in front of them for reference.

One of the research questions concerned the student's understanding of the role of formal mathematical definitions. In the analysis of the data the researcher formed her understanding of how students seemed to understand the role of mathematical definitions both on the basis of what students said about the role of definitions as well as how they used particular definitions when designating examples and counter-examples of a given concept. The researcher used occurrences of "misconceptions" in a student's evoked concept image to aid in her determination of the effects of a student's understanding of the role of formal definitions on their use of definitions. In the instances in which there was a discrepancy between these two elements, students could grant authority to either their concept image or the external formal definition. It was the researcher's view that, in the task of choosing between examples and counter-examples of a given concept, students who chose the external formal definition as the authority exemplified a mathematically acceptable use of mathematical definitions. When a student's evoked concept image and the external formal definition matched, such a determination by the researcher was not possible.

Results

Data from the interviews indicated that only one student showed evidence of what could be considered to be a mathematically acceptable understanding of the role of formal definitions. The remaining seven students seemed to have some level of misunderstanding about the role of mathematical definitions. Further, there were instances in the interviews when misunderstanding of the role of mathematical definitions promoted or propagated conceptual misunderstandings.

For example, Jesse[1], a third-year honors student, in his first interview, referred to formal definitions as "a lot of jargon." In the second interview, which was conducted before continuity had been discussed in the course in which the participants were enrolled, Jesse was asked to discuss point-wise continuity. He gave a definition similar to the following "simplified" (but mathematically correct) definition.

> *Definition.* A function f is continuous at x_0 if the limit of f as x approaches x_0 exists; if f at x_0 exists; and if the limit of f as x approaches x_0 equals f at x_0.

The researcher also provided a more formal "e/d" definition for point-wise continuity, and Jesse seemed to be able to understand this definition and to use it successfully to identify most of the list of examples and counter-examples provided in the interview. However, among the examples of functions that Jesse was given to discuss was the function $f(x) = |x|$. Jesse had access to printed copies of both his definition and the researcher's more formal definition for point-wise continuity, but he thought he remembered that the absolute value function was not continuous. As he thought about this he said,

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[1] The name "Jesse" is a pseudonym.

> No holes, jumps, maybe peaks, but I know cusps. [Jesse points to the graph he has drawn of the function.] Is that a cusp? I don't know. Cusps, or there were a whole bunch of things that were not continuous.[2] And, I think this is one of them. Although it looks pretty continuous. I'm pretty sure I remember that this is not continuous and my definition [the "simplified" definition] isn't cutting it, so I'm looking at the, at a real one.... [He goes back and forth between the two definitions for several minutes, reading and re-reading.] But that, but I know cusps, and sharp peaks are not... but from the definition, if we're saying that the limit of these two equals that, and f of c equals that, then that would be continuous. [Short pause.] But it's not. (Jesse, Interview 2)

Although Jesse correctly interpreted the meaning of the continuity definitions, because he believed that formal definitions were "jargon," he was unwilling to declare the absolute value function continuous. He trusted more his intuition or memory that it was not.

Discussion

Vinner (1991) describes various scenarios in the interplay between concept image and concept definition when a student is asked to do a task involving a particular concept. In his description a student might consult both the concept image and the concept definition, or he or she might consult only the concept definition (a purely formal deduction) or only the concept image (an intuitive response). In the example from this study, it is clear that Jesse consulted both his concept image and a mathematically acceptable concept definition for continuity which was know to him (and apparently included in his concept image). In addition he had external information in the form of two correctly worded definitions for point-wise continuity. He decided, however, that even though $f(x) = |x|$ fulfilled the conditions in both his and the researcher's continuity definition the function was not continuous at $x=0$. Something more is needed to accurately characterize Jesse's consultation processes.

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[2] It is apparent that in the interview, Jesse confused continuity with the notion of differentiability. Two days after his interview he reported realizing this fact.

The results of this research suggest that the notion of formal mathematical definition and its role is a concept in its own right – one for which an individual may have a concept image and a concept definition. In every case in which mathematical definitions are an issue, a student's concept image/concept definition of mathematical definition comes into play. The robustness of a student's understanding of this concept seems to have an influence on the student's success in his or her use and understanding of any formal mathematical definition. Jesse seemed to view mathematical definitions as one more piece of evidence in the decision process, but not the most important piece of information.

During the study other students sometimes seemed to make similar decisions based upon weak conceptual understandings of the role of mathematical definitions. Jesse essentially suspended his decision for several days and only decided that the absolute value function was continuous after he "cleared up" his concept image of continuity. At other times during the study, when students decided to give authority to their evoked concept image in instances when it was inconsistent with the formal mathematical definition, they often eased any cognitive conflicts that might exist for them by saying that the example must be an exception to the definition or by altering their interpretation of the definition slightly so that both agreed.

Therefore, although it is probably frequently true, one cannot always assume that an "incorrect" response during a mathematical task is necessarily due to a student's consultation of only the concept image or a concept image with a mathematically unacceptable concept definition. It seems that there is at least a third possibility – that during the task the student also consults a weak concept image of mathematical definitions, causing him or her to disregard what may have been a mathematically correct definition of the original concept in question.

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A DIDACTIC ENGINEERING RESEARCH PERFORMED WITHIN A COURSE ON ORDINARY DIFFERENTIAL EQUATIONS WHERE STUDENTS USE THE TI-92 CALCULATOR

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Abstract: This is a report showing several results from a research that is still in process, actually taking place within an educational engineering program (Artigue, 1995), in classes where students are learning ordinary differential equations (ODE). Our investigation was done with a group of 30 students at the Electrical Engineering Department of the Madero City Technological Institute, Mexico. The computing tool used by all of them is the Texas Instruments 92 (TI - 92) calculator.

Introduction

Nowadays, there are several reports dedicated to ponder on the teaching/learning methods on the subject of ordinary differential equations, in which emphasis is basically made on the modification of the current teaching scheme (algorithmic-algebraic) to be changed by other systems into which there is an incorporation of graphic and numerical approaches (Artigue 1989, Hernández, 1995, 1997). At the same time, there is a proposal for the use of models from the beginning. Under these series of works, the computing tools play a central role for putting into effect the different proposals (Hubbard & West, 1991, Blanchard, et al, 1996, Lomen & Lovelock, 1996, Borrelli & Coleman, 1996).

This time we are reporting on the results obtained by the use of the Texas Instruments Calculator, model 92 (TI-92), within the setting of a course on ordinary differential equations which is taken by 30 students of the Electrical Engineering career, at the Madero City Technological Institute, Mexico during the first six months of 1998.

Taking into consideration the previous results, (Hernández & Hitt, 1994; Hernández, 1995), the course was given by following an algebraic, graphic and numeric approaches. One calculator was used by every two students.

Features of the course

The topic ordinary differential equations is part of several linear algebra assignments that are taught at Mathematics III, within the Study Plan for Electrical Engineering, which is part of the Technological Institutes System. The course is given on a 5-hour per week basis, and the contents follow a traditional approach of the ordinary differential equations. This time, the selection of the group was chosen at random, and the students were not informed that the subject would have different characteristics.

The calculators were distributed two weeks after starting the course (having taught them the linear algebra topics). A total of 15 calculators were provided, one for each two students, which were kept by them until the end of the class period. A view screen and an overhead projector were also provided at the classroom.

The sessions using the calculators were not programmed, they were only used whenever necessary.

The subjects of differential equations were dictated as follows:

- Introduction: What is an ordinary differential equation?
- Differential Equations of First Order: The Algebraic (linear and non-linear equations), Graphic and Numeric Settings of Resolution.
- Linear Differential Equations of Second Order: The Algebraic Setting (variation of constants and the Laplace Transform). The Numeric Setting.
- Differential Equations Systems.

A didactic engineering research

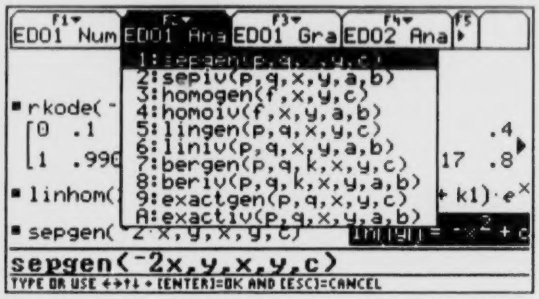
On previous articles, we reported some results obtained from the teaching/learning system of the ordinary differential equations that use the Derive computer software. Those investigations are structured around a teaching proposal with the use of the notion setting introduced by R. Douady (1991), and with the numerical, graphic and algebraic approaches (Hernández, 1997). Then we used the methodology of the didactic engineering in order to analyze *the relationship between the investigation and the action applied on the teaching system* (Artigue, 1995, p. 34).

In the present case, the investigation was divided into two phases: On the first one, there were a series of materials previously written according to a teaching proposal (Hernández, 1995, p. 129). Under this part, we basically navigated trough the three resolution settings incorporating the commands, functions and adequate programs in order to use the TI-92 calculator. At the same time, a folder (ODE) was created and used, in order to ease the handling of the three resolution settings. On the second phase, we experimented our proposal, having as an orientation, the methodology of the didactic engineering, under a level of a *macro-engineering*. We completed this one through external methodologies such as the design of questionnaires.

Following, we will describe some of the characteristics of the materials. In case of the classical algebraic setting of resolution, there were some examples given on how to construct commands in order to solve some differential equations, and it was requested to build some others. For example, for the linear type differential equation; in the graphic setting case, it was requested, with the use of the Line order of the TI-92, to build a program which would generate the direction field of a differential equation. It is worth mentioning here, that only a small group of students could perform it. At the same time, all the students were provided with some programs already developed by other authors, in order to compare them with the one they had written by themselves.

The numerical setting, with the use of the algorithms of Euler, improved Euler, the Taylor (Three Terms) and Runge-Kutta, was worked out under two levels: with the use of the sequence characteristic from the calculator, and with the generating of programs that would allow the implementation of such algorithms.

At the ending part of the differential equations of first order, they were provided with two folders, one containing the adequate commands and functions in order to ease the handling of the three resolution settings (figure 1). This folder also contains the commands related to the resolution of linear differential equations of second order, as well as applications for the electrical circuits and mechanical vibrations. The second one was known as the Adv folder, which contains more generic commands that allow the performance of all the previous tasks.



```
sepgen(⁻2x,y,x,y,c)
```

As we already mentioned, we used external methodologies, such as the design of questionnaires in order to measure the work in the classroom. Such were as follows:

1. Solve the differential equation: $y' + cos(x)y = sen(2x)\, y^2$
   a) Find the general solution with the use of the bergen command and HODE1Gen(), compare your results.

      Write your answer:

   b) Find the particular solution satisfying the initial condition y(0) = 1, of two different forms: i) using the general solution obtained

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in a), and ii) using the command HODE1IV(). Do graph the obtained solution, by a qualitative description of its behavior (describe any asymptotes, periodicity, etc.).

Write your answer:

2. Sketch out the direction field from the differential equation 1, by using 18 for the number of horizontal segments and 14 for the one of verticals, over the same screen do graph the solution obtained in 1b. Save the graph as a file (Picture).
3. Use the RK algorithm in order to find an approximate solution of the differential equation of problem 1 which would be attached to the initial condition 1b, under the [0, 4] interval.
   a) Use the delmey variable in order to visualize the approximate numerical solution.
   b) Use the option of sequence seq (function, var, start, end, step), in order to compare the approximate numerical solution with the exact solution.
4. An RL electrical circuit, which is described by the differential equation.

$$L\frac{dI}{dt} + RI = E(t)$$

solve this equation for the case in which R = 1 ohm, L=10 henrys and E = 6 volts.

a) Find particular solutions for case I(0) = b, for b = 0, 2, 4, 6, 8, 10.

b) Find the general solution, and calculate the limit of this expression when t Æ •.

c) Consider now a source of alternate current given by E(t) = A sen (at)

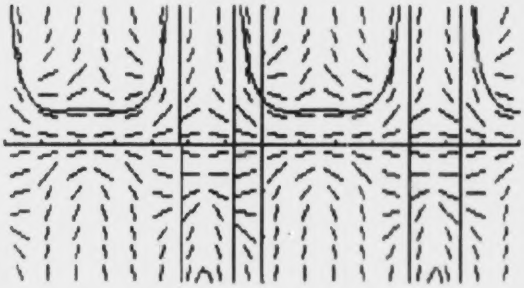
d) $10\frac{dI}{dt} + I = Asen(\alpha t)$

e) Determine the general solution for the following value doublets: i) A = 1, a =1, ii) A =2, a = _. Determine in both cases the c integration constant that complies with the initial condition I(0) = 0. Sketch the graphs of these two solutions in different windows. On these windows, do graph the corresponding E(t) entrance function, and describe the quality behavior of the solutions, ¿what is expected when t Æ •?

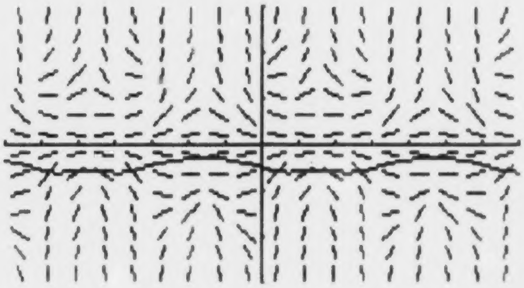
The reports of the tasks were delivered by doublets, using the option: "Lab Report" of the TI-92. Nevertheless, we must mention that there were technical problems, and most of the tasks were to be submitted in written by capturing the information directly into a microcomputer with the TI-GRAPH LINK. The basic problems were due to a lack of memory in the calculator and because of the great amount of graphics to be inserted into the report.

We must also mention that in some of the calculators, all the stored information was lost due to unknown problems.

We coded the results accordingly with the articulation between settings, which was exposed (Hernández 1995, Chaps., IV, V). We also added other codification.

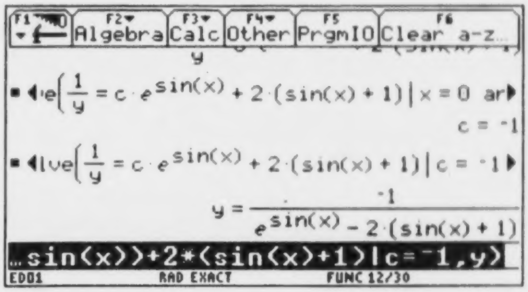


For example, on the previously exposed task, Lourdes and Iliana obtained, for problem 2, as a result, the graph shown (Fig. 2) which indicates that the problem was solved correctly, because the graph of the particular solution is compatible with the direction fields.



Carlos and Gaby, obtained the shown graph (Fig. 3), which suggests that they did not obtain the correct answer for the particular solution.

As a fact, if we use any of the two hode l iv & bergen commands for problem 1, these do not bring out a solution in a closed way. This is due to the processing of the trigonometric expressions of the calculator. The suggestion is to use the trigonometric identity *sen(2x)* = *2sen(x)cos(x)*.



y

■ ◂e$\left(\frac{1}{y} = c \cdot e^{\sin(x)} + 2 \cdot (\sin(x)+1)\right|$ x = 0 ar▸

c = ⁻1

■ ◂lve$\left(\frac{1}{y} = c \cdot e^{\sin(x)} + 2 \cdot (\sin(x)+1)\right|$ c = ⁻1▸

$$y = \frac{^-1}{e^{\sin(x)} - 2 \cdot (\sin(x)+1)}$$

```
…sin(x))+2*(sin(x)+1)|c=⁻1,y)
EDO1          RAD EXACT          FUNC 12/30
```

The solution in a closed form is expressed on (Fig 4.)

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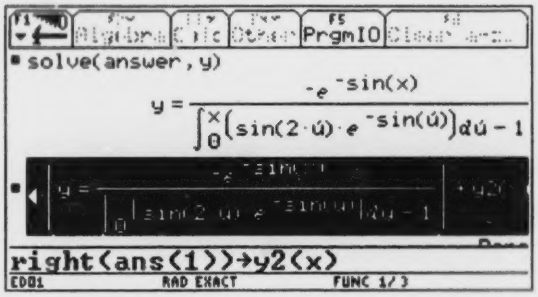

```
■ solve(answer, y)
```

$$y = \frac{-e^{-\sin(x)}}{\int_0^x \left(\sin(2\cdot \acute{u})\cdot e^{-\sin(\acute{u})}\right) d\acute{u} - 1}$$

```
right(ans(1))→y2(x)
ED01          RAD EXACT          FUNC 1/3
```

The above is coded within the articulation of the graphical and algebraic settings. Just as what happened to the team formed by Carlos & Gaby, there were also two other teams which did not get the correct solution. An interesting response was given by the Kova team, who even if not obtained a solution in a closed way, they graphed the formula expressed by the integral, and proved therefore, that it was compatible with the direction field.

Conclusions

Even if the Mexican educational system jumps from the calculation rule up to the computer without going through the calculator, in the sense that it was never integrated into the different curriculums, nowadays, with the evolution of the TI-92, it is necessary to look towards the future, by evaluating its viability in the classroom.

Considering the use of such a calculator in the classroom, we are hereby classifying our final arguments into advantages and disadvantages:

Advantages

- The algebraic manipulation with the calculator is of great help, as it provides enough time for the analysis of concepts and models.
- The resolution methods for the coverage of the resolution of the differential equations can be applied relatively easy for the solution of complicated problems (integrals by parts, partial fractions, numerical calculation, graphs, etc.)
- The study of models: electrical circuits, mechanical vibrations, logistical problems, etc., is much more enriching as it allows the variation of parameters.

Disadvantages

- The students may start having a low motivation in order to perform by themselves the algebraic manipulations.
- The students may become dependent on a calculator, even when they must have to perform relatively simple operations.
- The fact of varying parameters in a model may bring critical consequences for the response. Even the calculator may become "leveled-off" and loose all the information.
- The student may consider incorrect a response given by the calculator if he is only used to

- The variety of responses and strategies given within the calculator level is of great help in order to obtain different representations for the answers.
- The environment of the calculator allows the design of different menus (CUSTOM), which increases its potential.
- The introduction of the three resolution settings is possible due to the calculator.
- The creation of programs and functions is very versatile.

the representation of these in only one format.

- The students may have a deficient mathematical knowledge, which will not allow them to have an adequate use of the calculator.
- Without the technology tool is practically impossible to modify the course of the differential equations.
- There is a lack of capacity on the standard model in order to save enough information.

At present, the Plus module of the calculator, as well as the TI-89 model are being evaluated. Both calculators solve some of the above mentioned disadvantages. The new models improve on one side, at their memory capacity, and by the other, on their potential related to the differential equations concern.

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THE INFLUENCE OF TECHNOLOGY ON SOCIO-MATHEMATICAL NORMS IN A DIFFERENTIAL EQUATIONS COURSE

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The purpose of this research report is to describe how technology influences the sociomathematical norm of explanation constituted in a differential equations class. The data for this research comes from a semester-long classroom teaching experiment in a differential equations class at a US university. The students and teacher in the class interacted to constitute the sociomathematical norm of explanation in this classroom. The focus of this paper is on how the introduction of technology influenced this norm and allowed the enhancement of explanations. These technology-enhanced explanations can further how the mathematics education community understands the way in which the introduction of cultural tools, such as technology, may affect the norms of a classroom.

Reform efforts exist differential equations that follow from changes in the field that now stress dynamical systems and reforms in its prerequisite, calculus. Another major factor influencing these changes is the advent of technology, particularly portable, hand-held versions that allow students to explore more realistic problem types. Technology also allows students to reason about differential equations using qualitative strategies in addition to the traditional analytic strategies. This paper explores how the introduction of technology in a differential equations class affects the sociomathematical norms, particularly that of what constitutes an acceptable mathematical explanation.

Theoretical Framework

This research grows from the work of Cobb, Wood, Yackel, and McNeal (1992) on constructs of social norms and sociomathematical norms they developed working in elementary school classrooms. To distinguish, social norms are norms that can exist in any classroom, regardless of content. For example, that a student is required to provide an explanation can be a norm of an English class or a history class, as well as a mathematics class. Sociomathematical norms, however, are particular to mathematics. What constitutes a mathematical explanation is particular to mathematics and is therefore a sociomathematical norm. This paper examines how technology influences what constitutes an explanation in a differential equations class where TI-92 hand-held computer algebra systems were available to all students.

The significance that these norms have for student learning stems from the reflexive relationship between the social and psychological aspects of a classroom as described by Cobb and Yackel (1996). Through interaction, the individual students' and teacher's beliefs about their roles and the roles of others in the classroom and their beliefs and values about mathematics influence the ways these norms are constituted. However, the interaction also influences these beliefs and values (Yackel & Cobb, 1996).

Research Methodology

The methodology of this research is that of a classroom teaching experiment as described by Cobb (in press). A research team of three mathematics educators conducted the teaching experiment with one member of the team teaching the class of twelve primarily engineering students. The team videotaped each class meeting and collected all student work. Also, the team met before and after these class meetings to plan for future classes and discuss observations made during the previous meeting. The classroom social and sociomathematical norms and ideas for future instructional activities were explicit topics of discussion during these meetings. The analysis follows that of examining videotapes of whole class discussions to describe student and teacher participation in the classroom discourse. This classroom teaching experiment was situated in a program of developmental research within the Realistic Mathematics Education framework (Gravemeijer, 1998).

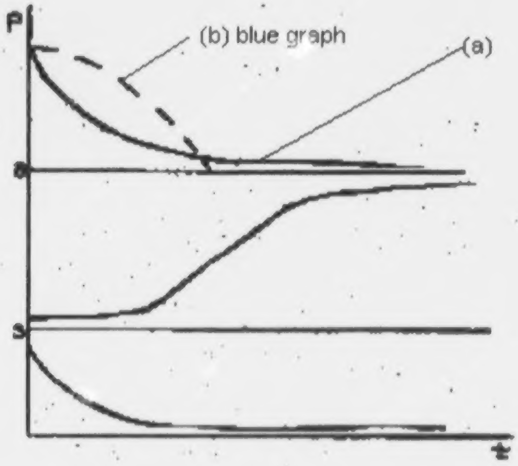
Results

To outline the results I focus on two areas taken from sessions on first-order differential equations. Briefly, I describe the typical nature of mathematical explanations in the project classroom. Following this description, I illustrate in detail on how the introduction of technology allowed students to enhance their explanations.

The Nature of Explanations

During the first two weeks of the differential equations class, the class focused on building a chain of symbolization for the slope field as an inscription symbolizing the rates of change in a differential equation (see Rasmussen, 1999). Students did not have access to technology during this time. The classroom interaction began to constitute a normative understanding that a mathematical explanation in discussions about differential equations should be made in terms of taken-as-shared, experientially real mathematical objects. In the context of differential equations, the students refer to rates of change in providing explanations. To illustrate, I draw on the following example from a whole class discussion.

On the previous day, students worked in groups to determine the long-term behavior of solutions to the differential equation dP/dt=0.5P(1-P/8)(P/3-1) by reasoning from the differential equation. The teacher provided students with TI-92s programmed to draw slope fields and sketch Euler approximations. All observed groups used their TI-92s to draw slope fields and make conjectures, but eventually began to reason about the behaviors of the solutions directly from the differential equation. On the next class day, during whole class discussion, the teacher drew a sketch of hypothetical solutions provided by students on the board, without a slope field. He then drew a hypothetical solution in blue that did not follow the path of the other solutions and asked the students how they knew that the blue graph was not a reasonable depiction of the behavior of solutions (see Figure 1). A student provided the following explanation drawing on rates of change represented in the differential equation to justify his claim that the blue graph is invalid.



Seth: According to his blue line, his blue line... His blue line shows an accelerating decreasing change in the population. But according to that equation, the acceleration of the decreasing of the population decreases with the change in population. Meaning the closer you get to eight, the slower the change is going to be.

Seth's explanation shows how he relies on a taken-as-shared mathematical object, the rate of change, depicted in the blue graph and the differential equation to justify his claim that the blue line is not an appropriate representation of a solution to this differential equation. This type of explanation is paradigmatic of the use of the rate of change to make explanations in the differential equations course.

The Nature of Technology-Enhanced Explanations

As illustrated above, a stable norm became constituted in the project class that a mathematical explanation relies on an experientially real mathematical object. However, these types of explanations were not always easy for others to follow. While many students followed it, others found Seth's explanation confusing and asked for further clarification in an effort to understand one another's thinking (see Rasmussen & Yackel, this volume, for an explanation of the social norms and general sociomathematical norms that became stable in this classroom). Another student provided an explanation that was enhanced by his previous use of technology.

All observed groups of the students used their TI-92s to create slope fields while working on the problem the previous day. As previously referred, the class developed the slope field as a symbolization of the rates of change in the differential equation. In this excerpt, Bill uses this symbolization of rates of change, the slope field rendered using technology, and the notion of steepness as a graphical re-presentation of rate of change, to provide an explanation.

Bill: I don't think the blue graph is right because it doesn't follow the slope field.

Instr: OK. Tell us about that. Why?

Bill: Well, if you get ... if you get close to eight, the slope fields are different between the two graphs. One is like not as steep but the blue graph is steep.

I conjecture that the chain of symbolization developed earlier in the class allowed Bill to make an explanation relying on rate of change, and the results of technology to remain within the normative aspects of the understanding that explanations rely on rate of change. His fellow students were able to understand this explanation better than the explanation provided by Seth. Without technology, constructing a slope field is not reasonable for everyday classroom discussions. Because students had technology at their disposal, they had the ability to base their explanations on experientially real mathematical objects rendered in imagery gained through technology. Thus, the additional interaction with the technology enhanced the constitution of a normative understanding of a mathematical explanation.

Conclusion

These examples begin to show that the nature of what constitutes a mathematical explanation can be expanded by the addition of a tool, technology, that allows discussion of symbols only feasibly produced with technology. This enhancement would be true no matter the tool. Imagine the classroom without pencil and paper and how this omission would effect students' ways of explaining their reasoning. While Bill did not explicitly construct a slope field at the time of his argument, I conjecture his previous construction allowed him to provide a mathematical explanation that would not have been reasonable without the technology. Other tools that might influence the development of what constitutes an explanation include manipulatives used in a range of classes. An analysis of chains of symbolizations demonstrates that these tools or representations of these tools signify mathematical meanings. An analysis of what constitutes an explanation and whether technology or other tools can enhance this normative understanding can provide information on how teachers use these tools in the classroom.

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MISUNDERSTANDING OF IF-THEN AS IF AND ONLY IF

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Abstract: The study addresses two common reasoning patterns that suggest a misunderstanding that may prevent students from properly reasoning in mathematics. That is, the misunderstanding of "if-then" as "if and only if." These common reasoning patterns are wrong responses to the illogical argument forms "affirmation of the consequent" and "negation of the antecedent," showing that the subjects do not realize that the conclusions are uncertain. It was hypothesized that the subjects who were not aware of the uncertainty of the illogical argument forms would not distinguish between these forms and similar logical forms where an if and only if condition has been replaced for the if-then condition. In such a case, not only their answers under both conditions would be similar, but also their justifications would show the same pattern. The findings support this hypothesis.

Misunderstanding of if-then as if and only if

Mathematical reasoning is based on the understanding of the "if-then" relation. Most of the theorems students learn at school are actually necessary and sufficient conditions. For example, the relationship between the sides of a right triangle given by the Pythagorean theorem is also a sufficient condition for determining that a triangle is a right triangle. But textbooks usually provide a deductive proof for the necessary condition and justify the sufficiency with only some empirical evidence (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1988, pp. 27-33). Moreover, in everyday language, if-then sentences often mean if and only if. For example, when parents say, "If you get good grades, we will give you a bike," the child knows that they mean that he/she will get such a present if the grades are good, and only in this case, that is, it means, "If you get good grades, and only if you get good grades, we will give you a bike." So, we may expect that students, based on both their classroom experience and their understanding of common everyday sentences, may believe that a property implied by some hypotheses or premises is also a sufficient condition for proving the premises and, therefore, they will misunderstand the conditional if-then as the biconditional if and only if (IFF). Evidence of this misunderstanding was found, among others, by O'Brien and Overton (1980) and by Knifong (1974).

Let us consider the possible entailments of the two illogical forms of arguments "negation of the antecedent" (NA) and "affirmation of the consequent" (AC). NA is: "If A then B, and Not A." It may be also stated as: "All the A's are (or do) B, and X is a C." It is not possible to state whether B is true or false (or whether X is, or does, B). However, English (1993) found that a high percentage of the subjects in her study, when asked whether X did B, would answer "No because X is a C, not an A." This answer would be right for "All A's, and only A's, are (or do) B, and X is a C", or equivalently, for "If A, and only if A, then B; and Not A."

The argument form AC is: "If A then B, and B" or "All the A's are (or do) B, and X is (or does) B". It is not possible to state whether A is true or false (or whether X is an A). Markovits (1995) found that most of his subjects, when asked whether A was true, would answer "yes." Again, this answer would be right for a biconditional: "If A, and only if A, then B; and B."

So, the most common mistakes about these illogical argument forms, answer "no" to NA and "yes" to AC, may be explained as implicitly suggesting a misunderstanding of if-then as if and only if. I will call them "implicit" answers, borrowing the name given by English (1993).

My study tested two hypotheses. First, students who provide the implicit answers to NA or AC will show the expected misunderstanding by providing the same kind of justifications for similar IFF questions, suggesting that they do not discern these two conditions. This hypothesis challenges English's (1993) claim that the implicit answer to NA showed that "children were aware, at least implicitly, of the unconnectedness of the premises" (p. 402). Byrnes and Overton (1986) found evidence suggesting that the comprehension of conclusions of certainty emerges earlier than the understanding of uncertain conclusions. So, if students who provide implicit responses to NA or AC have already reached this awareness, they should be able to provide right justifications for similar problems at IFF conditions, that is, when the conclusions are certain.

Second, it was hypothesized that students who provide the right answer "I cannot know because of insufficient information" for the illogical argument forms would provide the right answers and justifications to similar forms under if and only if conditions. It would show that these subjects are aware that the premises are unconnected only for the if-then conditions.

Method

Participants

A total of 25 seventh graders and 27 eighth graders, from a public school, in a small city in central Chile participated in this study. Six of them were not considered for the study because they skipped some questions, leaving 11 boys and 12 girls at each grade level. The subjects were low to middle class students with ages between 12 years 10 months and 16 years 2 months. They were asked to answer the questions of a booklet and allowed to withdraw at any time.

Materials

The materials consisted on a booklet with four items about different imaginary creatures on an island. The first page provided instructions and two examples of questions showing their correct answers. Each of the next four pages presented an item about one species of creatures and had as a title the name of this species. This format was supposed to prevent the subjects from mixing up these names with the names of the particular creature on each question. The subjects were supposed to know about these creatures only what a tourist guide would tell them. For each species, she would say two statements written on the top of the page. The first statement was an if-then sentence about some characteristic of the species. For two items the second statement provided the only if situation, pointing out that no other species had this feature (IFF condition). For the other two items the second statement was irrelevant to the questions asked (if-then condition). For each item four questions were asked corresponding, respectively, to a different argument form: the logical argument forms "modus ponens" (If A then B, and A; therefore, B) and "modus tollens" (If A then B, and Not B; therefore, Not A), and the illogical forms NA and AC. They were presented in a different order on each item. Two examples follow.

For the if and only if condition, an AC question about creatures called "fitis" is:

"The guide says:

1 Fitis have blue eyes.

2 None of the other animals has blue eyes.

Tobi is an animal with blue eyes.

Is Tobi a fiti? Yes No I cannot know Explain."

For the if-then condition, a NA question about creatures called "undis" is:

"The guide says:

1 Beware; undis bite.

2 They bite really hard!

Epy is a tori.

Does Epy bite? Yes No I cannot know Explain."

The fantasy context was chosen by considering both Watters and English's (1995) claim that real-life knowledge influences the strength or extent of the inferential rules that people combine, and Markovits' (1995) findings suggesting that by imbedding premises in a fantasy context they can be insulated from interference by empirical knowledge. So, the subjects, would be more likely to answer relying only on their understanding of both the premises and inferential rules.

Design and Procedure

Each subject received the booklet in a normal classroom setting during the last week of the school year. The instructions written on the first page were read aloud by the researcher who provided a short explanation. For each question, they were asked to circle one of the answers "Yes," "No," or "I cannot know" and to provide a justification based on the given premises. There was no time limit and most of the subjects finished in less than 30 minutes.

Results and discussion

Though the subjects were asked to justify their answers by referring to one or both premises, some provided other explanations. For example, a pattern of answers often found on the modus ponens questions was, "Yes because he is an...", suggesting the underlying assumption that the fact that the creature was of a particular species was sufficient to justify that it had the features of this species and that no further mention of the first premise was needed. Therefore, both for the if-then and the if and only if conditions, the answers to NA, "No because he is not...," and to AC, "Yes because he has (does)..." were considered as evidence of the assumption that no other species had the same features and were classified as "implicit" responses. The same pattern was often suggested by the explicit statements that a feature was *only* for these creatures for NA and that *any* creature with this feature *must be* of this particular species for AC (consequent implies antecedent). I extended the name "implicit" to the IFF conditions for responses that provided the right answer (yes or no) but referred only to the first premise for their justification.

The answers were classified, considering both the answers circled and the justifications provided for them, in four categories: (1) I = implicit; (2) C = correct answer and justification; (3) I' = same answer (yes or no) as I, and wrong justification not referring to any of the premises; and (4) W = wrong, neither I nor I'. On the statistical tests, numbers of answers under the if-then conditions and numbers of answers under the respective IFF conditions were compared through Kendall's rank correlation coefficient. An alpha level of .05 was used for all statistical tests.

Table

Number of subjects for each combination of answers and Statistics

		Number of if-then answers/Number of IFF answers)									Statistics		
		(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	r	z	p
(I, I)	AC	5	1	0	5	6	3	7	13	6	0.260*	2.55	.0107
	NA	17	5	0	8	9	0	3	4	0	0.317**	3.10	.0019
	AC&NA										0.293**	2.87	.0041
(C, C)	AC	19	10	3	7	2	2	1	1	1	0.077	0.75	.4511
	NA	11	4	2	9	7	1	5	3	4	0.196	1.92	.0551
	AC&NA										0.197	1.93	.0539
(I, C)	AC	3	2	1	9	3	2	15	8	3	-0.013	-0.13	.8956
	NA	12	6	5	9	7	2	4	1	0	-0.025	-0.25	.8063
	AC&NA										-0.050	-0.49	.6207

* $p < .05$. **$p < .01$.

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My first hypothesis predicted that if under the if-then conditions the answers provided for NA or AC were "implicit," the answers to the respective questions under the IFF conditions would also be "implicit." To test it, the numbers of implicit answers were compared. The findings support this claim. Statistically significant positive correlations were found for AC, for NA, and when considering AC and NA together (see Table).

English's (1993) claim about the subjects' implicit awareness of the unconnectedness of the premises was also tested by considering its logical consequence, that implicit answers for the illogical argument forms predict right answers under the corresponding logical conditions. The number of implicit answers on AC and NA and the number of correct answers under the respective IFF conditions were compared. Not statistically significant negative correlations were found (see Table). So, the evidence obtained by this test neither supports nor rejects this claim.

My second hypothesis predicted that subjects who provided correct answers on AC or NA would provide correct answers under the corresponding IFF conditions. It was tested by comparing the numbers of correct answers. Not statistically significant positive correlations were found for AC, for NA, and for AC and NA together (see Table). So, this hypothesis was neither supported nor rejected by the evidence.

The evidence suggests that subjects who provide implicit responses to AC and NA are likely to provide the same kind of answers under the corresponding IFF conditions and, hence, do not discern the difference among these two conditions. That is, they seem to misunderstand if-then as if and only if. This finding may be considered as evidence against English's claim.

A question that remains to be answered is, At what age or school level, if any, do most subjects begin to discern if-then from if and only if?

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THE STUDY OF THE INSTANTANEOUS RATE OF CHANGE CONCEPT SITUATED IN THE CLASSROOM

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Our goal in this study was to document what types of concepts, reasoning, and strategies students use when solving instantaneous rate of change problems, while working in cooperative groups. The experiment was in a classroom, with a regular group. We wanted to document also what types of participation were shown by the students. We found that the unit rate as a symbol led some students to confuse variable rate with constant rate, and the instantaneous rate with average rate. The words "average" and "instant" had meanings for the students that interfered with their conceptual understanding. Many times the rate was seen as only a numerical operation not as a relationship between quantities. The students wanted to give conceptual explanations in their participation, but frequently lacked the resources to do it.

Introduction

The logic of this teaching experiment is to use instruction as the primary site for probing students' comprehension and for gaining insights into their constructions (Thompson, 1994) of their instantaneous rate of change concept, and to understand the type of participation shown in the classroom. The learning acquired is related to the kind of instruction received. In this study, we investigated what type of thinking and reasoning related to the instantaneous rate of change is generated by the students when they receive instruction in which a problem-solving context is used (Santos, 1998). This context refers to a situation in which students can discuss a function and its derivative intuitively without a previous calculus course. We want to know to what extent this kind of teaching promotes cooperative learning and students' appreciation of inquiry, understanding, and reasoning (Greeno, 1997), as well as their use of abstract representations, such as tables, graphs, and formulas as ways of directing attention to the invariance of the situation.

Conceptual Framework and Procedures

The implementation of this type of instruction is based on some ideas from Greeno (1997) who considered participation in the practices of inquiry, understanding, and mathematical reasoning as fundamental in the process of learning mathematics. Participation serves as the frame of reference in analyzing and understanding learning given the work demonstrated by students as a result of their activities during the course. For this work it was decided to focus the analyses on two activities which appeared constantly during the development of the class: (I) the participation of the students in discussion teams and (II) the participation of the students during group discussions. It is fundamental to mention that the learning activities implemented in the classroom touched on such aspects as the students' being aware of their responsibilities as members of a small community, of the team and of the class as a whole, so they knew that it was important to express their own ideas and to listen to the ideas of others.

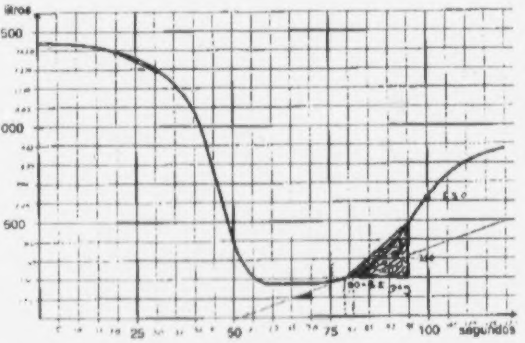
This study took place in a twelfth-grade introductory calculus class of 50 students. The group was divided into teams of 3 or 4 students who were given problems that required them to investigate the relationship between the function and its derivative (instantaneous rate of change) in a daily life context with the function's data represented as a table and as a graph. Students had to discuss with teammates in order to reach solutions to these problems. After this, the results obtained in the teams were discussed by all the students in one large group.

In order to document the students' participation, the classes were videotaped as students from the teams went to the blackboard to write their answers which would be discussed by the larger group. Also, the teams' interactions were audio taped. The students' written answers were then analyzed. The analysis focused on how students reasoned, comprehended and inquired when solving the problems of instantaneous rate of change working cooperatively.

Results

One team of four was audio-recorded. Students were given the data in the table. Their task was, making use of the table, to estimate the flow of water into the tank at 80 seconds and calculate the average flow between the 80 and 85 seconds. They were asked the question: What data do you need besides the data given in the table in order to make a better approximation of the flow at 80 seconds? You may use the graph.

Time (seconds)	Volume (liters)
80	205
85	265
90	350
95	490
100	620



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Part of the team's dialogue was the following:

> Sergio: we need to find a point around here, about 75, 85
>
> Julio: but, one that's closer
>
> Sergio: here, in the graph from 79 to ...
>
> Julio: to 79 corresponds ... 200
>
> Sergio: its 200 more or less, for 80 are ... 200 ..., in 85 are 265 ... closer to 80, in 79 and 81
>
> Araceli: but ...
>
> Sergio: here with the graph
>
> Araceli: but ... how?
>
> Sergio: in 79 there is 200; in 81 there is 200 ... about 210

We can observe that in their reasoning to find the instantaneous flow approximation of 80 seconds they used the graph. In the first approximation toward finding the flow they mentioned the interval 75 – 85, that is, 5 seconds below and up from 80, and afterwards the interval 79 – 81, also one second up and below from 80. The first interval seemed to be suggested by the table interval, and the graph grid, in which time was marked every 5 seconds. In a second attempt, in which they came closer to the solution, they chose the integers 79 and 81, also one below and up from 80, as observed in the dialogue. They faced the problem of estimating the approximate volume in the graph at 79 and 81 seconds because of the difficulty of distinguishing the precise measurement. When the instructor asked if it was possible to have a closer approximation, they proposed the rational numbers 79.999 and 80.001. As expected, they were not able to read the volume increase (rise) corresponding to such a short interval of time (run) in the graphic representation, and instead, even though they still had problems in distinguishing the volumes that corresponded to the points of time on the graph, they estimated an increment of accumulated volume between 79 and 81 seconds. We can see that reading graphical representations is a limitation when students try to find instantaneous flow. In other teams, this limitation led them to set up and solve a proportion (*V/80.1 sec=200 liters/79.9 sec*), in estimating, wrongly, a volume. In addition, in spite of an explicit question, nobody in the group could explain why, by taking smaller increments of time, a better flow approximation at 80 seconds can be obtained.

A student, Alejandro, calculated the slopes of tangent lines using two very close points of time. He divided the increment of the volume by the increment of time. Choosing small intervals of time to calculate the flow, he said that the flow was valid for any time of small intervals. In this manner, he made the concept of instantaneous flow and continuity less clear for himself. Frequently, when the students had to calculate flows, they did not distinguish between instantaneous flow and average flow; they used the formula $F= (V_2-V_1)/(t_2-t_1)$, with intervals of time of, for example, 5 seconds, although they did not mention for which periods of time or instants those calculated flows were valid.

Another student, Armando, made different flows correspond to different moments and used the procedure of tangent line slope to calculate the flows, but in attempting to give an explanation using the graph, he lost sight of the instantaneous flows and at the same time of the considered intervals. In the end, he did not mention to which moments the flows corresponded. Another difficulty frequently presented during instruction was that when they obtained the result of an instantaneous flow, e.g. 10 liters/second, they thought of the flow of 10 liters a second as a constant not a variable flow.

In general, formulas were taken as recipes and not as relationship between quantities, particularly, the flow formulas $F=(V_2-V_1)/(t_2-t_1)$ and $F=V/t$, the average volume and the average time formulas $(V_1+V_2)/2$ and $(t_1+t_2)/2$, respectively, and the formula for proportion $V_2/t_2=V_1/t_1$. It should be noted that frequently, in calculating the average flow, they divided $(V_1+V_2)/2$ by $(t_1+t_2)/2$. On some other occasions, in calculating the instantaneous flow in time t, they read the ordered pair (t, V) of the table or the graph and substituted it in the formula $F=V/t$. It can be seen that all these formulas include the arithmetic operation of division, although it seemed to be difficult for them to take it as an relationship between quantities or quantitative operation (Thomson, 1994). But when they worked with subtraction, they were able to give a quantitative meaning as an increment in quantity. On the other hand, many times they mistook or confused the quantities, e.g. flow instead of volume or time.

Regarding graphical reasoning, in general the concepts were used in the tasks as they were taught: the run and rise were taken as representations of the increases in time and volume; a straight line represented a constant flow and a horizontal straight line meant that there was no flow. Nevertheless, sometimes they represented the flow as the length of an arc, and they compacted the run and rise in the arc. This became apparent when we told them to, given the volume vs. time graph, find two moments in which the flows were the same. To solve the problem, they took two arcs of the same length and shape, and after reasoning and discussing with the whole group, a student arrived at the conclusion that it was better to consider two equal slopes of lines tangent to the curve to decide at what points of time the flows were equal.

The team work promoted dialogue, and in the communication established during the dialogues transcribed, they used no pre-established procedures. But because there was a more conceptual content, they arrived at a process of approximation to the limit: Sergio proposed two points of time that were useful for closer estimation of the flow at 80, 75 and 85 seconds by reading the respective volumes on the graph; Julio asked for a closer estimation, and Sergio accomplished this in saying 79 and 81 seconds, and then they explained to Araceli what they had meant. Through this kind of participation they were essentially refining their comprehension of this concept. But for some students, when they substituted an ordered pair (t, V) of the table in the formula $F=V/t$, and their intuitions about flow did not match their results, their belief in the formula was stronger than their belief in their intuitions. It seems that the participation of these students was not focused on understanding relationships. On one occasion, some students, not very sure of having used the flow formula properly $F= (V_2-V_1)/(t_2-t_1)$, called the teacher. Apparently, this lack of confidence in the use of the formula was due to their misunderstanding of the relationships among quantities. Even though the students had already discussed the matter in their team, they were not able to explain why the formula was useful. When they were told that what they had done was incorrect they changed their mind without reasoning or discussion. During the instruction it was intended that, rather than focusing on procedures, the students would gain an appreciation for exploration, understanding, and reasoning, but that did not always happen, possibly due to participation they had had in previous school experiences and to their lack of conceptual resources.

Discussion

Instantaneous rate of change, as a case of variable flow, is really only the result of attempts to quantify our sensorial impressions of continuous quantities and variability. This concept is present in daily life, and many people reason according to this idea, even though they cannot give it an explicit name, nor can they reflect upon this type of thinking. The continuity of the instantaneous rate of change from this point of view is not understood necessarily in a mathematical sense, but in the sense of an interrupted relationship between two dependent magnitudes. This is the case of the estimation of flow in this task of the experiment where the students used their intuition about the continuity of time, volume and flow. The students' responses suggest that they were not conscious of the thought processes that led them to infer that the closest points of time are t_1 and t_2 in the formula of the average flow $F=(V_2-V_1)/(t_2-t_1)$ which is a better approximation to instantaneous flow. It seems to be necessary that the professor intervene in order to clarify this very important idea. Students faced difficulties that distorted their intuitive comprehension of limit. These seem to have two sources. First, for some students the words "instant" or "moment" meant very short periods of time which can be of a second (more or less), which reflects the dictionary definition. In the Webster's School Dictionary, "instant" is defined as a very short period of time, while "moment" is defined as a minute portion or point of time. Thus, for them, a flow in a given moment or instant occurs during a period of time. The second source of their distorted conceptualization of limit seems to be the process of approximation they used to find the instantaneous rate of change, in which they took short increments of time to find the flow at a given instant. This seems to have reinforced their idea that an instant or a moment means a period of time and that during this short period the flow does not vary, an idea that contradicts their first intuitions that for each instant there is a different flow. Moreover, the students' understanding of the words does not concord with the mathematical concept of instantaneous rate of change that was intended in the instruction.

A problem that emerged is that, in order to quantify a variable flow as an instantaneous rate of change, we must visualize and symbolize it as constant flow, i. e. as a unit rate. For example, when a variable flow runs in a pipeline, to say that in the instant t the flow is x *liters/min*, meaning that if we keep the flow constant during a minute, x *lts* will flow. But many students for this situation think that during a minute necessarily x liters will flow as if a variable flow were the same as a constant flow. It seems that the way we represented the flow led some students to become confused about constant and variable flow and average and instantaneous flow. In reference to language use, there are two ideas and procedures related to the word "average". On one hand, some students think of it as a discrete arithmetical average, $(V1+V2+\ldots+Vn)/n$, when it is actually a continuous function. On the other hand, they may think of it as an average rate, $(V_2-V_1)/(t_2-t_1)$. Thus, in instruction the teacher should be aware of the two meanings students give to the word.

In the formulas in which division appears, this operation is generally deemed by the students a numerical operation, rather than a relationship between quantities. For them the formulas are recipes: This spawned many mistakes, such as taking an ordered pair (t, V) from the table or graph and substituting it in the formula $F=V/t$ to calculate the instantaneous flow. It seems necessary that the students explore the speed and flow formulas as relationships among quantities before instruction on the instantaneous rate of change (Thompson, 1994).

The students generally used the graphical representations given in the instruction, such as run and rise; nevertheless, they also used idiosyncratic representations, e. g., the shape and length of an arc to represent the flow, but these representations can evolve through (directed) discussion to the formal representation of the slope of a tangent line, for instance. It is important to point out that graphical representations have limitations when we want use them for calculating the instantaneous rate, especially when we take smaller and smaller runs and rises, making the lengths more and more difficult to measure.

Regarding the type of participation, the richness of the conceptual web determines their conceptual participation. Students often want to give conceptual explanations, but lack the conceptual resources; they especially lacked understanding of division as a relationship between quantities.

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NORMATIVE UNDERSTANDINGS REGARDING EXPLANATION: A STUDY OF ONE DIFFERENTIAL EQUATIONS CLASSROOM COMMUNITY

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The purpose of this report is to extend analyses of social interaction patterns that have been successful at characterizing elementary and secondary school classrooms to the learning and teaching of undergraduate mathematics. In particular, we document the social and sociomathematical norms regarding explanation in one differential equations classroom community and explicate how these norms were constituted in this specific case.

The research reported here grew out of a semester-long classroom teaching experiment conducted in an introductory differential equations class for mathematics, science, and engineering majors in a university in the United States. Throughout the semester, students in the class explained their reasoning without prompting, offered alternative explanations, and attempted to make sense of other students' reasoning and explanations, despite the fact that their prior experiences were with traditional approaches to mathematics instruction. A central premise of this report is that their ability to do so was enabled by explicit attention to classroom social and sociomathematical norms.

The purpose of this report is to demonstrate what these norms were and how they were constituted in this specific case. In the process of achieving this pragmatic goal, we also achieve our theoretical goal of establishing the usefulness of the constructs of social and sociomathematical norms for analyzing university-level mathematics instruction. The latter goal is important for two reasons. First, since these constructs grew out of investigations conducted in elementary school mathematics classes (Cobb, Wood, Yackel, McNeal, 1992; Yackel & Cobb, 1996) it is not readily apparent that they are immediately applicable to advanced university mathematics courses. Second, the explication of these sociological constructs complements the primarily psychological analyses of students' learning of advanced undergraduate mathematics and adds to the emerging body of literature on the learning and teaching of dynamical systems (Artigue, 1992; Nemirovsky, 1993; Rasmussen, 1998; Zandieh & McDonald, 1999).

Theoretical Framework

Our research employs a socioconstructivist perspective that seeks to coordinate psychological and sociological points of view, as elaborated in the framework offered by Cobb and Yackel (1996). As a first step toward coordinating these points of view, we bring to the foreground two constructs from this framework that fall within the social perspective—social norms and sociomathematical norms. Social norms refer to normative understandings regarding participation in mathematics instruction, such as the expectation that students will explain their reasoning, offer alternative explanations, and attempt to make sense of other students' explanations. Sociomathematical norms refer to normative aspects specifically related to mathematics, such as implicit understandings of what constitutes an acceptable mathematical explanation, a different mathematical explanation, and an efficient mathematical solution. An individual's beliefs about their role, other's role, and specific mathematical beliefs and values are assumed to develop concomitantly with classroom norms (Yackel & Cobb, 1996).

Research Methodology

The methodology for the teaching experiment falls under the heading of developmental research (Gravemeijer, 1994), which consists of a day by day cyclical process of analysis and instructional design, complemented by retrospective analysis of all data sources at the end of the teaching experiment. As part of the day by day analysis, planning and debriefing sessions were conducted to explicitly discuss classroom social and sociomathematical norms and instructional activities, the latter were developed following the principles of Realistic Mathematics Education (Gravemeijer, 1994). All mathematics lessons were videotaped for later analysis. The research team also conducted individual interviews twice during the sixteen-week semester with a majority of the students in the class. Regularities in social interactions were identified by analyzing videorecordings of class sessions, records of project meetings, researcher field notes, and the instructor's journal (Cobb & Whitenack, 1996). Selected excerpts are used in this report to exemplify normative aspects regarding explanation and to illustrate the interactive constitution of these norms.

The research team consisted of three university mathematics educators, one of whom taught the class and one of whom developed programs for the TI-92 graphing calculator that enabled students to generate numerical approximations, direction fields, and phase portraits. The class of twelve students also used a reform-oriented textbook for homework and reference purposes. The typical class session began with a brief introduction of a problem situation by the instructor, but no specific solution procedures were provided. Students then worked collaboratively to address the problem and then reconvened as a whole class to discuss their reasoning and analyses.

Results and Discussion

Social norms refer to those aspects of classroom social interactions that become typical or regular. For example, in the differential equations class discussed here, it was typical (or normative) that students explain their thinking, attempt to make sense of other students' reasoning, and offer alternative ways of reasoning. The following excerpt, which occurred early in the semester, illustrates how students could expect to be challenged by the teacher or other members of the class when they do not fulfill the obligation to provide an explanation, thereby contributing to the initiation of and interactive constitution of these norms.

The example picks up partway through a whole class discussion where students were explaining their previous analyses of the differential equation $\frac{dP}{dt} = 0.5P(1 - \frac{P}{8})(\frac{P}{3} - 1)$. This differential equation was offered as a way to model the rate of change for the population of fox squirrels in a mountainous region. Moving beyond the problem given, the instructor asked the class what would happen if the initial condition were negative. After an initial (correct) response, the instructor says, "Jerry says it will increase. What do you think Stan?"

Stan: Umm (15 seconds later). Yeah, I don't know.

Instr: How are you beginning to think about it? Our initial starting population is negative. Granted this doesn't make sense to have a negative number of squirrels, but maybe there is a different situation where a negative value for the dependent variable makes sense.

By asking Stan about his initial thinking, the instructor implicitly lets Stan know that there is an expectation that he will engage in the question posed and share his reasoning, however tentative. After a short wait, Stan begins to explain how he is thinking about the question posed.

Stan: (15 seconds later) I guess it would increase.

Instr: Tell us about it.

Stan: If you put a negative number in there, the first term, well, I guess it will decrease.

Instr: Ok, tell us how you are thinking about it. As a class we will think about it with you.

Stan: Ok. Well the first expression, 0.5P will be negative. The next one will be positive and next one, that one will be negative. So I guess you are going to increase.

Instr: So you are going to increase. What do you guys think?

Joe: So far it looks like everything is going to equilibrium.

After Stan explains his reasoning, Joe spontaneously frames Stan's remarks in terms of the behavior of solutions with initial conditions below and above zero without being specifically called upon.

The entire excerpt illustrates that the students and the instructor acted in accordance with the normative understandings that students were expected to explain their reasoning and that they were expected to try and make sense of other students' thinking. These two social norms, which were constituted early in the semester, contributed significantly to the climate of the classroom as one in which sense-making and meaning-making prevailed.

The above example illustrates a second critical aspect of classroom norms, namely their interactive constitution. By engaging in explanation and sense making of others' contributions, Stan and Joe not only acted in accordance with these as social norms, they also contributed to their ongoing constitution. Such norms are based on expectations and obligations that are constituted as participants interact with each other, rather than on a priori rules that are spelled out for students to follow. Whenever someone acts in accordance with an expectation, he or she contributes to the ongoing constitution of the expectation as normative in that situation. Thus, while it is the teacher who typically initiates the constitution of norms, all participants in the interaction contribute to their ongoing negotiation.

A construct that is closely related to social norms is that of sociomathematical norms. While social norms refer to normative interactions in the classroom in general, sociomathematical norms refer to those normative understandings that are specifically related to the fact that the subject of study is mathematics. In the differential equations class described here, procedural explanations typically did not constitute sufficient explanation. In particular for first order differential equations, explanations had to be grounded in an interpretation of the rates of change as expressed by the differential equation. To illustrate this claim, we draw on a whole class discussion about solutions to the differential equation that modeled the fox squirrel problem where students' initial explanations were very much in terms of procedural instructions. For example, Jerry justifies his conclusion that the fox squirrels would decrease if the initial population were greater than 8 as follows:

Jerry: Because some number greater than 8 over 8 is going to yield some number greater than one, which 1 minus something greater than 1 is going to give you a negative number and so something times a negative number is going to give you a negative number, so your slope is going to be negative.

In order to facilitate a shift in the nature of the explanation, the instructor probes further into the meaning of what Jerry has said in terms of the rate of the change and the context of the problem. Thus, similar to the way in which students could be expected to be challenged by the instructor or other class members if they do not offer an explanation, students in this class could be expected to be challenged to go beyond procedural instructions. As the discussion progressed, the instructor asked Dave how his group thought about the analysis. Dave's response reflects a shift from explanation as procedure to explanation in terms of the rate of change and the significance of the rate of change for the population.

Dave: Well, pretty much kind of same of what Jerry was saying but just the opposite. In this case, it says the fertile adults have to be able to find other fertile adults to be able to increase. Well if they don't, then the rate of change of that is going to be negative which makes everything else negative, so it's decreasing.

Although Dave's response was not at all in terms of procedural instructions (as Jerry's initial comment was), he prefaces his response with, "pretty much the same as what Jerry was saying." In so doing, Dave acts in accordance with the expectation that explanations will be conceptually oriented, rather than procedurally oriented. This example also illustrates how sociomathematical norms, like social norms, are typically initiated by the instructor and constituted in interaction. As students act in accordance with a norm, they contribute to its ongoing constitution.

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DETERMINING LINEARITY: THE USE OF VISUALIZATION IN PROBLEM SOLVING

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The purpose of this study was to examine the ways in which undergraduate students use visualization in their problem solving attempts to construct their meaning of linearity. The findings suggest that when confronted with tasks which have inherent geometric elements, undergraduate students may use visualization and analysis as two interacting modes to solve the problems. While students demonstrated flexibility in using visual and analytic processes, they also faced difficulties when making translations between the algebraic and visual representations.

The role of visualization in the construction of meaning by students of mathematics, as well as the use of visualization in their problem solving remains an active question in educational research. Previous studies which investigated students' use of visualization at the college level presented a broad spectrum of findings. Eisenberg and Dreyfus (1991), for example, indicated that students are often reluctant to use visualization to process mathematical information. On the other hand, Zazkis, Dubinsky and Dautermann (1996) in a study of learning beginning abstract algebra found that students use some combination of both visual and analytic strategies.

This study looked at the use of visualization in problem solving in undergraduate mathematics and specifically in situations involving linear algebra concepts. Linear algebra is an area of college mathematics in which the use of visualization comes in naturally. As Dorier (1995) indicates, linear algebra grew partly from analytic thinking about geometry and as such its study is facilitated by the interplay between the two. The few studies which pertain to the learning of linear algebra (Sierpinska, 1997) put a special emphasis on students' use of visualization by observing that the geometric and analytic coexist in the thinking of students and the flexible use of these promotes understanding in linear algebra. This study examines the ways in which undergraduate students use visualization in their problem solving attempts to construct their meaning of linearity.

Method

The task shown in the figure below was administered to eight undergraduate mathematics students who completed a linear algebra course with a grade of B or better. Students were individually interviewed and were asked to think aloud, and to explain their solution processes.

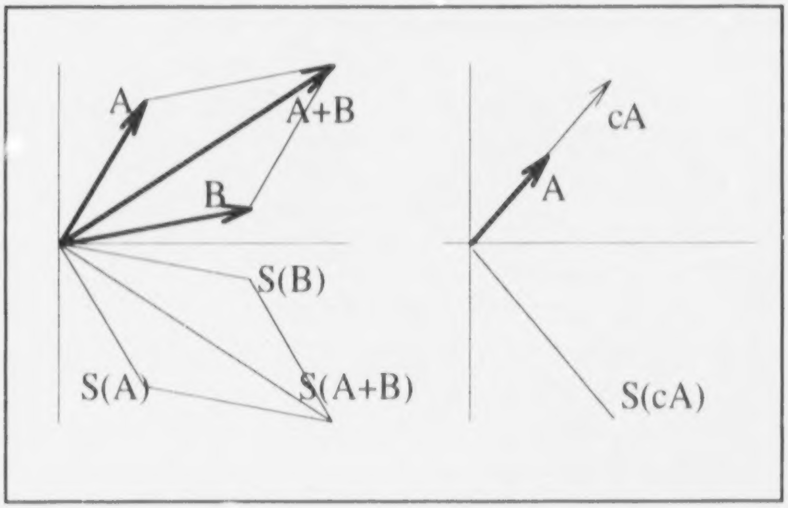
> Consider the functions given by the following expressions:
>
> (i) S: $\Re^3 \rightarrow \Re^3$ is the function that reflects a point through the x-y plane.
>
> (ii) L: $\Re^2 \rightarrow \Re^2$ is the function that moves a point over 3 units to the left and then 4 units up.
>
> (iii) F: $\Re^3 \rightarrow \Re^3$ is given by $F[X]=\text{Proj}_{(1,3,2)}X$.
>
> If A and B are vectors and c is a scalar, which of the above functions satisfy the following properties:
>
> - $f[A+B]=f[A]+f[B]$
> - $f[cA]=cf[A]$

Figure 1. The study task.

The task can be solved using either an algebraic or a visual approach. One may describe the three functions algebraically and then check whether the two properties of linearity (additivity and scalar multiplication) hold. Alternatively, one may construct a visual image of the action of a transformation and see whether this is consistent with a geometric interpretation of the two properties. For example, one may solve problem (i) by showing that the function S will reflect two vectors that determine one parallelogram to two vectors that determine another parallelogram, and the diagonal of the original parallelogram is reflected to the diagonal of the new parallelogram (Figure 2). Similarly, *S* will reflect a scaled vector to the scale of the reflected vector. The linearity of the translation (L) and the projection (F) can also be determined using either approach.

$$\begin{aligned}
S[A+B] &= S[(x_1+x_2), (y_1+y_2), (z_1+z_2)] \\
&= (x_1+x_2, y_1+y_2, -(z_1+z_2)) \\
&= (x_1, y_1, -z_1) + (x_2, y_2, -z_2) \\
&= S[A] + S[B] \\
S[cA] &= S[c(x_1,y_1,z_1)] = S[(cx_1,cy_1,cz_1)] \\
&= (cx_1,cy_1,-cz_1) = c(x_1,y_1,-z_1) \\
&= cS[A]
\end{aligned}$$

(a)



(b)

Figure 2. The algebraic and the visual approach to the first task.

Analysis of the data involved identification of situations which allowed us to make inferences about students' thinking, and thus provide information on the nature and development of understanding the concepts involved. Responses were coded for the use of algebraic and visual approaches and for the role that visualization and algebra played in students' understanding of linearity. Each protocol was coded for actions which indicated use of visualization and analysis.

Theoretical Framework

For the analysis of our data we used a framework which was suggested by Zazkis, Dubinsky and Dautermann (1996), the Visualizer/Analyzer (V/A) model, which views visual and analytic thinking as complements, rather than opposites. The model assumes that visualization and analysis, although distinct forms of thinking, inform one another and work together in the process of mathematical problem solving. More specifically, the V/A model describes this pattern as an interplay between successive acts of visualization and steps of analytic reasoning.

Zazkis et al. conducted their study in the context of learning beginning abstract algebra, but the model appears appropriate for linear algebra as well. Sierpinska (1997) also observed that in linear algebra the geometric and analytic modes of thinking coexist in the thinking of students and the conscious and flexible use of them all promotes students' understanding in this area. For this reason, the V/A model was chosen to provide a framework for the analysis of our data.

Results

The eight students demonstrated substantial variety with respect to their approaches to the problems and the ways in which they used visualization and algebraic reasoning when solving the problems. Figure 3 presents a description of the students' approaches.

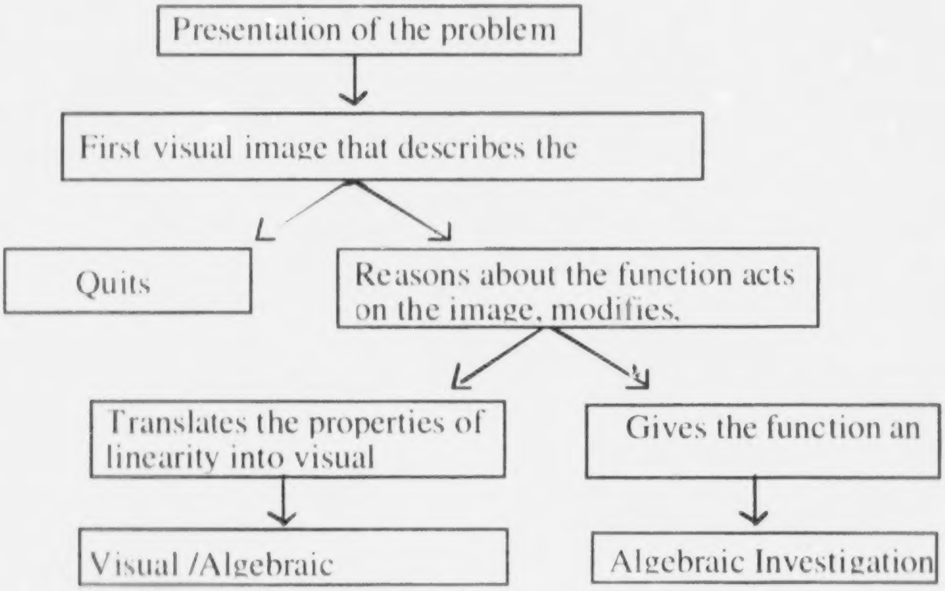

Presentation of the problem
First visual image that describes the
Quits
Reasons about the function acts on the image, modifies.
Translates the properties of linearity into visual
Gives the function an
Visual /Algebraic
Algebraic Investigation

Figure 3. Students' solution paths.

After they read each problem, all 8 students described a visual representation of the function. At this point, one student gave up the investigation of the problem. The remaining 7 students reasoned to some extent about the function primarily using their visual images. Their reasoning lead them in either of two directions: First there were those students who chose to solve the linear transformation problems they were given using a visual approach. In fact, all students chose this direction for at least one of the functions they were given. In general, students translated the properties of linearity into visual/geometric actions and used them to determine the linearity of the functions. Second, there were two students who, after an initial use of a visualization step, chose to use an algebraic approach for some of the functions they were given. These students used the visual representation of each function to help them express the function algebraically and used the algebraic description of the two properties of linearity to determine the linearity of the functions.

The first visualization step (V_1): The first reaction of all our interviewees was to create a visual image of each function either as a drawing or by some physical motion that they performed with their hands in a way that described an image as the following protocols illustrate:

DR: All right, so a geometric interpretation would be good for (i), I suppose, which reflects a point through the xy-plane...

MK: Hmm.. So, here's the xy-plane...[holds up a sheet of paper to represent the xy-plane] Yes. Here's your point....

The above students reacted to the problem by making a visual interpretation of the reflection function. The reasons for choosing their first step to be visual varied among students. Often they expressed a need for something concrete to work with and attempted to reduce what they perceived as complexity and unfamiliarity in the question by creating a visual image. They apparently viewed the visual representation as a more tangible way of dealing with the problem, and as a tool that would help them reach a solution. Others stated that their purpose for creating the visual image was to gain insights which would help them construct an algebraic description of each function and, therefore, proceed to solve each problem in a purely algebraic/symbolic manner as they usually do. However, despite their clear preference for algebraic approaches, they still constructed a visual image of the problem and repeatedly referred to that image. The construction of the visual image was necessary for the construction of their understanding of linear transformations

The first step of analysis (A_1): After constructing the visual image, the students proceeded to reason about it, opening the way to the investigation of the properties of linearity–the second part of the problem. However, students had varying levels of success with this step. For one student, the construction of the first visual image was as far as he could get; he did not see in the picture he drew anything that added to his understanding of the problem, and abandoned the effort.

At this point those students who succeeded had to make a choice: either translate the visual image they constructed into a symbolic one by giving the function an algebraic description and proceed to make an algebraic investigation, or translate the two properties of linearity into visual actions and proceed to make a visual investigation by reasoning about the visual images. Either choice required them to think *both* visually *and* algebraically. The choice each student made determined the path they would follow in their subsequent investigation of the problem.

Choosing the visual path: Each student chose the visual path to solve the problem for at least one of the 3 functions. After they created a visual image, students translated the properties of linearity into visual actions and proceeded to determine the linearity. Students' strategies varied with respect to their sophistication: some students reasoned with the use of specific examples of the functions or with general arguments.

Choosing an algebraic path: A second option for students was to do an algebraic investigation. Two students chose this path, for the investigation of one of the three functions. In this case, students used their visual representation to describe the function algebraically. Typically, students described the matrix of the transformation and used it to demonstrate (or reject) the linearity of the functions. Nonetheless, in each investigation the interplay of visualization and analysis was still present, mainly through references the students made to their first visual images. Also, for one of the two students who used algebra, this was only a formal step which confirmed his visual investigation. Specifically, TW, when investigating the reflection function, sketched a 3D system and after drawing a vector he pointed at how each coordinate would be transformed. Writing the matrix of transformation served only as his formal solution. As TW explained,

TW I'm thinking geometrically... That certainly doesn't prove anything, but that's again the geometric way...

Int: Why doesn't it prove anything?

TW Mathematicians don't accept pictures for proof! But that's how I would look at it. ... Anyway, this is my picture to see it ... to convince myself I'd have to do some math and actually write down some numbers.

Discussion

This study adds to the literature on the use of visualization in problem solving by college students. Our results suggest that when students are presented with problems with inherent geometric elements, they may use visualization to solve the problems. In this respect, this study supports the theory suggested by Zazkis et al. (1996) in that visualization and analysis can be considered as two interacting modes of thinking which support each other in the development of students' understanding of mathematical concepts, rather than as a dichotomy.

Students faced the most difficulties when they had to make translations between the algebraic and visual representations. The students who were comfortable in translating between representations were able to advance further in their investigations. In fact, the critical point in their investigations was during their first step of analysis (A_1), when they attempted to reason about the image they constructed and to connect it with the algebraically described properties of linearity.

Although students in this study demonstrated that most are able to take advantage of both visual and analytic modes of thinking when doing an investigation in linear algebra, they also demonstrated substantial variation in their performance. Some students used approaches that led them to known "traps" of visual thinking such as not considering whether the features of their diagrams are generalizable or not, and to "traps" more specific to linear algebra, such as thinking that the conjectures that hold for sketches in 2 or 3-dimensional cases extend to higher dimensions.

Overall our results suggest that students do not always avoid the use of visualization in their problem solving. While other studies on students' understanding of advanced mathematics concepts suggested that the algebraic mode is more common, in our study students tended to use an interplay of visual and analytic-algebraic methods in their approaches to problems involving linear transformations. One possible reason for this tendency can be related to the tasks that were used; the reflection, projection and translation functions have inherently strong geometric representations and students may have prototypical images (Presmeg, 1992) of the actions of these functions in their minds with which they feel comfortable to work. Most studies which investigated students' use of visual tools when solving problems, used problems which had weaker visual representations. It may also be true that the interview setting "allowed" the use of visual investigations more so than a written test would. Students who are taking a written test may feel a time pressure that drives them to conventional methods (or precludes taking exploratory steps). Further, students might be more inclined to frame their work in conventional ways, e.g., doing any sketches on scrap paper and turning in only the "official" work. The reasons for the choice of visual approaches appear to vary. It remains an open question as to whether the greater use of visualization in this study was mostly due to a difference in the participants, in the tasks they were given, or in the observation procedures (interview vs. written test). However, our results suggest that the use of visualization may in fact play a more prominent role in students' development of understanding of linearity than was thought in the past and further study is needed to determine the extent of the use of visualization and the details of the cognitive processes involved.

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STUDENT UNDERSTANDING OF EQUILIBRIUM SOLUTION IN DIFFERENTIAL EQUATIONS

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We explore how students' concept images of equilibrium and solution interact in their conception of an equilibrium solution in the context of a single ordinary differential equation. By examining segments of semi-structured student interviews, we provide instances which highlight that many students' concept images of equilibrium do not appear to be a subset of their concept images of solution in this differential equations context. Finally, we highlight several curricular and cognitive issues which may influence students' understanding.

Building on the reform efforts in calculus and the pedagogical possibilities from technological advancement, there is a reform movement taking place in the teaching of differential equations at the collegiate level. To be effective in this reform, we need to better understand how students learn the important concepts in differential equations. There has been little work in differential equations learning research (Artigue, 1992; Solís, 1996; Rasmussen, 1998). This paper reports on our progress in addressing these general concerns.

Theoretical Framework

We approach this research project using a constructivist perspective following from work on student understanding of function (Vinner and Dreyfus, 1989; Monk, 1992; Sfard, 1992; Breidenbach, et. al., 1992; etc.) and derivative (Zandieh, 1997; Clark, et. al., 1997). We use Tall and Vinner's (1981) notion of *concept image*, "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152) to organize our data on student conceptions of an equilibrium, a solution and an equilibrium solution in the setting of single ordinary differential equations. The notion of a concept image is a useful, but broad way of characterizing students' understanding. To build a more specific skeletal structure within the concept image, we build on Rasmussen's (1998) research on students' understanding in differential equations.

In his work, Rasmussen frames students' difficulties in terms of interrelated social and individual cognitive dilemmas. The interrelated cognitive dilemmas include the "function as solution dilemma," students' tendency to over-generalize, and interference from informal or intuitive notions. We briefly highlight his discussion of these dilemmas. Unlike in algebra where solutions to equations are numbers, solutions to differential equations are functions. There has also been a great deal of research into students' difficulties with functions. Rasmussen refers to this general difficulty with the function concept along with the leap from solution as number to solution as function as the "function as solution dilemma." Rasmussen highlights students' tendency to over-generalize in several ways. One way students tended to over-generalize was by thinking there would be equilibrium solutions any time the differential equation was zero, even if the equation was not autonomous. Other over-generalizations he refers to arose when students approximated solutions using numerical methods. Rasmussen's discussion of the third dilemma, the interference with appropriate mathematical reasoning from informal notions, focuses on the concept of stability.

Methodology

The two researchers collected parallel interview data from two differential equations classes, one at a large public university and a second at a small liberal arts college. The former classroom was observed by the first author and the latter was taught by the second author. During the semester, the researchers conducted three individual task-based interviews with 13 students from the public university class and 10 students from the liberal arts college class. These interviews were audio-taped and transcribed, and students' written work was recorded.

The analysis for this paper focused on student responses to the following three interview segments taken from the larger set of interview data: (1) Given eight first order differential equations and four slope fields, students were asked to make appropriate matches. (2) Given a single slope field (it was for $y' = y + 1$) students were to draw representative solutions on the slope field. (3) Open ended questions such as "What is a differential equation?" or "What is a slope or direction field?" were asked prior to the above tasks, and "What does it mean to be a solution to a differential equation?" asked as students discussed solutions in several places in this interview including after drawing solution curves in the task above. The researchers analyzed these segments of the interviews in order to examine students' conceptions of equilibrium, solution and equilibrium solution.

Results

Mathematically an equilibrium for a single ordinary differential equation must be a solution to that equation, in particular a constant function.

However, students in the two classes did not always associate equilibrium with solutions, much less with functions. The interview protocol did not intend for the interviewer to use the word equilibrium or equilibrium solution at any time and this only rarely occurred. However, all but two of the students in the two classes did refer to this notion. The most popular references were "equilibrium," "equilibrium point" and "equilibrium solution." Other phrases, each used by two or fewer students, included using the word equilibrium followed by one of the following: line, curve, state, value, slopes, areas, portion of the field. We can see that phrases such as "equilibrium solution" or "equilibrium line" seem to indicate notions related to a function, whereas "equilibrium point" or "equilibrium value" seem to refer only to numeric values. Seven of the 23 students over-generalized the notion of equilibrium to include all values for which $dy/dt = 0$, even when those values were not part of an equilibrium solution. However, there was no consistency in students' use of the terms equilibrium solution, to refer to a function, versus equilibrium point, to refer to a non-function. Students who over-generalized the notion of equilibrium as stated above might refer to a line $t = -1$ as an equilibrium solution even though t was the independent variable. Other students who only used the term equilibrium solution appropriately (such as stating that $y = -1$ is an equilibrium solution to the equation $y' = y + 1$) might nonetheless call this solution ($y = -1$) an equilibrium point. These two examples are consistent with Rasmussen's (1998) findings.

Another example where some students' understanding of equilibrium seemed to interfere with a correct understanding of solutions (and in particular equilibrium solutions) was seen when students were asked to draw representative solution curves on the given direction field. Three of the 23 students did not draw in the equilibrium solution, $y(t) = -1$. One student was very explicit in his rejection of a possible solution at the equilibrium. After drawing solutions both above and below the equilibrium, the interviewer asked him if "there are any other solutions … that don't … follow the same pattern." The student responded that "there's nothing for here … for the equilibrium point." Later in the interview when they were discussing the long term behavior of the solutions, the interviewer asked the student what would happen if the initial condition were at "y is equal to -1." The student responded "there isn't any values for that … that's the equilibrium point … that slope is zero."

Student comments regarding a solution to a differential equation were different than their discussions of equilibrium. As noted above, students' notions of equilibrium often had nothing to do with their notions of solutions to differential equations, much less with functions. However, when asked directly about solutions, students' comments tended to follow the pattern: A solution is a "function" that "has a certain relationship to a differential equation". For "function", students did not necessarily use the word function, in fact few did, but students mentioned some notion that one would expect a student to have associated with function, i.e. that might be included in a concept image of function. Examples included referring to a solution itself as a curve, expression, equation or using phrases that described the solution as a dependent variable given in terms of the independent variable or as a process that allows one to determine the future population or position at a given time.

When students described the relationship between differential equations and solutions, i.e. "has a certain relationship to a differential equation", they often included an implicit notion of function or at least expression or equation. There were three principal relationships described by students in the two classes: A solution is something created from the differential equation by integrating or performing a solving technique; a solution is the "original function" that has been used some time in the past to create the differential equation; or the solution is "something that fits the equation" such that one can "take all the derivatives and plug it into the equation [and] get an equality."

Discussion

Mathematically, in the context of single ordinary differential equations, one would expect a student's concept image of equilibrium to be a subset of his or her concept image of solution. However, for many students in the two classes involved in this study, the notion of equilibrium (or for some students even equilibrium solution) was not a subset but an overlapping set. All but one student in the study showed some indication of thinking of equilibrium as a point or place or numeric value rather than as a function satisfying the differential equation. These indications included less troubling issues such as the use of the phrase "equilibrium point" and a focus on the equilibrium as a position on the slope field rather than a solution. However, they also included more troublesome issues such as the use of equilibrium solution to refer to *any* collection of values for which dy/dt = 0, or a refusal to draw an equilibrium solution when asked to draw all possible solution types on a slope field. Student difficulties with equilibrium solutions seemed to stem from conceptions connected to the notion of equilibrium that were not overlapping with their conceptions of solution. We present several curricular and cognitive issue which may influence students' understanding. By addressing these issues directly, we may help students refine their concept images of both equilibrium and solution in a differential equations context.

One curricular issue is the fact that students do not always need to think of the variable y in a differential equation dy/dt = f(t,y) as a solution or function to solve problems in differential equations. Rasmussen has also raised this issue (personal communication). For example, to complete the matching activity described above, one need only consider numeric values for t, y and dy/dt to be successful. Students may also simply see y and t as symbols they are manipulating even when they are solving the differential equation analytically. Although students need to be able to perform these simple actions when necessary, we need to help them construct an understanding that y is the solution to the differential equation and that it is a function.

Another curricular issue which arose as students worked on the slope field matching problem is the usefulness of having a name to refer to the collection of points where dy/dt is zero (or another constant). Most students focused on these points when examining the slope field, and without having such terms as isocline or nullcline (the latter not used in reference to single ordinary differential equations in either class involved in this study), students may reach for some reasonable word to use and find equilibrium. By providing students with an appropriate label for these important points, we may also help them focus their concept image of equilibrium in a differential equations context.

There are times in differential equations when the use of the term equilibrium as referring to a single point is appropriate. Many texts use the phase line as a short hand for capturing the information on a slope field for autonomous differential equations. In addition, when examining systems of two autonomous equations, mathematicians usually represent the equilibria of the system as single points on the phase plane. Thus a third curricular factor influencing students' concept images of equilibrium in the context of a single ordinary differential equation is the appropriate use of this word in other differential equations contexts. We must explicitly help students distinguish appropriate and inappropriate uses of this word.

A cognitive issue which may influence students' understanding of equilibrium solution is that it is a very special case of solution, in exactly the way that a constant function is a special case of the set of real valued functions. Such non-paradigmatic cases are sometimes considered by students as non-examples of a concept (Fischbein, 1987; Vinner and Dreyfus, 1989). Such issues must be dealt with explicitly in helping students shape their concept image of equilibrium in the context of single ordinary differential equations so that it is more consistent with mathematicians' concept definitions of equilibrium in this situation.

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ON THE REPRESENTATIONS USED IN THE LEARNING OF THE LINEAR ALGEBRA

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Observations on students' learning linear algebra show that they can solve systems of equations perform the Gaussian elimination on matrices, they can compute determinants and inverses matrix, they know some properties and relationships about lines and planes en R^2 y R^3. But they have difficulties for recognizing and exploiting structural relationships between systems of linear equations, their geometrical interpretation and their matrix representations; so that to understand the generalizations about these concepts and its properties.

The acquisition of knowledge is a continuous process, in which are creating and transforming cognitive structures so to arrive a formal definition of the concepts more or less stable. During this process, the individual must create representations of the concepts. These have a symbolic function, so they are used like substitutes of the concepts. The operations on the representations leave to construction of action's schemes, which are components of these cognitive structures.

This is a research project to study the role of the representations in the learning of concepts and processes in linear algebra, particularly, those related with the solution of systems of linear equations, the study of lines, planes and hyperplanes in spaces n-dimensional, and use of the matrices. The aim of this study is to understand how a student deals with the representations algebraic, geometric and matrix and which reasoning do during this processes.

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STUDENT´S UNSUITABLE CONCEPTUAL STRUCTURE OF THE CONCEPT OF FUNCTION, LIMIT AND CONTINUITY: A CASE STUDY

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In this study we show that an unsuitable internal conceptual structure of the concept of infinity and that of function induces an obstacle in the construction of the concepts of limit and continuity of functions. Based on the idea of cognitive obstacle. We designed a general project which has been running since 1992, related to the construction of the concepts of function, limit and continuity. The project involves high school mathematics teachers and students of that level.

In this investigation, analyzing the work of one teacher, we found that this teacher tended to fix some strategies focusing on algebraic methods to calculate limits, favoring informal approaches to communicate a mathematical idea. His errors show that the teacher has got an unsuitable conceptual structure when calculating the limits of his examples. The teacher's process of limit is reduced to a substitution. In an interview with one student, it is observed that the schema he has got is related to "a continuous curve and because of the irregularity of the graph, a notion of discontinuous function". His idea of continuous and discontinuous function is similar to that of Euler (1748).

We suggest that a new approach of teaching of the concept of infinity (potential and actual) is required and also that of the concept of function to develop in students a suitable schema, where the idea of function and the infinite could play a better role in the construction of other concepts like that of limit and continuity of functions.

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LIMITS, FRACTALS, AND PARADOXES OF INFINITY: MAKING SENSE OF INFINITE PROCESSES THROUGH COMPUTER-BASED EXPLORATIONS

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In this report I present some results from a study whose main concern was to investigate students' developing conceptions of infinity and infinite processes as mediated by a computer-based environment or microworld. The general findings of the study indicate that the environment and its tools shaped these students' understandings of the infinite in rich ways, allowing them to discriminate subtle process-oriented features of infinite processes; it also permitted the students to deal with the complexity of the infinite by assisting them in coordinating the different elements present. The corpus of data is based on case studies of 8 individuals, whose ages ranged form 14 to mid-thirties, interacting with the microworld as pairs of the same age group. Here, I focus in particular on how, through the explorations, the students participating in the study made sense of infinite processes by looking at the *behavior of the processes* and by *coordinating the multiple elements* involved in those processes.

The microworld provided a means for students to construct and explore different types of representations æ symbolic, graphical and numerical æ of infinite processes *via programming activities* (in this case using the Logo programming language). It was designed as an exploratory setting providing tools and means with which to explore infinite processes, particularly iterative/recursive processes such as sequences and series, and 'limit objects' like fractals. The central topic of the microworld was the convergence (and divergence) of infinite sequences and series, and limits, through the use of recursive geometric figures.

Infinite processes, as seen in the traditional teaching of calculus, are often presented mainly through symbolic 'static' representations with focus on the "final" state or limit. A further difficulty, as is noted by Nuñez-Errázuriz (1993), is that multiple sub-processes and elements are found in many infinite processes, which can give rise to confusions and paradoxes. Here I illustrate how through the computer explorations of our microworld, students focused on the *behavior* of the processes (instead of only looking at the limits), uncovering and analyzing multiple sub-elements of a process. This gave the students a means to investigate *the relationship between* *different elements present and find ways to coordinate these elements*, thus assisting them in resolving possible confusions and paradoxes.

I illustrate this with an example of how a pair of 17-year old students encountered and dealt with the Koch curve paradox: where an infinite perimeter seems to be formed by an infinite number of "zero-length" segments! It is a case where multiple elements or processes were found (the *number* of segments, and the *measure* of the segments) within an infinite process, leading to a paradoxical situation (analogous to those found in Zeno's paradoxes). These students solved the paradox by analyzing the behavior of each of the two competing processes.

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BASIC CONCEPTS IN LINEAR ALGEBRA AND STUDENTS' ABILITY TO ARTICULATE REPRESENTATIONS: AN EXPERIMENTAL STUDY WITH UNDERGRADUATES

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In this study we discuss student conceptions related to linear space theory basic concepts. The work is focused on the analysis of students' capacity to articulate between graphic and algebraic representation when working with vector problems. Using a conceptual map a sequence of fourteen vector elementary problems was constructed either in a graphic or algebraic representation. The problem sequence involves the following interrelated concepts: linear combination, vector dependence and independence, set spanning, span, basis, dimension, vector space and subspace. In order to conduct the study five undergraduate students of science and engineering areas were selected. These students had already taken the first course of linear algebra.

All the sessions were video recorded and transcribed. Several conclusions are attempted drawing from the present research. We can suggest that undergraduate students can define most of the basic space theory concepts. However, they have some difficulties to interconnect and to change representations of these concepts. These difficulties are more evident when dealing with infinity sets, e.g., span and subspace. We also observed that some students had problems to use the concepts of vector dependence and independence when solving related problems, even though they were able to manage these concepts in different registers. Additionally, we observed that all the students have a remarked tendency to identify the object vector with his representation graphic or numeric in two or three spatial dimensions. The implications of these work for teaching basic concepts of linear algebra are further discussed.

MATHEMATICAL REPRESENTATION OF MONEY INTEREST

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The focus of this study is on helping students more conceptually understand mathematical situations, so that I start primarily from a question about how to mathematically represent financial situations such as money interest. There are a lot of different choices of banking out there so that we need to pay some attention to decide which one is better.

In most common way the rate of interest is usually represented in algebraic way to a large extent in order to make it easier to get the answer by putting numbers into a formula. But we can improve students' conceptual understanding about confusing situations of money interest by geometrically representing and approaching to problems. Compound interest can be characterized by exponential function while simple interest by arithmetic function. When exploring geometric or arithmetic characteristics of money interest, one uses a computer software, *Function Probe*. With Function Probe, one can more deeply understand the mathematical structure of interest in terms of both algebraic and geometric representation. Effective annual yield then can be generated by using the same software, and its features will be discussed.

It does not seem to be easy to conceptually understand the various and complex features of money interest, compared to its importance in our life. The rate of interest can or cannot be considered as important; It depends on the amount of money or personal choices. But I think that the features of money interest can be more effectively understood by using Function Probe and approaching from geometric (functional) point of view to compounding interest and its limitations.

Algebraic Thinking

AN ARITHMETIC-BASED ENVIRONMENT TO DEVELOP PRE-ALGEBRAIC NOTIONS: A STUDY WITH 11-12 YEAR OLDS USING CALCULATORS

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Abstract: Results from an ongoing project using calculators are reported. The study was carried out as part of the regular mathematics course within a comprehensive school with 25 children who have not had any algebraic instruction. A schematized discussion of theoretical and methodological issues is held. The study draw promising results that suggest to exploit equation solving to enrich children's arithmetic background as well as to help them make sense of algebraic expressions. The strategies developed by children while working with the still unknown within a number-based environment seemed to allow them go forth and back from the general to the particular so as to developed general methods.

Background

That study is aimed at investigating the extent to which a specially designed arithmetic learning environment supported by the use of calculators may help young students develop pre-algebraic notions. The classroom activities were based on questions where arithmetic operations were not the target goal but the means through which students confront number challenges on the basis of their own reasoning. The research was carried out during twelve weeks, two 50-minute sessions per week, with a class of twenty-five 11-12 year olds. The research was carried out in a comprehensive school. The children who took part in the study have not had any previous algebra instruction and show a wide range of mathematical abilities.

The results suggest that the experience gained by the children through handling the "still unknown" within an arithmetic-based environment, allowed them to successfully make sense of a wide range of different type of equations and find solutions through numerical approaches. As well, children's work shows that they were able to go back and forth from the particular to the general, particularly when arguing for the generality and efficiency of their own methods.

The use of calculators played an important part in the study. Each student in the class was given a calculator in loan, this fact provided support both to design the tasks and set up the classroom environment. The role assigned to the calculator, the design of classroom activities will be informed while discussing the theoretical and methodological issues in the following sections. The research results will be presented in a separate section and wider discussed during the presentation.

Theoretical issues

The theoretical approach for this study is based on the assumption of conceiving arithmetic as a sign system, which serves us not only as a code to perform arithmetic operations, but also as a language that allows students to express their reasoning mathematically. Underlying such assumption is Bruner's research on language acquisition (1980, 1982, 1983, 1990). Roughly sketched, Bruner's research found that natural language is not only a sub-product of intellectual development (Piaget, 1985), nor a result of child's imitation of adult's language. Bruner's results show that natural language is taught. The amazing way in which children learn such a complex thing as the mother tongue so rapidly -and apparently effortlessly- is explained by the caretaker's intervention. Bruner puts forward that the adult artificially arranges the environment so that the children learn the natural language by using it, without any need to previously know grammar rules and definitions.

Those principles and results were recast in order to set up a classroom environment for the teaching and learning of pre-algebraic notions on the basis of using number facts to make children handle the still unknown. The teacher and the calculator play a central role in such an environment. The calculator played the role of a "mathematical world" which requires students to express their mathematical ideas trough producing arithmetic expressions under the mathematical rigor imposed by the calculator code. The teacher's role will be discussed in the next section

Methodological issues

Research questions and method. The main research question was to investigate what and how children learn when arithmetic calculation is fully supported by a calculator to face number-based challenges. A qualitative data analysis approach was used to observe and analyse children's achievements using a case-study methodology (Miles and Huberman, 1984).

Sources of data. Three main sources of data were used: (i) The students' written work throughout the fieldwork, (ii) A task-based individual interview carried out with six case-study children at the end of the field work. The case-study children were chosen according to their mathematical attainment during the study: a boy and girl of below average attainment, a boy and a girl of average attainment, and, a boy and girl of above average attainment; (iii) Notes taken by the teacher (researcher) during each classroom session.

Data gathering. The written work done by the students was reviewed, marked and recorded in special sheet formats after each session. Each individual interview was tape recorded and transcribed for further analysis. The notes taken by the researcher during the classroom sessions were used to refine observations both from children's written work and interviews.

Tasks. The tasks were prepared in a worksheet format; this allowed the teacher to pay attention to individual questions asked by the students. The worksheet's format also allowed children to work at their own pace. To do that, an envelop with 10 worksheets was given to each student at the beginning of the session, they were encouraged to do as much as they can in each classroom session, the only restriction for them was not to get to the end of the session having done nothing. If somebody could not work out a worksheet he/she must ask the teacher or some fellow pupil for help. When students appeared to be working fluently the teacher individually asked some questions to inquire for details about the strategies the students were using. The tasks used in the study can be exemplified by the following sample questions: (i) "Find out three numbers that add up 0.321" (this included common fractions and the other arithmetic basic operations); (ii) "Can you add 489 and 358 without using the plus key?", and (iii), "Can you find out the missing number in $4^x=28$?" (included also linear, quadratic and rational equations).

Individual interviews. The aims of the interview were to closely observe the strategies children used to face the most difficult mathematical assignments given during the study, and what pre-algebraic notions, if some, they developed while learning through exploring with numbers.

Results

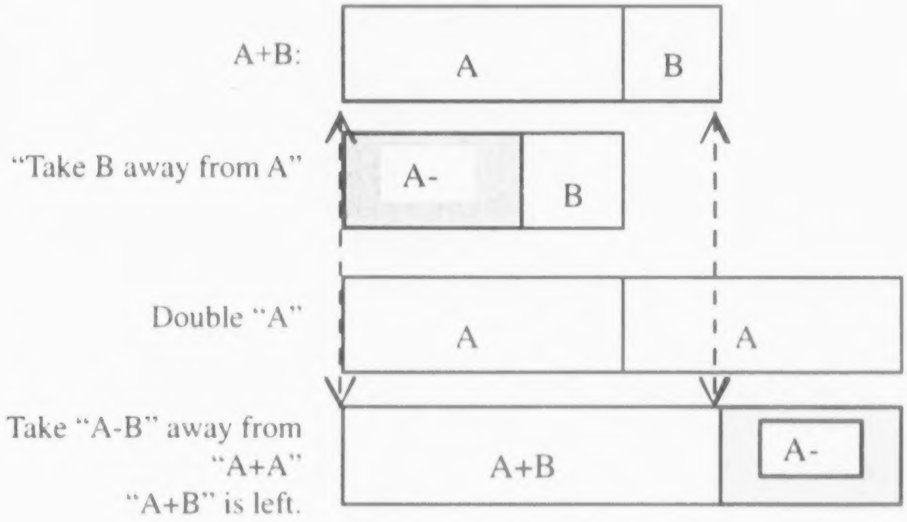
Number and operation senses. The children developed relevant notions about order with decimal numbers. That notions were explicitly showed when they were trying to find the missing number in equations of the type $4^x=28$. The calculators were set to work with six digits in the floating point format. Their first was to say that there is no value for x so that it fits that puzzle. After a while a pupil came to the front an using the view screen showed that $4^{2.5}=32$ and said, *"that's not the number we want but I'm getting closer*". After that they challenged themselves to come to the front and show a better solution. Finally they got a "solution" but a pupil set her calculator to work with 12 digits and found that this was no a solution. This led to a discussion on how the calculator works depending on the number of digits it is working on. Some of them found out which criteria the machine "uses" to adjust number results. It is worth noticing that at the beginning of the activity a good number of the children wanted to stop going on because they believed there were no numbers between 2.4 and 2.41 ($4^{2.4}=27.8576$ and $4^{2.41}=28.2446$). A pupil finished that discussion showing that 2.4=2.40 because $4^{2.4}=4^{2.40}$. Of course that number fact was not new for them, they were taught about this in earlier school levels, however all them seemed to be more comfortable trusting on the empirical argument provided by the calculator and enthusiastically go on working till they get an approximation with 28 followed by eleven zeros before a non zero decimal number or 27 followed by twelve nines. They could hardly get such a good approximations without handling well formed notions about order with decimal numbers and having made sense of the role played of arithmetic operations as means for validating their conjectures.

Equation solving. The twenty-five students were able to successfully solve the equations posed in the worksheets, most of them using trial and refining methods. Despite that equations as sophisticated as the ones described above were deliberately introduced with no explanation, the children were able to face the task quite naturally once some of their fellow pupils found that "the only thing you can do with this is to find out the missing number". Children's work provide empirical evidence for a number-based approach to equation solving, even in the case of equations involving the unknown on both sides of the equal sign. This suggests that if symbolic manipulation is not part of the learning scenario, neither the abrupt introduction of letters nor that kind of equations should be an obstacle for children as Rojano (1992) and Küchemann (1981) reported within paper and pencil environments. Silvia's explanation help us explain this fact (a 12 year old average attainment girl): "*It was easier than the activities before where we had always to find out missing numbers ... as when finding three numbers that add up 2/3 ... that was harder ... in the others* (equations) *the letter tell us that there is a missing number ... You just have to find it out ...Though it may be harder as well the calculator helps*".

The most common strategy used by the children was trial and refining. However, the above average students generally used inverse operations in the case of equations of the form ax+b=c. This strategy seemed to be supported by activities of the type "find two numbers that multiplied give 8.956". As we will see next these tasks helped students attempt general strategies. For example, Jorge explained: "*you choose one of the numbers, the one you wish, then divide the other number by the number you chose ... the result of this and the number you chose are the two numbers that multiplied give you the requested number ... That's it*". Some pupils showed more sophisticated strategies implying a sort of mental algebraic transformation. For instance, in order to solve an equation of the form a/x=b, two students were able to transform it into an equation of the form a=bx. Mariana explained how she solved the equation 1.267/q=100.412 as follows: "*I had to find a number which ... if you divide 1.267 by it the result is 100.412... I could not find it by guessing with several values, then I realized that if I knew what number q is, when multiplying it for 100.412 it must give 1.267 ... that is to say q¥100.412=1.267... This way I found that the number I wanted results from dividing 1.267 by 100.412, that gave me 0.0126180... I checked it out with the calculator and it was OK*".

Transition from the particular to the general. The teaching activities focused on number manipulation, and for the same thing they are centered on working with specific cases. However, the strategies used by the children showed a notorious trend toward generalization. This finding suggests that working within such arithmetic environment may be an important antecedent in the step from arithmetic to algebra. Our data both from individual interviews and classroom activity suggest that there was a decisive factor in encouraging children to develop strategies that have to do with generalization processes: the activities can be confronted using many different strategies and have no unique solution. During the classroom work they witnessed how their fellow pupils managed to solve the same problem situation using different strategies. These types of activity -and the support offered by the calculator- seemed to lead them to explore as many methods as possible without that drained their efforts. This seems to have favored that they frequently found more than one form to solve a problem. This fact helped the children break the fashion of unique answer and encouraged them to search for more general and more efficient strategies. Selected episodes of the children's work are presented next in order to provide evidence for this.

The "broken key" activity required the children to find out forms to perform arithmetic operations without using a supposedly "broken key". When trying how to add two numbers without using the plus key, Atahualpa (12 year-old above average student), found through numeric explorations the following strategy: "*First, double the bigger of the given numbers. Then take away of it the result of subtracting the smaller from the bigger*". His explanation can be described by means of the identity 2a-(a-b)=a+b. It is worth reminding that Atahualpa did not have any algebraic elements that allow him to express and manipulate those relationships by means of that identity. Though he was not able to clearly explain why that method works, the argument he used to defend his idea is interesting. He challenged his fellow pupils to show a couple of numbers that could not be added using his method. Such an argument resembles a general procedure to show the non-validity of a mathematical proposition. Some children made an effort to find an example against Atahualpa's strategy. When not being able to make it they intended to explain their validity and obtained the following diagram, where the rectangles A and B represent the numbers they want to add "without adding".



Final remarks

The results of this study show promissory findings that suggest that the calculator can be exploited to help children develop notions and numeric strategies that seems to be an important support in the transition from the arithmetic to the algebra. The notions and strategies developed by the children strongly encouraged the idea of using the approach to equation solving used in the present study both, to enrich children's arithmetic background, and help them in making sense of algebraic expressions through number-based exploration. On the basis provided by this stage of the study a replica will be held incorporating teachers who volunteer to take part and a larger students population.

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IDENTIFICATION OF DIFFICULTIES IN ADDITION AND SUBTRACTION OF INTEGERS IN THE NUMBER LINE

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Abstract: This article reports the results of a clinical study where the number line model was used. The main interest in modeling in this analysis is not its usefulness in teaching but as a resource, which exhibits the different cognitive tendencies of secondary school students when faced with the numerical domain of integers. Results reveal that position and interpolation of points in the number line become intricate when they are used in different numeric scales. These facts point out the difficulties students will face when using graphic representations with two or more dimensions. Likewise, the ignorance of zero as a number may obstruct the extension of the numerical domain of natural numbers to integers in school tasks.

Introduction. When reviewing research literature, we found authors such as Glaeser (1981), Freudenthal (1983), Janvier (1985), Bell (1986), Peled (1991) and Gallardo (1994b) who express the relevance of the use of teaching models in the addition and subtraction of integers. This work appeals to the number line model as a research tool for the analysis of conceptual difficulties when operating with integers, showed by second grade students in secondary school. This study is leaned on the three conceptions of line, described by Ernest (1985): 1) as an aid to order integers; 2) as a model to perform addition and subtraction operations within this numeric set; and 3) as an essential inclusion of number line within mathematics curriculum.

The study. Two questionnaires were applied to 38 second grade students in secondary school. The first was a diagnostic one applied at the beginning of the study, and the second at the end of the study, after students received an instruction phase with the number line model. From the information requested from questionnaires, seven students were selected to participate in individual clinical interviews, which were video-tape recorded and analyzed. These students were those who obtained the maximum and the minimum of successful answers in the posed questions. The blocks of items in the interview dealt with the following topics:

1. Position and interpolation of integers in the number line.
2. Link between an integer and a verbally described situation.
3. Operations with integers at a syntactic level and its representations on the number line.

4. Resolution of word problems.

In this article we report results on clinical interviews regarding the first topic. For the analysis of interviews, we appeal to Filloy's theoretical proposal (1991). This author claims that students show cognitive tendencies when new mathematics concepts and operations appear. These tendencies arise in teaching situations in which concrete models are used. With regard to the numerical domain of integers, the Chinese model as used by Gallardo (1994a) reveals the existence of cognitive tendencies, which exhibit the various levels of acceptance of negative numbers by secondary school students. We can say that such cognitive tendencies may appear in one individual or even in the whole population of students that is being analyzed. In fact, these cognitive phenomena are different depending on the concrete model used in the teaching of concepts. Several cognitive tendencies, different from those expressed with the use of the Chinese model, arise when the number line is operated on. These cognitive tendencies are the following:

Creation of intermediate meanings. Codes, whose meanings proceed from personal syntactic rules, are introduced within these processes. Some expressions of the use of these codes emerge in the following situations: Record of signs[1] ; confusing the number with its symmetric; interpolation of numbers; assigning value to points in number line, and zero as middle point.

Centering of readings. Students mainly focus on graphic representations in number line, which does not allow them to solve the problem posed. This happens in the following cases: Different types of counting, identification of scale, and different scales.

Inhibitory mechanisms. The lack of meanings impedes to solve the task. The facts are: Absence of minus sign; loss of zero; do not assign a position to zero on the number line, and lack of spaces when crossing by the position of zero.

Evidences of the study

Related to position and interpolation of integers in the number line, some examples of the aforementioned cognitive tendencies are the following:

1. Record of signs (the way of writing the sign to indicate a positive or negative number). Students write the sign at the right upper side of number (b , a^{+}).

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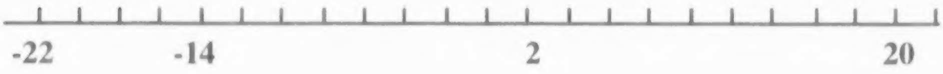
[1] Diverse cognitive tendencies exhibited by students are analyzed in the following section of this paper.

2. Absence of minus sign (forgetfulness of minus sign while mentioning it or writing it). Two cases:

a) Students leave out the minus sign of the number searched when doing a counting of numbers positioned at the left side of zero. For example:

Item: Position of points on the number line; scale goes two by two:

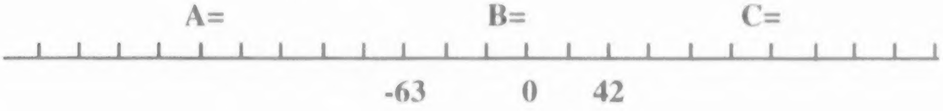
a) +15 b)–17 c) 0



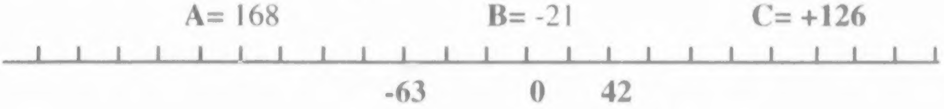
With respect to b), the student affirms: *I started counting two, four, six and here* (she points out the point corresponding to –14) *would be 14*. She adds, ... *So, there* (she indicates the point that is one place at the left side of -14) *goes 16, and between 16 and the following, which is 18, goes 17, doesn't it?*

b) Students forget to write the minus sign to the negative number very distant from the origin, even when there are negative numbers close to it. For example:

Item: Given the zero, assign values to points in the number line. Scales goes 21 by 21:



Student writes the following values:

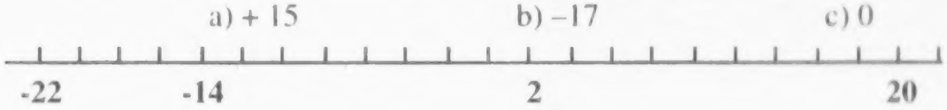


Note that the value given to A has no sign.

3. Use of symmetric (Position of values on number line, using the symmetric of a number). Students confuse a positive number with its symmetric.

4. Interpolation of numbers (Position of a number between two other given numbers). Two cases: a) To place the searched number in the closest given value (scale goes two by two). For example:

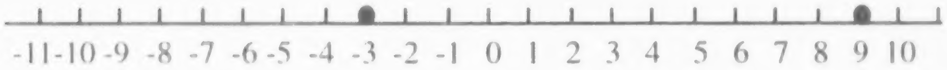
Item: Find the points on the number line. Scale goes two by two:



I: *Now, where would you place the negative number 17?*

S: *Here* (the student puts a mark and writes –17 on the point at the left side of 14).

b) When counting integers amongst two other numbers, an integer is added at both extremes. For instance:

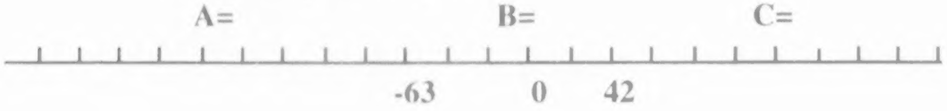


Item: How many integers are there between –3 and 9?

S: *From here to there* (he puts some marks on number –3 and 9 and counts). He adds: *There are twelve.*

5. *Assigning values to points in the number line* (Used algorithms to assign a number to the points in the number line). Three cases; scale goes 21 by 21: a) Additive strategy. Answers obtained do not correspond to those numbers mentally added. b) Students obtain two different results for the same operation when multiplication is used in a vertical way; and combine both additive and multiplicative strategies within the same procedure: instead of multiplying by the scale, students multiply the value obtained from a previous operation by the number of spaces. For example:

Item: Given the zero, find the values for the points in the number line. Scale goes 21 by 21:

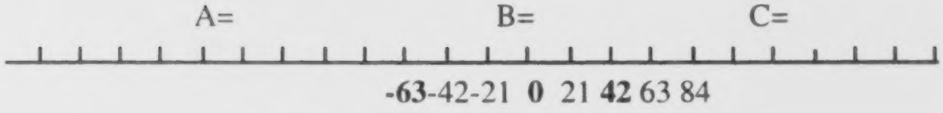


S: *It is 84.*

I: *It is 84, ok.*

S: *And then, 84 …* (the student multiplies 84 by 5 and obtains 420 as result; she multiplies again 84 by 5 and obtains 400 as result).

$$\begin{array}{r} 63 \\ -42 \\ \hline 21 \end{array} \qquad \begin{array}{r} 21 \\ \times 4 \\ \hline 84 \end{array} \qquad \begin{array}{r} 84 \\ \times 5 \\ \hline 420 \end{array} \qquad \begin{array}{r} 84 \\ \times 5 \\ \hline 400 \end{array}$$



c) A tendency to change the given values in the number line rather than modifying the scale found. For instance,

Item: Given the zero, find the values for the points in the number line. Scale goes 21 by 21:

A = **B =** **C =**

I: *Which is the value for A?*

S: One hundred and forty

I: Which is the value for B?

S: Twenty.

I: Twenty, and for C?

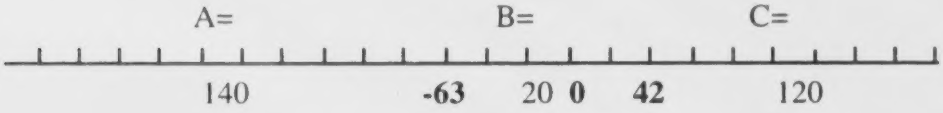
S: One hundred and twenty,

I: One hundred and twenty, ok ... How did you find these numbers?

S: Because counting goes 20 by 20, doesn't it?

I: Oh, 20 by 20.

S: *But here* (she refers to number 42 on the number line) *it seems that it is exceed two centimeters, it is forty two.*



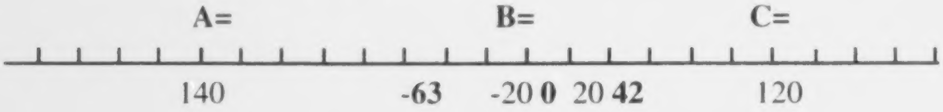
6. *Types of counting* (Different ways of counting to give a value to points in the number line). Two cases, scale goes one by one. To place zero: a) given two values in the number line, students carry out an ascendant counting. They consider a given negative number as origin and start counting from left to right. b) Students make an ascendant counting of integers as natural numbers on both sides of zero. Two cases, scale goes two by two. To find the points in the

number line. c) Students go back to the origin each time the counting is interrupted. d) They make an ascendant counting without considering the order and without giving an unique value to each point in the number line. This situation occurs each time that students operate with numbers of two digits.

7. *Identification of scales* (Students find the pattern of segments forming the number line). Two cases. Scale goes 21 by 21). a) Students use a multiple of 10 as a pattern (20). To find the pattern, students subtract absolute values of integers given in the number line.

8. *The use of different scale in the same number line.* (Construction of different patterns for segments in the number line). Scales goes 21 by 21. a) Students utilize a different scale for positive numbers and another for negative numbers. b) They use different scales in consecutive segment of the number line. Students do not imagine equidistance amongst segments. For instance:

Item: Given the zero, assign values to the points in the number line. Scale goes 21 by 21:



The student gives the value of –20 for point B.

I: *Now, How many points are there from here to* here (interviewer points at to segment of number line that goes from 20 to 42)?

S: Twenty two.

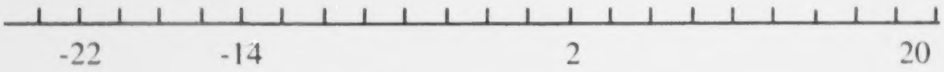
I: Twenty two but, could that be possible in the number line?

S: This is , if the…, if the line crosses by here, it could be, couldn't it?

9. *Denotations of Zero* present in three situations: Loss of zero as origin (students do not assign a particular point in the number line for zero) Two cases. Scale goes two by two. a) Students consider the first positive number given as point of symmetry. This difficulty emerges in the following item:

Item: Find the points in the number line. Scales goes 2 by 2.

a) +15 b) –17 c) +11 d) -4 e) 0 f) +11



Student considers point 2 as if it were the zero.

b) Students modify the scale to avoid give a point to zero in the number line. This occurs in the aforementioned item, in which the first positive number given is considered as point of symmetry. c) When scale goes 21 by 21, student consider as origin the given number closer to the value searched: *Zero as middle point* (Students see the number line as a finite segment and locate zero at the middle). For a scale one by one, students locate the zero by counting segments of the number line as well as those marks that indicate the points in such number line. *Crossing by the Zero*. (Students showed difficulties to assign negative values, after they have obtained positive values). In a scale 21 by 21, students failed in the counting at the side of negative numbers, leaving out spaces or considering two segments as one.

Conclusions of this study

The most relevant conclusions of the study are the following:

- One of initial problems with integers that students face is the way they write them, because students write the signs at the right upper side of number, and this place corresponds to the valence of chemical elements. This fact reveals a lack of clarity between arithmetical and chemical languages (Castillo & Gallardo, 1996).
- Omission of minus sign when writing or mentioning a negative number may indicate that a concrete referent similar to that of natural numbers is missing, such as counting.
- Loss of zero as origin, the position of zero as middle point and mistaken situations when crossing by zero in the number line, make noticeable the difficulties that students have in considering zero as a number. This fact may avoid the extension of the numerical domain from natural numbers to integers.
- The use of different scales makes complex the position and interpolation of points in the number line. This fact is an evident precedent about those difficulties that students will face when operating with graphic representations with two or more dimensions.

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TALKING MATHEMATICS IN SMALL GROUPS: COMPARING STUDENT TALK AND STUDENT TALK IN AN ALGEBRA I CLASS

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This study examines and compares transcripts of audio tapes of teacher-student dialogue with those of student-to-student discussion as students work in small groups on a variety of mathematics problems in an Algebra I class. Topically related sets were compared and analyzed and discourse patterns were identified. Student turns were categorized as using language *About*, *In*, *With*, or *Beyond* mathematics as well as for the purpose of the language: to question, to describe work in progress, to explain, to evaluate, or to affirm or acknowledge. Comparison across the two sets of transcripts provides evidence of the importance of providing opportunities for student-to-student talk in mathematics classrooms.

The national reform effort in mathematics education calls for a constructivist approach to teaching mathematics. Recently developed curricula, including the program used in this study, are based on the theory that students construct their own understanding of mathematics and that teachers and materials can be prepared to better serve students in helping them to develop this understanding. These materials generally include problems designed for small group work so students can talk about their work as they are doing it.

Recent articles have urged researchers to study classrooms in which teachers are attempting to help students develop their own understanding through the social negotiation of meaning as they work together and talk with each other and the teacher (Cobb, 1996, Lerner 1996). However, very little research has been done in relation to language use in mathematics classrooms at any level, let alone high school. Very few researchers in mathematics education work as regular teachers in classrooms and report on their work with diverse groups of children in that setting. Lampert (1985), Ball (1996), Parker (1993) and Romagno (1994) are exceptions. Lampert at the middle school and Ball in a third grade classroom have focused their work on generating thoughtful whole group discussions. Parker at the fifth grade level and Romagno in a ninth grade basic mathematics class, team taught with the regular teacher and focused on developing alternative curricula and teaching methods to engage students in thinking more deeply about mathematics. There are not long term studies of high school classrooms with the researcher as teacher, nor has any study besides Brenner's (1995), which focused Spanish speakers in an Algebra 1 class, dealt with the discourse that occurs between students and teacher and between students as they work in small groups in a mathematics classroom.

The purpose of this study was to examine the discourse in two different small group contexts in a diverse Algebra 1 class in order to learn about the ways in which students working in small groups use language as they complete mathematics problems. The central question in this research was one of six addressed in a year-long study. How does student talk within their small groups differ from their talk with the teacher?

Because the question addressed in this paper was one of six questions that were about what happens in a classroom where students work in small groups, I used an ethnographic approach. I arranged to teach an Algebra 1 class from October to June, in an inner-city school with a mixed population of Asian, African-American, Latino, and White students. The class was using the College Preparatory Mathematics (CPM) Algebra 1 materials (CPM, 1992), a replacement course for Algebra 1, designed to enhance students' problem solving, reasoning, and communication skills and to use the other areas of mathematics, geometry, graphing and functions, probability and statistics, as a basis for understanding algebra.

As the teacher, fully responsible for all aspects of the class including attendance, grading, and talking with parents, I could be an insider, but because I was only teaching one class I would still be an outsider in the some important respects. While I did not live through a regular teacher's long and demanding five-period teaching day, my other full time responsibilities did not allow me any more preparation time for teaching than a regular teacher would have for such a class, so it was easy to stay with my plan of using the materials as recommended, not supplementing or changing them significantly. Using the materials as they were written was important to my goal of working in a classroom situation that might be considered close to a normal class where I would experience many of the same pressures and dilemmas as a regular teacher (Ball and Lampert).

I gathered data in the four categories described by Eisenhart (1988): participant observation, interviews, collection of artifacts, and reflections including "emergent interpretations, insights, feelings, and the reactive effects that occur as the work proceeds." (p.106) To supplement my observation as a participant, I used an audio-tape recorder during class in two ways. To record teacher-student exchanges I wore a small tape recorder and an external microphone as I moved around the classroom from group to group. I also recorded groups of students by placing a tape recorder with an external microphone in the center of a group of three to four students on one of the student's desks. Artifacts gathered included written work and records: all the assignments the students did throughout the year, my gradebook records, their mathematics grades in previous and following courses, and information from the counseling office. The analysis in relation to this question was based on the transcriptions of 28 classroom tapes. The transcripts include discourse markings to show pitch, pauses, volume, breathing, drawn out words, openings, interruptions, and splices. A discussion of the analysis and findings follows.

Initially I separated the dialogue into two categories, on-task or off-task. Off-task included all school or classroom business, any social conversation, and any disruptions from inside or outside the classroom. On-task meant the exchange was related to the mathematics we were working on, and on-task turns were categorized as *About, In, With,* or *Beyond*, based on an idea from Brenner's work (1995). The *About* category includes language about doing the problems that does not use any mathematical vocabulary that relates to the problem. Talk is entirely in standard or non-standard English (or Spanish). Examples include questions such as, "How do we start number six?" or "Could you check to see if this is right?" *In* mathematics means using words from what Brenner describes as the mathematical register to directly describe an operation or procedure. *With* mathematics means the student uses a combination of mathematical vocabulary and standard language to consider or decide on a problem solving strategy, to develop an algebraic representation of a problem presented visually or in standard language, to make a generalization about a problem, to connect or relate two previously separate mathematical ideas, to discuss a group of problems, to relate an example to another example, to generalize an explanation of how to carry out a set of procedures. *Beyond* is again mostly standard or not so standard language used to describe a connection between ideas or to generalize or to ask a "why" question. After coding each student turn in the teacher-student and student-group transcripts, I looked within each category to identify questions, descriptions, reading or repetitions, explanations, and other communication purposes for using language.

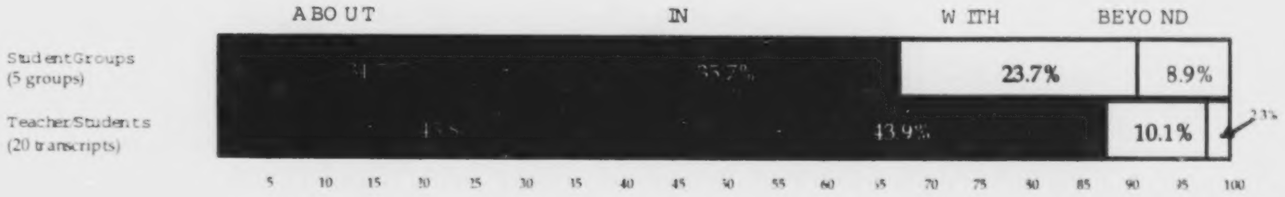
Another part of the analysis included identifying topically related sets, TRS's (Cazden, 1988) in the on-task talk and looking for discourse patterns in those sets. I did this first for the teacher-student transcripts and then for the student groups.

That there would be differences between the teacher-student talk and the student-group talk is not surprising. What is interesting is what an analysis of those differences showed. Unfortunately the brevity of this report precludes including samples of excerpts that could serve to clarify the descriptions and that would represent the evidence for some of the findings.

Topically related sets (TRS's) were relatively easy to identify in the transcripts of the small group teacher-student dialogue because the beginnings were usually marked by the arrival of the teacher and the endings by her departure. For teacher-student exchanges there was a very clear question-answer-verify (QAV) pattern, that was reminiscent of the Inquire-Respond-Evaluate (IRE) pattern described by Mehan (1979), in the teacher student dialogue. However with the small groups, control of the dialogue started with the student and shifted back and forth between student and teacher depending on who was asking the question. Differences in who controlled the TRS varied in relation to the strength, confidence, and current state of applicable knowledge of the student and had little apparent relation to the type of problem. Girls initiated a greater number of TRS's and took a significantly greater number of turns per TRS.

While the basis for the teacher-student TRS was the particular question the student had, topically related sets for student groups were generally based on the problems, starting when the lead student started a new problem and not ending until the last student had his or her say. Student-to-student TRS's often overlapped each other. When the problems were routine exercises or reasonably straightforward applications, groups would settle into the pattern: read the problem, continue working, check results, verify, and move on. On developmental problems, problems that required explanation, and investigations the discourse did not follow a pattern. What became clear in the group tapes was that each individual had to talk through questions and solutions for him or herself. If a part of a group moved on to another question before one student was finished, that student would chime in later with a declaration of his or her solution or a request for help or verification. In listening to the tapes and re-reading the transcripts I came to realize that what I had at first thought of as a frustratingly inefficient and repetitive process was actually providing an opportunity for each student to articulate what he or she understood.

When I tallied the categories of language use *About*, *In*, *With*, and *Beyond* for students on the teacher-student transcripts and compared with student-group transcripts I found that within the groups students used language *In*, *With*, and *Beyond* mathematics a greater percentage of their on-task time than they did with the teacher. Much more of their talk with the teacher was *About* mathematics. Figure 1, which is based on 2177 student turns in the teacher-student transcripts and 697 turns in the student-student transcripts shows the percentages over all of the transcripts.



ALL	Student Groups		Teacher/Students	
ABOUT	221	31.7	953	43.8
IN	249	35.7	955	43.9
WITH	165	23.7	220	10.1
BEYOND	62	8.9	49	2.3
total	597	100.0	2177	100.1

■ ABOUT
■ IN
□ WITH
□ BEYOND

Overall Comparison of Students' Use of Language with Each Other and with the Teacher

Figure 1

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The type of problem students were working on had a direct effect on use of language *With* mathematics. For example, when students worked on word problems that required them to write equations they would be working *With* mathematics in order to represent the problem. When students worked on summary problems there would be more talk *Beyond* mathematics because the questions required that they give general descriptions of groups of problems. Because I was worried that the balance of types of problems represented in the two complete sets of transcripts might not be comparable, I looked at the comparison of the teacher-student and student-group transcripts from the same two days, January 3 and 5 so I could compare use of language when the students were working on the same problems. The difference is even greater for a same day comparison. The box and whisker plot in figure 2 represents the percentages for the teacher-student transcripts and shows the student-group percentages for all groups, for the Jan. 3 and 5 groups combined, and for the Jan. 3 group alone. The three group percentages would be outliers in relation to the teacher-student percentages.

In both settings students' main use of language was to describe the work they were doing. When the teacher was present, students asked more questions, but when students asked questions within their groups, a higher percentage of those questions were asked using the language of mathematics, *In* or *With* mathematics. Students did more explaining when the teacher was not present, and they made more statements evaluating both their own understanding and the work they were doing.

In relation to another question in the larger study I found that stronger students responded to the teacher in sentences more often than weaker students who more often answered with phrases. When they responded in sentences the stronger students were using opportunities to articulate mathematical relationships that the weaker students were missing with their one to three word answers. I then re-classified student turns in the teacher-student and student group transcripts as statements, phrases, or questions and found that students (in general, weaker, stronger, and in between) articulated their thinking in statements almost twice as often in their groups as with the teacher, where they were more inclined to use phrases.

This study provides evidence that work in small groups can provide opportunities for students to verbalize their mathematical thinking, and that given that opportunity the students in the groups that were taped took advantage of it. The finding that students used a higher percentage of mathematical language with other students in their groups and spoke more often in sentences with each other than with the teacher validates the importance of providing the opportunity for small group work in mathematics classes. The results of this study have implications for teachers'

Box and Whiskers for Teacher/Student Transcripts
IN WITH BEYOND

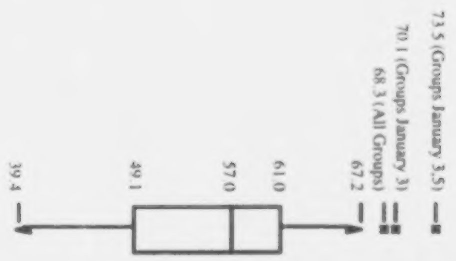

Figure 2

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expectations in relation to working with small groups and for their roles in facilitating small-group work.

Clearly further work in examining the dynamics of small group work and talk is needed. This effort is an early step toward seeking a better understanding of some aspects of teaching and learning mathematics through work with small groups.

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RE-THINKING SLOPE FROM QUANTITATIVE AND PHENOMENOLOGICAL PERSPECTIVES[1]

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The purpose of this study was to investigate a foundational aspect of understanding slope that has received little research attention, namely the *creation* (as opposed to the *use*) of ratios as measures of attributes (like motion, steepness, or sweetness). Nine high school students enrolled in an Algebra 1 course participated in the study. Not surprisingly, the results of the study showed that creating a ratio as a measure is very difficult. Of greater interest is the identification of two major sources of difficulty that have their roots in quantitative reasoning and cannot be explained solely by appealing to additive thinking: 1) conflating various attributes that could be measured in a given situation, and 2) not understanding how various quantities in the situation that affect the attribute of interest. Furthermore, everyday experience with rates like speed did not help in formation of ratio, but rather presented a possibly misleading image of understanding.

Background and Purpose

Slope is not an isolated concept but is part of the multiplicative conceptual field (Vergnaud, 1983). Consequently, slope lives at the intersection of three major research areas: 1) functions and graphs; 2) proportional thinking (including ratio and rate); and 3) algebraic thinking. Researchers studying functions have begun to move from a "representational approach" (i.e., an examination of the set of connections that one makes across representations, typically tables, equations, and graphs) to a "quantitative approach" (i.e., an exploration of the sense one makes of the quantities in situations to which one applies numbers and calculations)[2].

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[2] We follow Thompson's (1994) definition of quantity as a conceptual entity that involves an object (situation or event), an attribute or quality of the object that is conceived in such a way that it can be measured directly or indirectly, an appropriate unit, and a process of assigning a numerical value to the quality.

Our work builds on studies that explore connections between situations and graphs (Nemirovsky, Kaput, & Roschelle, 1998), students' conceptions of rate (Thompson, 1994), and rates of change as emerging from an understanding of covariation (Confrey & Smith, 1995; Saldanha & Thompson, 1998).

If we treat slope as a measure of the rate of change of covarying quantities (as opposed to the ubiquitous treatment in schools of slope as a counting technique used to determine the "tilt" of a straight line in a coordinate grid system), then students' construction of rates and ratios are of critical importance. However, identifying a ratio as the appropriate measure of a given attribute (like motion, steepness, or sweetness) appears to be a complex process that has received little research attention (with the notable exception of Simon and Blume's 1994 study with preservice teachers). Simon and Blume named this process "creating a ratio-as-measure." Our study is also framed by a phenomenological perspective, which broadens the psychological approach by taking into account how people experience quantities in the world and how those experiences are represented in everyday language (Nemirovsky, Tierney, & Wright, 1998).

In everyday experience, language encodes some intensive quantities[3] as entities, (e.g., speed or gas efficiency). Other intensive quantities are named by referring to the two quantities or "parts" that comprise the ratio, (e.g., candies per bag, or girls to boys). Note that in these latter two examples, attempts to name the intensive quantities in a similar manner seem awkward or contrived, (e.g., "candy density" or "girlness"). Hereafter these two linguistic categories for intensive quantities will be referred to as "entity" and "parts" categories respectively.

One hypothesis is that students are more likely to have greater difficulty creating a ratio-as-measure in the entity case because the natural language does not make explicit the derived or composed nature of the intensive quantity. Students may also feel that they experience these quantities directly and thus, should measure them directly, e.g., with an inappropriate extensive measure, like the slant height of a ramp as a measure of slope (as a 10th grader in our pilot study did). An alternate hypothesis can be developed from the research of Kaput & West (1994), who argue that our everyday cultural experience provides frequent exposure to particular instances of certain rates (like price), and in the case of speed, "the biology of perception may enable us to encode this as a rate without needing to build on particular instances[4]. In both price and speed, the language also helps by providing a ready word that encapsulates the ratio as an entity (p.241)". Hence, students may be more likely to create appropriate ratio or rate measures in the entity case.

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[3] Schwarz (1988), distinguishes extensive quantities (like distance or age, which can be counted or measured) from intensive quantities (like speed or gas efficiency, which are composed as a ratio).

Our goal was to explore these two hypotheses, examine the nature of the measures students created, and identify difficulties other than the well-documented case of additive thinking.

Methods

Nine students from a large urban high school were chosen from an Algebra 1 class just prior to slope instruction. The students were selected by the teacher to represent a range of performance levels (as measured by letter grades) in his class of about 25 students. Each student was interviewed individually for about 45 minutes, and the interviews were videotaped. The interviews were semi-structured, and the interviewer's responses varied based on the students' work. Tasks involving intensive quantities from both entity (see the speed and ramp tasks in Figure 1 below) and parts categories (see nutrition bar task) were utilized. Furthermore, the speed and ramp tasks differ across another dimension. Students are likely to have experienced both motion and the quantification of motion (namely speed) but not the quantification of steepness (other than as an angle). A coding scheme was developed to determine whether or not a ratio had been formed. Revised categories emerged as a result of cross-task analysis.

1.**Speed task**. How can you determine how fast the mouse is running? What about a train?

2. **Ramp Task**. Suppose you're a manufacturer or builder of wheelchair ramps [a drawing of a ramp with a platform was provided], and you want to put them in a catalog. You want to be able to communicate to people how steep the ramp is so they know what kind of ramp they will be buying. How would you go about determining the steepness of any ramp?

3. **Nutrition Bar task**. Some people eat nutrition bars [5 bars were presented]. Suppose you work for a fitness magazine and you want to review every nutrition bar on the market. You think it

Figure 1. Three "ratio-as-measure" tasks

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[4] Kaput & West (1994) and Thompson (1994) make similar distinctions between ratio and rate. Rate is a reflectively abstracted constant ratio, and ratio is a static instance of a more general rate.

Results and Discussion

Creating ratios-as-measures was difficult for all subjects. Only one student even came close to forming a ratio and then only for the bar task. The finding from the pilot study of treating intensive quantities extensively did occur but did not arise as an organizing theme to explain students' behavior across tasks. Other issues were more prevalent and critical. Interesting differences emerged across various types of quantities, but they were not the ones we expected. Three major findings follow:

1. Focusing on the attribute to be measured was a problem with tasks from the entity category. In both ramp and speed tasks, there was evidence that students conflated other attributes (e.g., "difficulty in climbing" in the ramp task) with the particular attribute named in the problem statement (e.g., steepness). Very often the subject and the interviewer had to negotiate the meaning of the attribute before they could proceed with the interview. However, images of other attributes continued to play an important role in some students' thinking. For example, in the speed task three students identified the length of the mouse and the distance across the room as the two quantities used to determine how fast a mouse runs. One student's thinking provides insight into what all three might have been thinking. Iris's work suggests that she may have conflated speed with another quantitative relationship in the situation, namely the number of steps one takes to maintain a particular speed. Iris mentioned that mice "scatter, they run fast... cause I mean they are smaller, and they run fast." The notion that small things have to be faster to cover the same distance in the same time may be related to the common experience of children taking more paces in order to keep up with a parent. A student may therefore consider the child to move faster than the adult. Iris talked about a small mouse needing to move three times as fast as someone who walked across the room in 7 or 8 paces. It is possible that her image was one of using body-length as a fundamental unit of speed. The small mouse can be placed end-to-end three times to fit in one pace, which could explain her conclusion that the mouse traveled three times as fast.

2. Identifying which quantities affect the steepness of the ramp presents a major conceptual difficulty for students. A more striking pattern of task differences emerged when we compared students' responses to the ramp task versus the other two tasks. In both nutrition bar and speed tasks, 7 of the 9 students eventually settled on two appropriate quantities (which could be used to comprise a ratio). In contrast, only one student was able to tease out which quantities affected steepness. In the speed task, familiarity with miles per hour seemed to aid in the selection of time and distance. Similarly, in the bar task, the explicit reference to two goals—increasing protein and minimizing whatever makes one gain weight—seemed to help students select two quantities (most typically protein and fat). In contrast, in the ramp task, students had to decide which of many possible quantities affected the steepness of the ramp. Three major sources of difficulty emerged.

First, nearly half of the students were uncertain about the role of the platform. As one student put it, he wasn't sure if increasing or decreasing the length of the platform would change the steepness of the ramp.

Second, most students had a much harder time understanding the effect of changing the length of the base (minus the length of the platform) on steepness than the effect of changing the height. For example, Eugene argued that if you make the length shorter then you will make the ramp less steep but if you make it longer than it's steeper, because "longer makes it steeper." Perhaps he was thinking that increases in one quantity always result in increases in another. Iris's work is interesting because, unlike Eugene, she came to see, over the course of the interview, that changing the length affected the steepness. Yet, despite this insight, she continued to grant greater status to height; thus, illustrating yet another quantitative complexity of creating ratios-as-measure. At the beginning of the task, Iris equated height with steepness and reported 5 ft as her measure of the steepness of the ramp. Furthermore, she stated that it was impossible to make the ramp steeper other than by increasing the height. Then the interviewer posed a problem that appeared to put Iris in conflict. She was asked to create a second ramp, one that had the same steepness as the first ramp but had different dimensions. Iris realized that changing the height would make it steeper, so she thought she needed to keep the height the same. As a result, she considered other quantities in an effort to change some dimension in the ramp. After a great deal of thought, she claimed that making the base longer did not keep the steepness the same (as she evidently had expected) but instead would make it easier for someone to wheel up the ramp and thus would be less steep. Later on, the interviewer once again asked Iris to consider the effect of changing the length on the steepness. Iris claimed that making the base longer would make it a "little less steep". However, she also claimed that it wouldn't make that much of a difference, but that increasing the height would have a "drastic" effect.

Finally, conceiving of the ramp as a change situation as opposed to a static situation was difficult for some students. For example, Jeanne could create images of increasing the height of the ramp (and then lengthening the slant height to "meet" the height in order to reflect on the steepness of the ramp). However, when she was asked to consider shortening the length of the base, her image was simply one of "cutting off" the ramp (by drawing a line perpendicular to the base, about halfway across the ramp), thus leaving the steepness unchanged. The interviewer tried to help Jeanne think of stretching or shrinking the length of the base as Jeanne had done with the height, but the interview was unable to nudge Jeanne's image of truncating.

3. Familiarity with speed did not help with the formation of ratios. Seven of the nine subjects mentioned "miles per hour" in the speed task. At first, we attributed ratio formation to these students, primarily because of the use of the term *per*. However, when questioned further, students appeared to be using miles per hour in one of two ways. For some, it was simply a label. For example, Jody stated that a mouse could travel 5 inches in 2 minutes. Then she calculated 2 ÷ 5 to obtain 0.4 and said that the mouse's speed was 0.4 mph. Similarly, Ramona used a 12 inch mouse and a 54 foot room to obtain the measure of/speed as 12 x 54 (which she thought was 58 since 1 x 5 = 5 and 2 x 4 = 8). She reported the result as 58 mph despite the fact that she had not even considered time in her calculations. Furthermore, four students suggested using either a speedometer or a radargun to measure the speed of a train and were able to produce reasonable numbers like 75 mph, but as Eugene put it, "you couldn't figure out the speed if the speedometer were broken." The second meaning for miles per hour was simply as two numbers, e.g., 5 mph, meant that you let someone run for one hour and then you see how far they ran, here 5 miles. However, students were stumped when they were asked, for example, how far the person traveling at 5 mph went in 1/2 hour or how long it took to go 1 mile.

Conclusion

Neither hypothesis was supported in full by the data. Claims of forming rates (or even ratios) subconsciously because of extensive everyday experience with speed or steepness seem unfounded. At the same time, the hypothesis that intensive quantities in the entity category will necessary be more difficult wasn't quite right since students' association with speed with mph did seem to help in identification of two quantities. However, this association did not imply the formation of ratio. Furthermore, new quantitative complexities were revealed in the speed and ramp tasks, namely, the difficulty of focusing on an attribute to measure and then identifying which quantities affect that attribute.

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EXPANDING THE COGNITIVE DOMAIN: THE ROLE(S) AND CONSEQUENCES OF INTERACTION IN MATHEMATICS KNOWING

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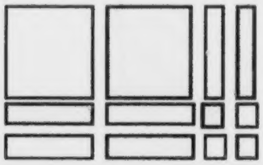
This paper is a contribution to the considerable current literature which is trying to account for mathematics knowing in interaction. The model which lies at the heart of this essay is based on an enactivist view of knowing in which embodied action is observed to bringing forth of a world of significance with others. In this paper we discuss how the model provides an observational tool which maintains the individual construction of knowledge (knowledge as a process) but shows such knowing as fully coemerging with the socio/cultural domain in which it is observed to occur. The model also is used to observe how co-recursive social interaction can lead to an expanded cognitive domain for individuals and the community in which they are observed. In this paper one use of the model is provided. We use it to observe and interpret how interaction changes the cognitive domain for a pair of students and to observe the teacher's pedagogical content knowing in action.



"I think this is an interesting polynomial because every other piece is negative."

Figure 1 Mallory's "interesting factorable polynomial

I. Prologue

The instruction to "create a factorable polynomial with both positive and negative terms that you think is interesting and say why," prompted Mallory, a tenth grade student to draw the image and make the comment noted above. The teacher/researcher (henceforth referred to as teacher) found Mallory's work so interesting that she was asked to present it to the class. The teacher drew the diagram on the board and Mallory comment, "I think

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Variable-entry (Simmt, 1996) and open-ended prompts are intended to be rich in terms of their potential to foster mathematical thinking and in so doing be useful in both individual problem solving and classroom research (Schoenfeld, 1994). Our research verified that prompts such as the one noted in the opening passage satisfy this criteria. However, the prompts offered by the teacher were only one kind of perturbation that fostered interaction and mathematical thinking among the members of the mathematical community (Cobb and Bauersfeld, 1995) with which we worked. As we will discuss, Mallory's response to our prompt triggered the mathematical thinking of others in this class, including a fellow student, the teacher/researcher and the co-researcher. Hence, the teacher provided the initial prompt, "create an interesting polynomial," but much of the mathematical thinking observed in this class was occasioned not by the prompt *per se* but in the interactions among members of the class.

II. Background

In a previous PMENA paper (Simmt, Kieren and Mgombelo, 1996) we discussed the nature of occasioning and portrayed how it occurs in action. We explained occasioning as the mechanism by which a student selects from the perturbations in the environment and transforms that selection for use in her or his mathematics knowing (von Foerster, 1981). But this is only part of the mathematics knowing observed in classrooms.

There is considerable current research and practical interest in the roles of interaction and communication in the mathematics classroom. Of particular interest to us are the attempts to find "ways in the middle" between applications of radical and social constructivism in studying mathematics learning (e.g. Cobb and Bauersfeld, 1995; Confrey, 1995). Such work focuses on mathematical meanings "generated through discourse in the regulation of practices" (Sierpinska, 1998). The work presented in this paper is a contribution to the search for ways of giving accounts of and explaining the roles of communication and interaction in mathematics knowing.

In this paper we ask, "how does interaction in the social domain figure in the mathematical knowing of individuals?" and the reciprocal question, "in what way does individual knowing impact the social and environmental domains?". In our discussion we elaborate on a model which shows the co-recursive nature of mathematics knowing in interaction. Further, we illustrate how interaction in a social domain expands the cognitive domain of the individual at the same time as the individual's knowing acts change the domain of interaction for herself and others (Maturana, 1988). We do so by concentrating on how this model can be applied in the study of classroom interaction involving the knowing of both students and the teacher.

III. A model for observing mathematical knowing in interaction

Our work is based on theories from second order cybernetics and systems theory (Maturana, 1989; Varela, Thompson and Rosch, 1991; von Foerster, 1981) which propose that the knowing of all living systems emerges from the plastic structure of system but is co-specified by, or coemergent with, the environment in which it occurs. Our work is related to the work of the radical constructivists who study the personal knowing of the individual (Steffe and Gale, 1995) and those who study mathematics knowing as a phenomenon that occurs in social settings such as the classroom (Cobb and Bauersfeld, 1995; Voigt, 1995).

Understanding all knowing as structurally determined yet observing the ways in which interaction feature in the actions of the individual has prompted us to seek a way in the middle that privileges neither the individual knower nor the social context. The perspective we take suggests that mathematics knowing is a fully embodied phenomenon which co-dependently arises in a person's interactions with the personal, social and cultural dimensions of his or her experience. In this enactive view of cognition (Varela, Thompson and Rosch, 1991), mathematical knowing is observed as the bringing forth of a world of significance, which includes mathematics, with others.

Enactivism explains that the environment does not instruct nor specify *particular* changes in an individual; rather, *the person's interactions with the environment* act as perturbations to trigger potential changes in both the person and the environment. Further, any changes to either the individual and its environment are co-determined by the structures of the living being and the environment. It is in this way that the organism and its environment are said to be co-emergent; they co-dependently arise. This fundamental circularity (Varela, 1989) between organism and environment has deep implications for social organisms such as humans. Because knowing always involves the persons' lived histories this circularity allows for recursion which constantly transforms our being and hence our knowing. Every act of knowing is both a result of previous acts of knowing and new knowing of those previous acts. For humans it is through recurrent and co-recursive interactions with other human beings in language that a world of significance (in our case including mathematics) is brought forth (Figure 2). Through the co-recursive bringing forth of the world the cognitive domain of the individuals and the community has the potential to expand.

The model anticipates that mathematics knowing has fractal qualities. Indeed the model has been derived from the study of knowing at many levels. First, we interpret our data at the macro-level with whole class data by discussing ways in which a prompt offered to students was taken up and

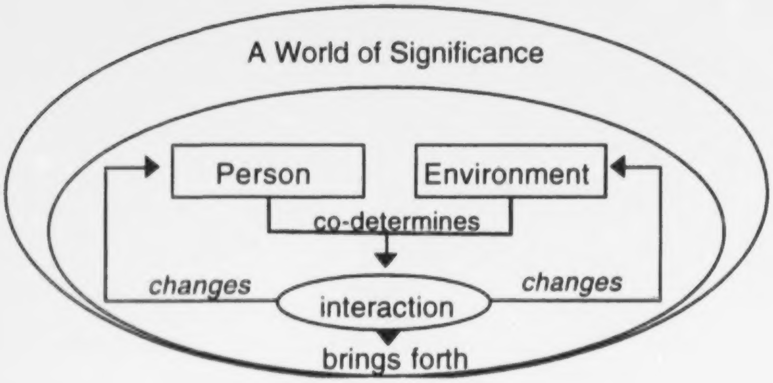

Figure 2 Model for observing interaction

transformed to create many other perturbations for both mathematics knowing and pedagogical content knowing. This is followed by a micro-level analysis and interpretation of student to student and student to teacher interaction in terms of how the cognitive domains of the participants were expanded. Using these interpretations we show the various ways in which interaction conditions and is conditioned by mathematical knowing of the both the students and teachers involved. In so doing we are able to deal with the issues which Confrey (1995) sees as emerging from such an interactive view of cognition (i.e., dependence on the environment assumed; knowing self is both autonomous and communal; diversity is anticipated; learning is reciprocal; emotional intelligence is acknowledged).

IV. Research sites and practices

To show how the model works we draw 'data' from one of a three of long term projects in which members of our mathematics education research group have been studying the nature of mathematics knowing in action. The particular site from which this data is taken is a six week long teaching experiment in a 10th grade non-academic mathematics class of 28 students with histories of poor academic performance in mathematics. Six months after the classroom work, four members of the class participated in clinical interviews. The artifacts from this research include field notes, student working papers; video tapes of classes and of clinical interviews.

V. Expanding the cognitive domain

One of the consequences of our model is that each act of knowing expands the knower's cognitive domain and at the same time changes the sphere of possibilities for those interacting in the community. Consider, what happened when Mallory's response to the prompt was shared with the participants in her class (see Figure 3, 1). Craig was provoked by Mallory's solution to think about the polynomial product and the simplified expression for it (2). "Wow," he said, "that would be a zero polynomial." He apparently reasoned that if for every positive term there was a corresponding negative term then this would equal zero. This shows that while he was perturbed by Mallory's work, he actually transformed what he "took in" from the environment for his own use. His own thinking then prompted him to notice and express that a zero polynomial was not nothing (3). This was a very new idea of zero for him (and one which an observer could see as algebraic). The teacher, who was listening for the "mathematics of students" (in Steffe's terms) was taken by Craig's comment. Reflecting on the comment the teacher/researcher went back to the factors of the polynomial product and immediately rethought the notion of multiplying by zero (4). At the same time that the teacher's mathematics knowing was prompted, the teacher's understanding of the nature and value of the prompt and what might be "expected" learnings by these "academically weak" students in interaction around this prompt was occasioned. This had a profound impact on the teacher. The next year when teaching the polynomial unit again, he purposely asked students to create and study factorable zero polynomials (5).

From observing and interpreting the situation noted above with the model we are able to understand just how a prompt occasions (but certainly does not cause) individual knowing (e.g. Mallory's knowing) and accounts, in part, for diversity in classes. But it also allows one to see just how the public re-presentation of one's knowing provides the possibility for occasioning the knowing of others (e.g. Craig's knowing) and leads us to interpret mathematical knowing as both autonomous and communal (as well as emotional features not discussed here). By thinking about even the brief interaction among Mallory, Craig and the teacher through the lens of the model we also observe just how learning is reciprocal. We are prompted to think about the teacher as one of many participants in the classroom influencing and being influenced by a large number of co-occurring mathematical conversations. Using the model we also observe the ways in which the teacher's pedagogical content knowing acts to occasion student knowing, which in turn occasions (again) the pedagogical content knowing of the teacher and so on. In summary, the model prompts us to observe knowing in the interaction among interactions.

VI. Consequences for mathematics education

The model we offer is useful as a research and theoretical tool for providing accounts of mathematics knowing in interaction. Like Simon

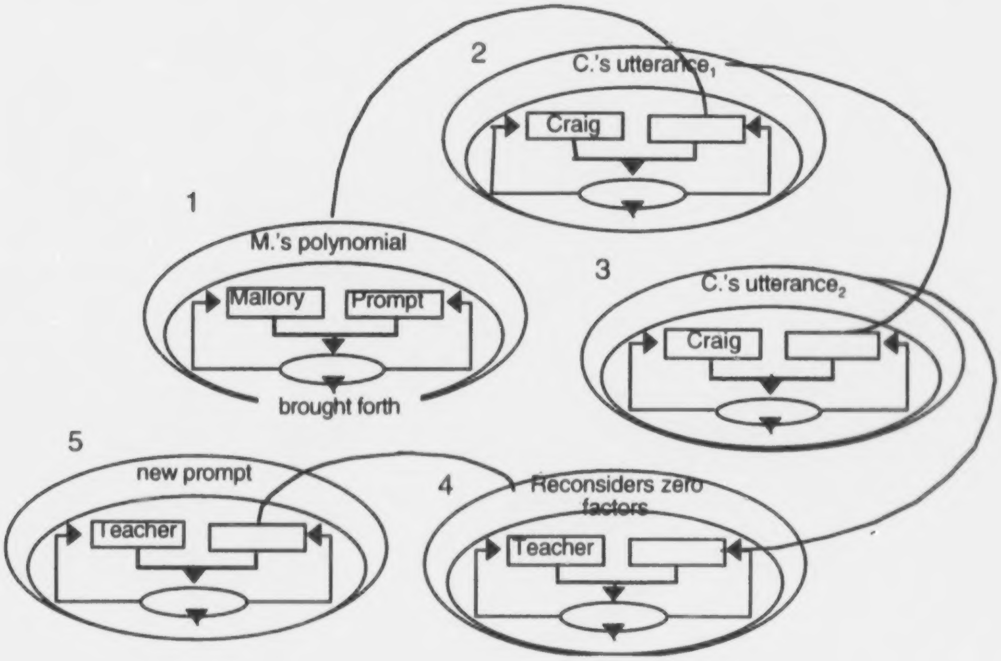

Figure 3 Model of interaction among students and with teacher

and Tzur (1999) have suggested, a model and its use coemerge with the situation being observed. Hence, the model (and the underlying enactivist and interactionist theoretical ideas) itself prompts us to look for certain things in the interaction and to provide an account in a certain manner. Although not fully illustrated here the model prompts us to look for evidences of how individuals select elements from the environment—which could include the utterances and work of others—and transform them; of how people express their own ideas into the interactive flow (e.g., to whom are such expressions addressed); and how such interaction acts recursively. At the same time, the very actions of the classroom participants and our modeled accounts of those actions influence our theoretical views.

The use of the model has occasioned our understanding of mathematical knowing as bringing forth a world with others in a sphere of possibilities and provided us a more precise way to explain this phenomenon in a wide variety of settings for mathematical knowing. The model provides a way of thinking about mathematical knowing which maintains the individual construction of knowledge (knowing as a process) while showing the function of the environment (physical, social, and cultural) in expanding the domain for such embodied knowledge.

construction of knowledge (knowing as a process) while showing the function of the environment (physical, social, and cultural) in expanding the domain for such embodied knowledge.

Finally, this work is significant in that it addresses issues of teaching directly. If teaching is thought of as the deliberate providing of possibilities for occasioning mathematics knowing and the conditions for expanding students cognitive domains, then this model points to alternatives to direct teaching. Further, because it provides a way of observing pedagogical content knowing changing in action it also provides a more detailed model for observing how curriculum becomes "wrapped around" the teachers and students in action in a mathematical community.

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THE NOTATION STUDENTS DEVELOP TO EXPRESS A RULE

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This paper presents one episode from a three-year study by Bellisio (1999) that explored the development of algebraic thinking in children in grades four through seven. The focus of the entire study was an examination of the notation and letters used by children who were given open-ended investigations that involved algebraic reasoning.

Students need time to build up an understanding of algebraic concepts before the formal study of algebra. There are benefits to the introduction of algebraic reasoning in the elementary grades, to serve as a natural foundation for the formalization and generalization required in a standard algebra course (Alston & Davis, in press). Davis and Maher recommend a classroom organization in which the students are encouraged to share their ideas, to challenge each other, and to explain and justify their solutions (1990).

Six students met with a researcher from Rutgers University for a discussion that lasted one and one-half hours. The students were members of a class that had taken part in a three and a half year longitudinal study by mathematics educators from Rutgers funded in part by a National Science Foundation Grant. Two years earlier, the class had been videotaped working on a series of problems involving fractions and operations on fractions. In order to study retention and growth, six students were chosen for a follow-up group interview. The students were brought together in a private room in their school and were presented with a series of explorations dealing with fractions and patterns. Videotapes of that interview session, student papers and researcher notes provided the data for this study.

Two problems from this session will be described in this talk. In the first problem, the students were asked to discover how many toothpicks would be needed to make a ladder with R rungs. For the second problem, the students calculated the surface area of R Cuisinaire rods that were placed on top of each other, offset by one centimeter, in a staircase formation.

The six students were able to use algebraic variables and write expressions using variables to demonstrate their ideas, generalizing from a specific situation. They developed notation and built representations that were at first concrete, using toothpicks or rods. They then represented their ideas using natural language and expressed them using symbolic notation. The students found two methods to calculate the surface area and due to an arithmetic error, the two rules they wrote were different. Eventually they discovered their mistake and showed that both their rules for the surface area were the same.

This research leads to a deeper understanding of the process that students utilize as they build representations, verbalize ideas and then write symbolic notation to express these ideas.

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EIGHTH GRADERS IN AN ALGEBRA COURSE: A FOLLOW-UP STUDY

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The purpose of this study was to track the progress of a group of high school seniors who were enrolled in a first-year algebra course when they were in the eighth grade. Four years ago, they were surveyed and interviewed regarding their reasons for taking an accelerated course and were asked to predict their future grades and possible career intentions. In the current study, the grades of those students in mathematics classes were accessed. Students were also surveyed to determine which courses they took in high school, how they performed, and their current college and career aspirations. The survey had a 34% response rate.

A repeated measures t-test showed that, as eighth graders, male students predicted their grades in mathematics to be significantly higher than they actually performed, while females more-accurately predicted their grades. A Pearson correlation r-value of 0.363 resulted when predicted grades were used as a predictor for actual grades — a statistically significant result. In the initial study, three out of four eighth graders reported that they intended to take four years of high school mathematics; however, the follow-up study showed the actual figure to be 91%. A somewhat disturbing result of the follow-up survey was that 25% of the eighth graders who took an accelerated first-year algebra course started their next year in a class that was designed for freshmen, such as Algebra I because their performance had not been strong enough the first time around. The survey also showed that 19% of the high school seniors do not plan to take any mathematics at all in college, or at most, one basic required course. When asked if they would take a first-year algebra class in eighth grade over again if they had the choice, 94% responded affirmatively, and most indicated that participation in the class allowed them to "get ahead" and to "test out" of other classes in high school. Students appeared to be driven by ego goals of outperforming peers more than learning for the sake of learning. In fact, mathematics was cited in the follow-up study as the students' "least favorite" subject area in 19% of the cases and was the favorite content area for only 43% of the respondents.

APOS THEORY, VYGOTSKYI AND NATURAL LANGUAGE

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The relationship between Mathematics and Language has been the topic of many papers and presentations. However, the majority of these deal with the question of the role of language in learning mathematics. Here, we address the reciprocal question, namely, the influence of the development of mathematical understanding on the development of the natural English language. The possibility of such an influence is suggested by the cognitive and linguistic analyses of the following programming task developed on the basis of the APOS theory of cognitive mathematical development (Vidakovic, 1996):

> Instruction: Give a verbal description of what the following computer code is doing.

```
length:=0;
for i in [1..3] do
        length:=length*r(i)**2
end;
length := sgrt(length);
length;
```

There maybe two different kind of responses to that task obtained from a student. A student may respond by describing the full routine of the computer in proper order, namely *that the computer first sets length to be zero; then it takes i to be one and finds the length as the sum of the previous length and the value of the first component of r squared; then it takes i to be two and adds to the of the length the second component squared; then it takes i to be three, squares the third component of r and adds it to the length; then it takes the square root of length and prints the result.* This is called an *action* level response by the APOS theory.

A student may also say*: The computer finds the length of a line segment from the origin to the point whose coordinates are given by the tuple r. It finds it as the square root of the sum of l components of r.* This represents a *process* level response - a response on a more advanced mathematical level, according to APOS theory.

At the same time, a closer look at the language involved in each of the responses reveals a significant, purely linguistic difference between them. The language of action response is closer to what applied linguistic calls the language of *paraphrasing* the information, while the language of the process response is closer to what is called *summarizing* of the information. This means that the reflection upon the mathematical content of the task may, possibly necessarily, stimulate not only the development of mathematical understanding but also the parallel development of the language in lifting it from the basic language of paraphrasing to the higher order language of academic proficiency. This simple observation relating the level of mathematical understanding with the level of the language necessary for that understanding has far reaching implications. The natural theoretical framework for the analysis of such a situation has been prepared by Vygotskyi, (1986) in his discussion of the dynamic relation between thought and language. The presented paper exploits Vygotskyi's framework to investigate these implications in the context of several teaching experiments.

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A DIDACTICAL APPROACH TO TEACH THE CONCEPT OF VARIABLE IN ELEMENTARY ALGEBRA

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In the last years there has been a considerable interest to understand the difficulties students face when dealing with the concept of variable in elementary algebra. It is important to explore new teaching strategies that take into account-available research results.

The purpose of this study is the development of specific activities to teach the concept of variable using programming with the computer language ISETL, and the follow up of the progress of students and the difficulties they face while using them. This will provide feedback in the design of new activities more suitable to the way students construct their learning.

The activities were designed using the theoretical framework developed by Ursini and Trigueros. In their study the authors emphasize the different uses of the concept of variable and they propose a decomposition of each of those uses in terms of the requirements of the comprehension of each of them. Specific activities for each of the uses of variable were designed in terms of the framework and these were complemented by other activities that aim to favor that students can make relationships between these uses and solve complex problems.

The activities were tested with a total of 90 students. Results obtained show that students in the study groups were more successful in dealing with conceptual questions where other students had difficulties. The feedback given by the computer and the possibility to visualize their results gives students the opportunity to reflect upon specific operations on variables, to solve equations and to deal with functions. Students showed some difficulties in using some algorithmic manipulations but were able to structure in a richer form the different uses of the concept of variable and gained a lot in their appreciation of mathematics.

SECONDARY STUDENTS' DEFINITIONS OF EQUATION: CONSTRUCTING MEANING IN A SYMBOL RICH ENVIRONMENT

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Representation is a significant component of mathematics. The use of mathematical models to represent complex phenomenon plays a critical part in secondary mathematics, particularly the concept of equation. Research indicates that students are deficient in understanding the complex mathematical ideas denoted through conventional representations (Greeno and Hall, 1997).

This study involved sixty-five public secondary students enrolled in Algebra, Geometry, or Introduction to Calculus. Qualitative methodologies were utilized to explore students' definitions of equation. Subjects wrote responses to the question "What is an equation?" This open-ended mode of inquiry allowed freedom for respondents to provide in-depth perspectives and insights. Students were not given a time limit for responding to the question and all completed the task within fifteen minutes.

The responses were analyzed in order to classify the definitions based on similarity of student perspectives. Additional analysis determined some of the major ideas found within the definitions. Inter-rater reliability of coding was .97 on a random sample of 30% of the responses. Two categories emerged for the classification of the definitions: conceptual and procedural as the definitions focused either on mechanical aspects such as symbol manipulation or on equations as representations or models of situations and phenomenon. A comparison of the types of definitions provided by course showed no statistical differences (chi-square 1.807, D=2, $p<.405$). Eighty percent of the students provided procedural definitions while 20% provided a conceptual definition. More than half of the responses indicated that equations were representations with equal symbols and variables. More than fifty percent also stressed the importance of operating on the equation by performing the action to both sides. Other perspectives included the idea of equations as problems and formulas. Slightly less than 20% of the responses indicated that equations are representations of problem situations.

Findings indicated that students do not have mature definitions of equation regardless of taking advanced math courses. The question emerges about how to best develop ideas for such concepts that are based in pedagogy emphasizing symbols and operations on those symbols. Since equations are taught and presented in texts by stressing technical rules and the use of these symbols, students may be "drawn to focus on the concrete aspects of the notation, although the mathematical ideas behind the notation may elude them." (Sierpinska, 1999)

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REPRESENTATION SYSTEMS IN SCHOOL ALGEBRAIC PROBLEMS

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We have carried out a research work to assess the skills and abilities of secondary school students (16 years old) in elementary algebra. We have designed an assessment tool, using ten verbal algebraic problems.

This assessment instrument was given to a sample of 160 students, divided in two groups: 80 students had recently finished the secondary schooling (16-18 years); other 80 were university students (20-24 years), who had not studied mathematics for the last three years o more.

The protocol assessment was made according to a classification of system representation, according Javier's sense (1987, 1996), Duval (1993), Hitt (1996) and other authors who have carried out research in this topic.

We have defined and classified the representation systems in five different categories: numeric (whole-part, guess-error), graphic, graphic-symbolic, and symbolic-alphabetical representation systems. The two sample strata fitted to this classification.

On the other hand, we have applied a cluster analysis, whic is a exploratory multivariate technique allowing to build categories of similar entities, and we have obtained four different types of solving categories: 1) Solvers who are versatile to use a variety of representation systems. 2) Solvers who use mainly numeric representation systems. 3) Solvers who use almost exclusively a symbolic-alphabetical representation system. 4) Solvers who reject representation systems that contain graphic or graphics-symbolic elements.

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Assessment

JUST RIGHT GROUPS: PERFORMANCE GROUPING FOR MATHEMATICS BASED ON CONTINUOUS ASSESSMENT

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During the 1997-98 school year five teachers used a set of assessment benchmarks they had developed during the previous two years to assess their students and sort them into three or four grade level groups based on their performance. These were flexible groupings in that assessment was continuous and teachers moved students from one group to another throughout the year. Based on early-late assessment tasks, problems and class discussions reported by the teachers, students' written responses throughout the year, and some standardized test results, students made more progress in learning arithmetic than similar groups of students these teachers had worked with in previous years. Students also did better than other students in the school both on the assessment tasks and on standardized tests.

Rio Vista Elementary School is one of many that California now designates as "low performing schools." It is an ethnically diverse school which serves a transient, low-income population. During the two years preceding the 1997-98 school year our teacher-research group had determined a minimal set of benchmark assessment tasks that would give us useful information about what our students knew. The tasks include one-to-one counting, conservation of number, instant recognition of missing addends to six, sums to ten, doubles to 20, place value, and a timed addition facts test to 20, a timed multiplication test of facts to 10x10, and an explanation of two ways to find 4x27. When teachers used these assessments they realized that their lowest and highest performers were outside the range of their best attempts to meet the needs of their heterogeneous classes.

After many attempts to group students for math within our classrooms and much debate, we decided to try grouping across classes, based on our assessment of student performance. Some would call this tracking, clearly not an acceptable practice in many places. We think our experiment was different from tracking in several important aspects. This paper reports our findings in relation to the question: How did the students respond to the grouping and what progress did they make in learning arithmetic? This is a short overview of a much longer report which documents our methods of assessing students, organizing and reorganizing the groups, our work with students in the groups, and the progress that students made individually and collectively. We think we have an important contribution to make to the discussion of a compromise between completely heterogeneous grouping and tracking for learning mathematics.

In December three first and second grade teachers met and decided to divide their 58 students into three groups for math: low, medium, and high performing. The assignment of students to groups was originally based on the teacher's assessment of each student's work as a problem solver using criteria we had developed two years earlier (Kari, 1996). At the same time four fourth and fifth grade teachers decided to group their students into low, low/medium, medium, and medium/high performing groups. They used a place value assessment developed by Kamii (1985) and an assessment of the student's response on two multiplication problems to decide on an appropriate group for each student. The upper grade classes were larger, ranging from 29 to 32.

The primary groups met for one hour four times a week, and the upper grades met every day for 45 minutes. Work focused on developing understanding and competence in arithmetic. The teachers continued to do mathematics in their regular classrooms as well. The 4/5 teachers continued to use the TERC units and district replacement units to teach mathematics beyond computation, and the primary teachers taught mathematics in the context of their art, social studies, and science lessons as well as in taking care of school business. The teachers drew upon Kamii's work (1989, 1985) and the work of the Cognitively Guided Instruction program (Carpenter & Fennema, 1994, 1993) in planning their group sessions to focus on mental math, games to practice skills, and word problems. The grade level teams planned together, used the same ground rules, the same games, and the same contexts for their problems. Students in the lower performing groups worked on the "same" problem or played the same game as those in the higher performing groups, but the numbers were smaller and the problems less complicated. This practice made transitions from group to group smoother for the students.

Students were allowed to go to other groups if they chose, but only one student did, and after a few days she decided to return to her first group. The teachers moved individual students from one group to another whenever it seemed appropriate. At spring break the primary group reassessed all of the students and reconstituted their groups.

Based on our understanding of the need for discourse and the social negotiation of meaning (Vygotsky, 1982) in learning mathematics, our goal was to pose problems that would engage all the students in the discussion. The teachers found themselves in a quest for "just right problems" for their groups, and we learned that what appeared to be small changes could make a major difference the students' response. The teachers recorded the problems they used as well as the students' oral responses. They also collected written responses to the word problems and assessment questions. At our research group meetings we established rubrics for assessing levels of understanding demonstrated in the written work.

Our analysis is based on the teachers' written records of students' responses and students' written work in their grouped sessions, beginning and end of year assessments (the benchmark assessments listed in the first paragraph) which were given school-wide, the district standard assessment, and teachers' observations of students when they returned to their regular classrooms. Based on their journal notes and students' written work each teacher selected students who represented the low, mid, or higher level performers within their groups and wrote a descriptive report including discussion of the students' solutions of specific representative problems at different times of the year. Some excerpts from those reports are included along with the overall observations so the reader can see some examples of the evidence that provided the basis for our general observations.

Overall observations of all five teachers:

Student response to the mathematics groups was positive, and students did not express any awareness of how the groups were determined. Students at all levels were eager to go to math groups and talked about their accomplishments with enthusiasm when they returned. Students in the lower and mid-level performing groups, who were passive participants (or distracters) in mathematics discussions at the beginning of the year in their regular classes, became avid participants in their groups. We kept track of the diversity of the groups and found that they remained ethnically diverse throughout the year.

Students showed impressive gains on our assessments from the beginning to the end of the year and in comparison with other students in the school at their grade level. Their standardized test scores also compared favorably with those of other students in the school, but it is impossible to know to what extent these scores are related to our grouping of students or whether they are a result of the teaching methods we used. For example, on the state mandated SAT9 achievement test, school wide Rio Vista second graders scored in the 23rd percentile. The research project second grade students from Amy's, Judy's and Catherine's classrooms collectively scored in the 33rd percentile.

Observations for the first and second grade groups:

The lowest performing primary students were generally not doing well in other subject areas, particularly in reading. At first they did not listen to each other and did not know the social rules of playing games, nor did they understand many of the expectations of school. The teacher spent the first month explicitly teaching expected school behaviors, and by mid-February students were participating in mathematics discussions. The following excerpt describes discussion in Judy's group in February.

> The first mental math problem for my group was, "I have 6 cookies, and I want to split them between my rabbit and me so we each have the same amount." When the students started giving me their answers, I thought that the problem was too difficult, but we ended up having a wonderful mathematical discussion. Eliana thought I would get 3 and the rabbit 2. Richard said we would each get 5. Jamar thought we would each get 1. When Darius said I would get 8 and the rabbit would get 7, another student, Lupe, countered "No, that can't be right. It's too many. It's more than six." Only Tevin said we would each get 3. I said to the class, "Tevin says 3 + 3 = 6. Do you agree? Show me how much 3 + 3 is." All the kids put their fingers up and started counting. Kayla miscounted and said it was 7. Eliana and Richard kept trying to convince her that it was 6. Luis and Lupe were discussing whether it would be fair or not. Cleveland said, "you can give 1 to you and 1 to the rabbit and like that until you've done all of it." Luis stated, "you can just give each person 2 and that would be fair." (Rummelsburg, 1998)

On May 26th, she gave the students this story problem:

"I bought 4 Jolly Ranchers and each one costs 3 cents each. How much money did I spend?"

> This problem was *just right* for the group, and many were able to solve it without assistance. We were working on how to approach a story problem by drawing a picture if necessary, and how to write an equation. I read the problem to the students, answered any questions and then passed out a piece of paper with the problem typed at the top. Luis was the first one done. It took him about three minutes to solve it. When I asked him how he got done so quickly, he said, "I was thinking about it before you gave us the paper."
>
> Luis's strategy: 3 + 3 + 3 + 3 = 12

> "I knew 3 + 3 = 6 and 3 + 3 = 6 and 6 + 6 = 12. So the answer is 12."
>
> By the end of the year, Luis was one of my best problem solvers. This was quite an achievement considering what he was able to do in September (On counting legs on a fox and a goose, Luis had written: 1 2 3 4 5 6 7). (Rummelsburg, 1998)

By the end of the year the middle group exceeded the teacher's expectations in their understanding of place value and their work with two digit numbers, and the assessments reflect this progress. The groupings were changed at spring break, and most students in Amy's group were still debating place value.

> The change also brought new energy, just because there were new dynamics to each group. On April 30, we had a great discussion about whether or not the digits in 11 represented 10,1; 10,10; or 1,1. This discussion emerged from a mental math problem 11+9. We got two answers 20 and 29. The strategy that produced 29 was that 10+10=20 (each 1 in 11 counted as 10), then 20+9=29. There was a heated debate over the value of the digits and although the majority seemed to have persuaded each other that 11=10+1, some remained skeptical.

At the end of the year we re-assessed all of the students on place value.

> In the fall, only one of the students in my group had demonstrated any understanding of place value. Camille responded to a counter suggestion and was therefore at a level 2. By June, however, sixteen students were at a level 2 (responding to a counter-suggestion), three students were at a level 3 (able to identify place value in number to 100), and only one child, Alex, was still convinced that the one in 16 was a one and he thought my counter-suggestions were silly. (Kari, 1998)

By late spring, the high performers, who were selected because of their understanding of place value and their knowledge of addition combinations, were working with numbers in the hundreds. This is notable because in the other first and second grade classes this year and in our own experience over the previous three years teachers did not reach the point of working with their students on problems with numbers in the hundreds.

Observations for the fourth and fifth grade groups:

Based on standardized measurements the lowest performing students made impressive gains in problem solving and reasoning as measured on the Woodcock-Johnson test and at least a year's gain in computation as measured by the WRAT-R1. Thirteen of the students in the lowest performing group who took the SAT9 scored between the 20th and 50th percentiles on mathematics reasoning, an unprecedented achievement in a school where the average percentile ranking is in the low twenties. As with the primary students, students in the medium performing group who very rarely spoke in the regular classroom were willing to take risks and contribute in the groups. Lindsay's report of Isaiah's work on 4x27 represents the growth of students from January to April in her low/middle performing group.

> Students seemed to accelerate in their mathematical learning after participating in the math groups. For example, Isaiah struggled with the problem 4x27 in early January. He came up with the answer 87. He explained that 2x4 equals 8 and that the 7 had to be added to the end. Isaiah's second method of getting 87 was to add the numbers 22, 22, 22, and 21. In contrast, in April Isaiah was able to successfully solve the problem two ways, and in one of those solutions he broke the problem into sub problems and used multiplication, 4x20 and 4x7. (Hunt, 1998)

The highest performing group was lively and surprised the teacher with its development of a spirit of competition, especially among the boys although the girls were equally competent. She did not observe the same level of competition during classroom mathematics lessons with her regular class. The students in the highest performing group were not shy about taking on challenges or about sharing their strategies as Connie reports.

> The students in my math group also demonstrated some enthusiasm for word problems. I did not see or hear any comments such as, "I don't get it!" as the problems were introduced. Students seemed confident in their ability to solve the problems and would go to work as soon as the problems were given. Most of the word problems were structured around multiplication opportunities. For example, "There were 36 children in the class and each child needs 9 pieces of paper for a report. How much paper does the teacher need to buy?" was a problem introduced in January. For this particular problem many of the students chose addition as the primary strategy for solving the problem. Latisha added "36 + 36 + 36 + 36

> + 36 + 36 + 36 + 36 =" and then bracketed the 36's into 72's and then the 72's into 144's, and then the 144's into 288 + the final ninth 36 for a total of 324. This was a common strategy. Students comfortable using multiplication, such as Rashaad made two problems - "4 x 36", and "5 x 36" and then added the products. Joshua multiplied "36 x 3", and then multiplied the product by 3 again. Rashaad and Joshua were definitely the exception and not the rule when it came to word problems. Most of the students used addition strategies, such as Latisha's when solving word problems. (Gillan,1998)

While our students scored better than students in the rest of the school on the standardized tests they took, we can not be certain the grouping affected their scores, because the teachers who were grouping were using a problem solving approach to teaching mathematics and the rest of the teachers were, for the most part, using a traditional approach. Our evidence for saying that the grouping helped in improving student performance is based on the descriptions of the problems students were able to solve by the end of the year compared to the level of the problems they were solving at the end of last year. It is clear that students in both the primary and intermediate high performing groups were succeeding in solving problems that teachers had never dared to present to their classes in the preceding years. It is also clear that students in the lowest performing groups made more progress than teachers had come to expect in the past. And while it appears that the mid-range students were able to participate more fully, it is difficult to know whether they progressed further than they would have in a heterogeneous class setting, but they did exceed the teacher's expectations.

> When I look back to my original goals, I find that I was underestimating the group's potential for success. I had no idea that we would be working with double digit numbers so well, or that so many in the group would be grappling in such sophisticated ways with our place value system. (Kari, 1998)

We have continued this study through the 1998-99 school year and are in the process of analyzing the data and preparing a follow-up report. We plan to continue through 2000-01 in order to follow the students and to gather more assessment data. In the meantime, we have lost our comparison group as other teachers in the school have become interested in learning more about constructive approaches to teaching mathematics and are teaming up to assess and group their students for math. Next year we will be working with another school where the teachers are interested in using our assessment tasks but will continue to use a more traditional program with heterogeneous groups.

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STUDENTS' PERCEPTIONS OF INSTRUCTION AND ITS RELATIONSHIP TO THE DEVELOPMENT OF QUANTITATIVE LITERACY

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Using data from the literacy component of the Third International Mathematics and Science Study (TIMSS), this study investigates the relationship between students' perceptions of classroom instruction and students' development of an "everyday" understanding and appreciation of mathematics or quantitative literacy (QLT). This study considers the level of inquiry-based instruction versus traditional drill-and-practice instruction found in classrooms across the U.S. and Canada and their relationship to QLT. Further, this study investigates QLT at the school level considering the effect of school compliance with reform-oriented methods of instruction (NCTM, 1989, 1991). Results suggest that both forms of instruction influence student QLT, however, student attitude toward math is more positively affected by inquiry-based instruction. Similarly, schools whose curriculum is influenced by national subject associations tend to have higher levels of QLT overall.

The United States, Canada, and most other major industrial nations, have recognized the critical role of an everyday understanding of math and its applications in its workforce and among its citizenry. This everyday understanding of number and its applications is what has been referred to as quantitative literacy ([QLT], Steen, 1997). QLT involves an appreciation of mathematics including: 1) Knowledge of math content; 2) The impact of mathematics on society; 3) The reasoning process of mathematics; 4) The historical development of mathematics; and 5) A positive attitude toward mathematics (Atkin & Helms, 1993; Mitman et. al., 1987; Orpwood & Garden, 1998). The type of mathematics assessed in the literacy component of the Third International Math and Science study (TIMSS) was the mathematics that "school leavers" have retained and are ready to apply to the everyday experiences of being a citizen and worker (Orpwood & Garden, 1998) and served as the data source for this study.

Several documents, including the influential report *Everybody Counts* (National Academy Press, 1989) have expressed the importance of possessing quantitative literacy in order to function in today's society. The National Council of Teachers of Mathematics ([NCTM], 1989; 1991) has outlined curriculum and instruction that can help reach the goal of "math for all." A major component of the instruction recommended by NCTM is that students be involved in inquiry-based learning. However, few studies have looked at the effectiveness of inquiry-based instruction on quantitative literacy. This study compares the effects of inquiry-based instruction to drill-and-practice methods in promoting QLT while considering the effects associated with being in a school system that tends to promote more reform-oriented curriculum. In addition, by comparing and contrasting the U.S. and Canada this study provides a deeper and better understanding of the teaching and learning of mathematics in North America.

Data

The TIMSS literacy study surveyed students in their final year of secondary school. The United States sample involved 5807 students within 211 schools, and the Canadian sample involved 5232 students within 338 schools. In the present study, schools for which no data were available regarding the influence of national or provincial subject associations on school curriculum were deleted. This resulted in a working sample of 4620 students within 166 schools in the U.S., and 4779 students within 308 schools in Canada.

Measures

At the student level, this study investigated four of the five components of QLT using three measures from TIMSS. Math content knowledge was measured using students' scores on the mathematics literacy test. Students' understanding of the reasoning process and social impact of mathematics were measured together using students' scores on the reasoning and social utility sub-test (RSU). A student's attitude toward mathematics was measured using a 4-item scale (Cronbach's __.89). In TIMSS, students were characterized as generalists (students who had not studied advanced mathematics) or specialists (students who had studied advanced mathematics). This characterization served as a proxy for curriculum coverage in the present study. Gender and SES (parents highest level of education) were included as control variables.

Students were asked to rate, on a 4-point likert scale, the frequency of 13 instructional activities in their mathematics lessons. Two subsets of these items were used to form two instruction scales, an inquiry-based scale (INQUIRY), and a drill-and-practice scale (DRILL). Face validity, internal reliability (INQUIRY: _=.73; DRILL: _=.72), and results of a confirmatory factor analysis justified the use of these scales in further analyses. The two scales were used to measure the amount and type of instruction occurring in mathematics lessons.

At the school level, principals were asked, "How much influence does the national (provincial) subject association have in determining the curriculum taught in your school?," to which they could answer, "none," "a little," "some," or "a lot." This variable was dummy coded to form four groups and was used to assess the intention of schools to implement reform oriented curriculum. An aggregate measure of school-level SES was included as a control.

Results

Descriptive statistics are presented in Table 1. Within- and between-country means were compared using t-test statistics. Students in both the U.S. and Canada report being asked to participate in more drill-and-practice type activities than inquiry-based activities. In both the U.S. and Canada, specialists reported receiving significantly more instruction (whether drill or inquiry) than the generalists. Canadian students were found to be more quantitatively literate in terms of math content and RSU, however, there was no difference across countries in students' attitudes toward mathematics. Within each country, specialists were found to be significantly more quantitatively literate on all measured components of QLT. The gender difference in QLT in the U.S. was minimal compared to the large difference found in Canada.

A two-level hierarchical linear model ([HLM], Bryk & Raudenbush, 1992) was used to investigate the effects of instruction on student development of QLT. A between-student equation regressed student QLT outcomes (math content knowledge, RSU, and attitude) on measures of classroom instruction (INQUIRY, DRILL) while controlling for background variables (GENDER, SES, CURRICULUM). A between-school equation was then used to predict any variability across schools in QLT.

Results from unconditional models suggest that levels of QLT significantly vary across schools in both Canada and the U.S. (see Table 2). However, in all cases, the majority of the overall variance is attributable to within-school differences. In both Canada and the U.S. (see Table 3), curriculum and SES were found to be strong predictors of within-school differences in math content knowledge and RSU, with males doing significantly better than females. After controlling for these variables it was found that overall both forms of instruction were positively related to increased achievement for these components of QLT. In addition, inquiry-based instruction was found to have a strong positive effect on students' attitude toward mathematics in both Canada and the U.S..

At the school level, SES significantly predicted school differences in math content knowledge and RSU in both Canada and the U.S. After controlling for SES, schools in the U.S. that reported "a lot" of influence from the national subject association on school curriculum were found to

Table 1

Means and Standard Errors[a] for QLT and Student Perception of Instruction for Canada and the United States

		Curriculum		Gender	
	All students	Specialists	Generalists	Male	Female
CANADA					
Instruction					
INQUIRY	2.37 (0.01)*	2.44 (0.02)*	2.34 (0.02)	2.39 (0.02)	2.34 (0.02)
DRILL	3.14 (0.03)	3.23 (0.03)***	3.11 (0.04)	3.10 (0.04)*	3.18 (0.04)
Quantitative Literacy					
Math Content	517.1 (2.84)***	585.7 (3.39)***	496.6 (2.79)	537.3 (3.57)***	498.7 (3.66)
RSU	526.9 (2.49)***	574.9 (3.36)***	512.5 (2.94)	545.3 (2.68)***	510.1 (3.55)
Math Attitudes	2.51 (0.03)	3.01 (0.03)***	2.36 (0.03)	2.60 (0.02)***	2.42 (0.04)
N	4779	1804	2975	2316	2463
USA					
Instruction					
INQUIRY	2.42 (0.02)	2.58 (0.03)***	2.37 (0.02)	2.43 (0.02)	2.41 (0.03)
DRILL	3.12 (0.03)	3.30 (0.04)***	3.06 (0.03)	3.07 (0.03)**	3.16 (0.03)
Quantitative Literacy					
Math Content	463.4 (3.60)	554.3 (5.70)***	437.4 (3.47)	469.5 (4.55)*	457.6 (4.03)
RSU	472.7 (3.43)	547.1 (6.09)***	451.4 (3.03)	479.3 (4.94)*	466.2 (3.74)
Math Attitudes	2.55 (0.02)	2.96 (0.03)***	2.43 (0.03)	2.58 (0.03)*	2.52 (0.03)
N	4620	944	3676	2250	2370

Note: $^{*}p<.05$; $^{**}p<.01$; $^{***}p<.001$; [a] Standard errors appear in parentheses. Because sampling weights were used, adjusted standard errors are reported instead of standard deviations. Asterisks in the first column represent significant between country differences for respective measures. Asterisks in the "Specialists" and "Male" column represent significant within-country differences between specialists and generalists and between males and females.

Table 2
Unconditional Models

	United States			Canada		
	Content	RSU	Attitude	Content	RSU	Attitude
Fixed Effect						
Average Sch. Mean	476.89	481.00	2.62	540.65	541.15	2.67
Random Effect						
Variance Component						
Variance Component						
Between School	2744.98***	2257.69***	0.0469***	904.46***	1076.17***	0.0189***
Within School	5674.76	5418.56	0.6073	7571.30	5828.30	0.5922
Attribution of Variance						
Attribution of Variance						
Between School	.33	.29	.07	.11	.16	.03
Within School	.67	.71	.93	.89	.84	.97

Note. *p < .05, **p<.01, ***p<.001.

have significantly higher levels of QLT than schools reporting "none." Schools reporting any level of influence were found to have significantly more positive attitudes toward mathematics than schools reporting "none." In Canada, school attitudes were not affected by this variable, however, there was significant influence on overall math content knowledge and RSU.

Conclusions

Results from this study suggest that drill-and-practice instruction still dominates mathematics lessons in the U.S. and Canada, however, when used, inquiry-based instruction was found to have a positive effect on students' development of QLT even after controlling for background variables. This finding was true for all components of QLT, but especially for attitudes toward mathematics, where the effects of inquiry-based instruction were found to be the strongest. Schools who tend to implement curriculum advocated by the national subject association (e.g., NCTM), on average, were found to have higher levels of QLT than schools who do not, especially in the case of attitude towards math found for U. S. schools.

It is important for students to develop an understanding of mathematics that can be applied readily on the job and in everyday life. However, when advocating instructional reform in order to better promote QLT it is important to test it against traditional methods. The findings from this study are encouraging as they suggest that inquiry-based instruction is as effective as drill-and-practice instruction, but more encouraging, that students' attitudes toward math are more positively affected through the use of inquiry-based instruction.

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Table 3

Final Hierarchical Linear Models

	United States			Canada		
	Content	RSU	Attitude	Content	RSU	Attitude
Fixed Effects						
Regression Coefficients						
Regression Coefficients						
School Mean	477.03***	481.07***	2.63***	539.60***	540.37***	2.67***
School SES	52.93***	49.58***	0.03	31.75***	30.02***	0.01
A Little [a]	17.42	14.30	0.21***	14.76*	10.08	0.08
Some [a]	10.82	8.65	0.23***	18.82**	13.80*	0.04
A Lot [a]	24.41***	20.18*	0.33***	16.17*	12.25	-0.01
Curriculum	95.43***	74.60***	0.60***	75.60***	56.42***	0.61***
SES	10.07***	10.40***	-0.04***	7.70***	8.02***	0.00
Gender	13.34***	13.41***	0.08**	33.33***	30.64***	0.13***
Inquiry	5.29*	4.26	0.14***	6.70*	7.05*	0.20***
Drill	8.98***	11.19***	0.05*	8.13**	9.01**	-0.02
Random effects						
Variance Component						
Variance Component						
Intercept	1232.81***	932.64***	0.0426***	777.32***	934.26***	0.0282***
Curriculum[b]	——	——	——	515.10	——	——
Level-1	3990.21	4212.00	0.5451	5503.47	4516.41	0.4864
Variance Explained						
Variance Explained						
Intercept	55.1	58.7	9.2	14.1	13.2	00.0
Level-1	29.7	22.3	10.2	27.3	22.5	17.9

Note. $*p < .05$, $**p<.01$, $***p<.001$. [a] These categories are in comparison to "none."

[b] The effect of 'Curriculum' was found to vary across schools only for Math Content in Canada; In all other cases the student-level variables were found not to vary across schools.

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MATHEMATICS COMPETENCE AT THE UNIVERSITY LEVEL: DO FIRST YEAR STUDENTS HAVE BASIC MATHEMATICAL RESOURCES TO ACHIEVE IT?

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Recent proposals on what mathematics should be included in pre-university level recognize that it is important to pay attention not only to the meaning attached to the content but also to activities that promote aspects of the mathematical practice. Thus, conjecturing, visualization, argumentation, and symbolization are some examples of activities that students should experience during their study of mathematics (Santos, 1997). To what extent the ideas endorsed by the recent proposal have influenced the students' learning of mathematics recently? It is possible to identify the students' mathematics resources needed to be successful at the university? Is there any continuity or mathematical bridge between the mathematics studied in high school and first Year University? These research questions were used to document what first year university instructors expect from their students (in terms of mathematical resources) and the actual mathematical competence that these students bring into the classroom.

Sixty-nine first year students participated in the study. A questionnaire including 10 questions which responses only required the use of basic high school mathematics and interviews (with students and instructors) were the main instruments to gather the information. Ten instructors were asked to judge the questionnaires and they also expressed their opinion about the type of mathematical resources they expect from their students. Results showed notable difference between what instructors expect from their students and what they showed in their response to the questionnaire. In addition, the type of problem included in the first exams given to these students by their instructor showed more emphasis on proving than on the application of the mathematical relationships. It seems that the mathematical resources students normally bring into de classroom are not sufficient to fulfill the instructors' expectations for their courses.

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IMPACT MATH: CASE STUDIES ON A GRADE 7 AND 8 IMPLEMENTATION PROJECT

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In June 1997, the Ontario Ministry of Education and Training in Canada released a new curriculum for Grades 1-8 Mathematics. Province-wide testing at several grade levels was also introduced, prompting a renewed attention on assessment methods. As the curriculum includes the same five content strands as in the NCTM Standards, with similar process components implied, the new curriculum is a philosophical shift for many teachers. Against this background, a project called Impact Math was undertaken to assist Grade 7 and 8 teachers with the new curriculum.

The Impact Math Project uses a train-the-trainer model to provide two sets of two-day workshops, inservicing over 600 teachers. A print module for each strand was produced, each containing background on the new content, instructional strategies, assessment method, exemplary activities and sample student work complete with rubrics and sample scoring guides. These activities are based on the notion that professional development opportunities need to change to encourage new learning on the part of teachers.

Preliminary data from 16 case studies of Impact Math workshop participants indicates that significant teacher change has occurred in some cases. Most teachers were especially pleased with the mathematics content information, the student activities and the samples of student work in the modules. A number of instructional strategies as modeled in the workshops were observed in the study classrooms by the researchers. Factors influencing teacher success include teacher beliefs and readiness for change, collaboration with a peer or peers, perceived administrative support, and a willingness to invest time in personal development. Experience in teaching and teaching mathematics plays a role: in some research sites, teachers with less than 10 years of teaching experience were the most open to change. In contrast, a number of very experienced teachers did not want an observer with a research agenda in their classroom.

Complete data from the 16 case studies, together with emergent themes from qualitative software analysis, will be available for the Poster.

Discourse

ADDRESSIVITY TOWARDS THE OTHER IN MATHEMATICAL INTERACTIONS

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Abstract: Mathematics educators continue to devote attention to whole class and small group discourse however, we have often neglected to consider the perceived purpose and expected response of the listener in these interactions. *Addressivity* is defined as the quality with which one turns to and addresses the other in interaction. *Addressivity towards the other* is an observational domain through which to observe and analyze mathematical interactions and to discuss possible implications for mathematics learning within various forms of discourse such as monologues, arguments or conversations.

Introduction: Present reforms in mathematics education continue to emphasize the discursive roles of the teacher and of students as we attempt to move from a 'teaching as telling' model to one that is more student-centred and dialogical. This research explores one aspect of the nature of mathematical interactions and the potential opportunities for learning arising within various forms of discourse. "Addressivity towards the other" arose from this work as a tool for observing mathematical interactions.[1]

Bakhtin (1986) asserts that the quality of "addressivity" within an interaction changes as the genre of communication changes. From the first words in an exchange, we guess at and cast our speech in the genre of the interaction while taking into consideration the actions and utterances as well as our relationship with the other. Rather than a focus on the speaker, as is common in discourse analysis, addressivity also takes the role of the listener into account, thus highlighting the topic of their interaction and the nature and emotional basis of the participants' relationship. Here, addressivity, or the quality of turning to and addressing an 'other,' orients the observer's gaze towards all persons involved in the interaction and provides a means through which to observe distinctions between various forms of mathematical discourse.

Theoretical Framework: The observational domain of *addressivity towards the other* was developed within an enactivist theoretical framework. Enactivism interrelates Maturana and Varela's biological perspective on cognition (Maturana & Varela, 1987; Varela, Thompson, & Rosch, 1991) with Gadamer's 'philosophical hermeneutics' (Gadamer, 1989) and Bakhtin's 'philosophical anthropology' (Bakhtin, 1981) (Bakhtin, 1986). These theoretical and philosophical positions provide a framework from which to observe knowing as it occurs in interaction. From this perspective we as humans are not seen to simply react to the environment around us, nor are we isolated and contained individuals who manipulate our surroundings. Instead, reality and knowing are said to be brought forth on a moment to moment basis through our actions and interactions with others and with the environment. Enactivism suggests that knowing is not stored in the environment or in the head, nor is it something one can appropriate and utilize within a specified time period; rather, knowing is observed as an event that occurs in interaction—at the intersection of personal histories, culture, environment, relationships, emotions, actions and utterances. Understanding and reality then do not preexist the interaction, but come into existence as an event of the interaction.

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[1] This paper is drawn from a larger study investigating the nature of mathematical conversations (see Gordon Calvert, forthcoming).

Addressivity towards the other in mathematical interactions provides a means for considering the interaction as a whole by asking, "What purpose does the other, the listener, appear to serve?" "What is the role of the other in the interaction?" and "What form of response is anticipated or demanded?" These questions may be asked as an outsider to the conversation or asked of oneself as an insider whether one is a student, a teacher or a researcher. We would expect to answer these questions differently depending on genre of communication that the utterances and actions are cast into.

Contrasting Addressivities: Keith and Joanne and Kylie and Tamera were two pairs of undergraduate students who participated in this study and their interactions provide an illustrative contrast of addressivity in mathematical interactions. Each pair was given a prompt which initiated a mathematical exploration. From a teacher's perspective both Keith and Joanne, and Kylie and Tamera may be observed to be working together, talking to one another and sharing explanations; however, the quality of addressivity within the interactions is markedly different when we consider the perceived purpose that the 'other' serves in the interaction.

As Keith and Joanne carried on an investigation initiated by the prompt provided, they were both observed to be expanding their own understandings of the mathematical environment brought forth. Many of their musings and explanations were stated aloud; however, they addressed each other as though speaking to a passive audience resulting occasionally in something akin to dueling monologues. The mathematical explanations provided appeared to be offered primarily to themselves, although it was expected that the other person respond in some way. For example, Keith and Joanne's interactions often included responses of agreement when similar findings and actions were raised; however, there was little attempt to understand actions or explanations different from their own individual conceptions. At one point Joanne offered a description of her activity to Keith who, rather than attempting to understand, said, "That's not what I did," and continued the interaction by offering a description of his own activities. Joanne appeared to listen and upon completion responded by uttering, "Hmmm," and then turned back to her work. They continued to address each other in this way throughout the session. The other person's utterances seldom altered the individual's own course of action and hence, rarely provided direct opportunities for expanding his or her mathematical understanding.

Tamera and Kylie's interaction, on the other hand, offers a contrasting form of addressivity which is more conversational in nature. Tamera and Kylie formed a mutual topic of concern which they addressed together. Actions and explanations were offered to one another to help further their understanding together and it was expected that the other person respond in some way either through questions, clarifications or extensions. There appeared to be an implied responsibility for trying to understand the other person's explanation and for offering an understanding gesture in return. Tamera and Kylie allowed themselves to be 'moved' by the utterances and actions of the other; that is, the potential for expanding their own mathematical understanding was possible as they addressed, listened and responded to one another. It is important to note that this does not imply that the nature of Tamera and Kylie's interaction allowed them to learn "more" mathematics but rather, their opportunities for learning expanded as they purposefully addressed one another in conversation.

Discussion: The brief descriptions of Joanne and Keith's and Tamera and Kylie's interactions provide a basis for understanding addressivity. When we ask, "What purpose does the other appear to serve?" and "What form of response is anticipated or demanded?" we observe contrasting relationships. Joanne and Keith served as a audience for one another, but it appeared to matter little who the audience was or what the audience did or did not understand. Similarly, in some interactions we as the listener are aware that the speaker continues to speak regardless of to whom he or she is speaking. As the 'other', we serve only as a body in the audience; it matters little who we are, as no response or only a minimal response is anticipated. Listening to a classroom lecture or presentation without an opportunity for interaction is an example of addressivity in which the speaker may turn *at* the other, but does not alter his or her actions or utterances by responding to the other. There are also times when a similar addressivity occurs even in more intimate settings such as small group discussions. Occasionally in our interactions with another we intuit that our gestures, or more significantly, we, as participants in the dialogue, are not accepted or even acknowledged by the other. A communication genre in which the other is not acknowledged and no response is expected may be defined as a monologue:

> Monologism, at its extreme, denies the existence outside itself of another consciousness with equal rights and equal responsibilities, another *I* with equal rights (*thou*). With a monologic approach (in its extreme or pure form) another person remains wholly and merely an object of consciousness, and not another consciousness. No response is expected from it that could change everything in the world of my consciousness_. Monologue manages without the other, and therefore to some degree materializes all reality. Monologue pretends to be the *ultimate word*. (Bakhtin, 1984, p. 293)

It was mentioned previously that Keith and Joanne engaged in part in dueling monologues. Both parties spoke without fully listening to the other and neither party then was wholly open to changing their own viewpoint. This form of discourse is a concern when a teacher structures opportunities for mathematical explanations in classrooms without considering the role that the listener plays as perceived by the speaker and the expected response of the listener.

By way of contrast, Tamera and Kylie participated in discourse more conversational in nature. The addressivity within a mathematical conversation is characterized by the quality of conversing with, of turning *with* another to generate explanations for the questions arising through their experiences together. In a mathematical conversation, a person *addresses* the other because he wants to share his present mathematical *understanding* with the other on some mutual topic of concern. It is expected that the other's responses continue and perhaps extend previously stated understandings of the topic (Gordon Calvert, forthcoming). A conversation presents an opportunity for participants to bring ideas into play, thus, putting themselves at risk, and being changed, not by force, but by questioning their own assumptions and opening themselves to the experience. Allowing one's understanding to change and expand in conversation, a characteristic of its addressivity, is an emotional choice that one makes. It is a choice in which each person accepts the existence of another consciousness and allows oneself to be 'moved' by the gestures of the other. It is each person's responsibility in conversation to try to understand the other's explanation and to offer an understanding gesture in return. The sense of responsibility is an ethical imperative implied in the addressivity within mathematical conversations: a person is responsive to, responsible for and respectful of oneself and the other. Emotional satisfaction within the conversation occurs in the perception that the person broadened his or her mathematical understanding through the interaction.

Recent research in mathematics education provides several examples of argumentative discourse in which the other serves a different but visible and purposeful role (e.g., Cobb et al., 1997; Wood, 1999). Addressivity within argumentation is such that persons *address* one another with the intent of attempting to *convince* the other of the truth of their own argument. This form of discourse demands a response from the listener. The response expected and listened for in the other's gestures is either *agreement* or *disagreement* with what has been proposed. The addressivity of argumentation may be characterized by the quality with which participants attempt to compel each other to change viewpoints as well as their listening for and making gestures of agreement or disagreement. Emotional satisfaction is achieved at times when one feels he or she has successfully presented the stronger, more rational argument, which at its best also includes changing the viewpoint of the other person. One parts from such interactions with a sense of victory or defeat having either won or lost the argument.

Concluding Remarks: An awareness of addressivity draws attention to the quality and emotional basis with which one turns to another in a mathematical interaction. When a student speaks, what purpose do the other students or does the teacher appear to serve? When the teacher speaks is he or she talking *at*, talking *to* or talking *with* his or her students? Are the listeners serving as a passive audience, as an opponent or as an ally engaged in the same task of understanding? What form of response is expected and provided by these others? Are they expected to provide the appearance of listening, offer agreement or disagreement through a counter proposal, or to ask for clarifications and offer extensions of what has been offered? The choice one makes is often dependent upon acceptance or negation of the other, as well as acceptance or negation that another viewpoint, another reality, is possible. In establishing social norms in the mathematics classroom attention to addressivity encourages teachers to consider the interactive relationship as a whole.

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UNDERSTANDING THE DEVELOPMENT OF CLASSROOM DISCOURSE THROUGH MATHEMATICAL CONTENT

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The purpose of this research was to investigate how mathematical content influenced the development of mathematical discussions in a college-level course on algebraic reasoning. The research was conducted over the course of one semester in an algebra class for preservice teachers. The instructor facilitated discussions of problem solutions on a daily basis. He encouraged all students to share their ideas and to question others in order to make sense of the mathematics involved in each problem. Verbal data from class sessions as well as interviews were captured in field notes and on audio tapes. Notes and audio tapes were analyzed using qualitative techniques of coding and pattern finding to determine the influence of the mathematical content on the nature of the discussions. In general, the content appeared to have some influence on the students' ability to reason algebraically. Even so, the development of the discourse over the semester seemed to be more closely linked to the students' growing sense of what was expected of them in terms of verbally justifying solutions and less connected to the mathematical content.

Introduction

Communication continues to be a central theme in the reform of mathematics education (National Council of Teachers of Mathematics [NCTM], 1989, 1998). In particular, many mathematics education researchers and advocates of the current reform efforts support the view put forth by NCTM that mathematics classrooms where students express ideas, challenge the ideas of others, and present convincing arguments are classrooms that facilitate students' development of mathematical concepts (Cobb, Yackel, & Wood, 1992; Hiebert et al., 1997; Lo, Wheatly, & Smith, 1994; NCTM, 1998; Pimm, 1987). Even so, there are many questions still to be answered about the role of discourse in the classroom and about the factors that contribute to the development of mathematical discourse (Ball, 1991; McCrone, 1997; Silver & Smith, 1996).

Mathematics is commonly regarded as an individually and socially constructed body of knowledge, and a specialized language for communicating about many aspects of our world (Cobb et al., 1992; Pimm, 1987). A constructivist theory of learning, the belief that the individual learner must actively construct her or his own understanding of the ideas with which they interact, complements this view of the nature of mathematics. Taken together, new mathematical knowledge (individual or shared) is constructed "through interactions and conversations between individuals and their community" (Corwin & Storeygard, 1995, p. 7). Thus, movement among one's personal sense of a concept and the taken-as-shared mathematical meaning of the concept is crucial for learning to take place (Bartolini Bussi, 1998). The teacher's and students' roles in this movement help to determine the learning that occurs. This view of the teaching and learning process emphasizes the importance of classroom interactions and the mathematical content being discussed. Hence, research into the teaching and learning of mathematics must investigate the nature of these interactions and how the mathematical content (or the taken-as-shared meaning of concepts) influences the development of discussions.

For the purposes of this research, I chose to focus on the mathematical content and the role content plays in determining the nature of interactions and the development of mathematical discussions . The study took place in a one semester (four month) college-level mathematics class for preservice teachers. The rationale for investigating this student population is that if preservice teachers experience success in discussion-based mathematics classes, they are more likely to establish similar learning environments once they become teachers. Since communication is thought to be an important process for the learning of mathematics, research on factors that influence students' abilities to communicate (particularly students who will soon be teaching the children of our communities) is crucial and timely.

Methods

Participants and Setting

The study participants were a university professor and students enrolled in a course on mathematical problem solving which focused on algebraic reasoning. This course is typically the first in a series of mathematics content courses for students intending to specialize in mathematics education at the elementary school or middle school level. The students were primarily in their second or third year at the university, and considered themselves to be good mathematics students since the majority had obtained fairly high grades in mathematics courses during high school. The mathematical content of this course was similar to the content encountered in a first algebra course at the high school. Although it was not necessarily new content for the students, it was presented within the context of problems which required a variety of reasoning techniques. This class format was new to most of the students.

One of the main goals of this course on problem solving and algebraic reasoning was to help students make sense of the mathematics they had encountered in high school in order to develop a deeper understanding of what it means to reason algebraically. In order to encourage students to make sense of the mathematics, they were expected to investigate problem situations, make conjectures, communicate ideas with classmates, and justify their solutions. Thus, the focus was not necessarily on the answers to the given problems but on the justification for the solution method.

Features of Classroom Interactions

Students were expected to work on problems outside of class time (independently or with classmates) and be prepared to share solutions and justifications during class time. The instructor encouraged students to work with ideas familiar to them in order to build on their prior knowledge and make sense of the concepts and procedures encountered in solving the problems. He also encouraged students to work together since he believed that sharing ideas would help students to be successful in solving the problems and that students could offer each other alternative methods to consider.

The instructor's interactions with students did not follow a standard initiation-response-evaluation (IRE) interaction pattern. Instead the instructor invited students to share solutions, to express ideas and to respond to each other's contributions. In fact, it was typical for the instructor to answer a student's question with a question in order to invite others to respond. Through questioning and encouraging students to question each other, the instructor encouraged students to take responsibility for making sense of the mathematics, for making sense of each other's ideas, and for determining the validity of solution methods.

Data Collection

Data collected consisted of audio tapes of classroom discussions throughout the semester, as well as field notes to supplement the audio tapes. Samples of student work were recorded as problem solutions were presented to the class. In addition, several students were interviewed at a few points during the semester to clarify their understanding of what was being discussed in class (mathematical content) and their sense of the usefulness of the discussion for making sense of the mathematics (discussion content). The problems assigned to the students were collected and the mathematical content of the problems was analyzed.

Organizing the Analysis

The mathematical content of assigned problems was coded by two researchers and recorded in a matrix. For the most part, the researchers' agreed on the nature of the content in the problems. The class instructor also aided in determining the mathematical concepts and procedures he felt students were encountering in the problems. Some of the concepts and ideas explored over the semester included: the concept of a variable; pictorial, tabular and graphical representations of problem situations; linear and non-linear equations and functions; direct and inverse variation; division of fractions; problem solving techniques.

The instructor's and the students' contributions to the discussion of problems were coded in two ways. First, this verbal data were coded according to the matrix of mathematical concepts and ideas. Second, the data were coded in order to identify the nature of discussions. Codes were analyzed to determine patterns in the ways students and the instructor interacted during whole class discussions. These patterns of interactions were checked against the concept codes in order to determine whether particular mathematical concepts influenced the discussions.

Results

Although the data analysis is not complete at this time, preliminary analyses indicate that mathematical content did influence students' contributions to the whole class discussions for at least some of the students. More specifically, the content appeared to influence the students' ability to reason algebraically. In addition, on occasion, the instructor structured the discourse in unique ways when the mathematical concepts appeared to cause difficulties for the students. Even so, the development of the discourse over the semester seemed to be more closely linked to the students' growing sense of what was expected of them in terms of the discussions and less connected to the mathematical content. For instance, across the semester the students' questions to the teacher were more likely to be questions about what he would accept as a valid argument rather than questions about the concepts and procedures that framed the argument. However, as the semester progressed, students and the instructor participated in discussions that were more focused on the mathematical concepts of the problems, suggesting that students had come to know what was expected of them and most were using discussions as opportunities to make sense of the mathematics.

A conversation is included below with some of the analysis and interpretation that illustrates the results noted above.

Example 1

The dialogue that follows occurred early in the semester. The problem being discussed asked students to determine the number of pupils in Class B if the average test score of Class A, with 20 pupils, was 80%, the average test score of Class B was 70%, and the average for both classes was 74%. A solution to this problem required students to write an algebraic equation with the unknown quantity represented as a variable. A student, Terry, presented two possible ways of expressing the problem situation but most students could not make sense of her reasoning, as evidenced by silence when the instructor asked someone to paraphrase her work. Since students were having difficulties writing such an equation or making sense of the equations presented, the instructor asked students to consider a simpler problem: "If 20 students took a quiz worth 50 points, how would you calculate the average quiz score as a percentage of the total points possible?"

Instructor: What would you do?

Diana: You add all the scores, multiply 50 by 20, then divide by x. X is the sum of all the scores.

Instructor: Would that give me average percent? Will this give me the class average?

Elaine: No.

Danielle: I don't think that's what it is... (inaudible).

Instructor: You're not sure Diana's method will give the correct number? Cynthia?

Cynthia: Take all the scores obtained...added together... divide by the number of scores. Take that and divide by the number of possible points.

Terry: Isn't that the same as Diana's?

Cynthia: I don't know.

Instructor: How can we find out?

Terry: Do an example.

The instructor agrees with Terry's suggestion and asks students to work with a neighbor or independently to try a few examples. He suggests they start with a number smaller than 20, such as 6 quiz scores.

One possible solution to the modified problem in example 1 required students to write an algebraic expression with a variable. In fact, both Diana's and Cynthia's suggestions contain unknown values or variables. However, whereas the original problem would require that students reason algebraically, the modified problem required only arithmetic reasoning. That is, the modified problem did not require that students discuss relationships between numbers or operations on numbers (Esty & Teppo, 1996). Rather, the simpler problem could be solved and justified by considering how to calculate a number. Since the discussion of the original problem brought silence while the modified problem seemed to elicit discussion, one can see that the content of the problems did influence the nature of the discussion.

Multiple examples of class discussions over the course of the semester can best illustrate the development of discussions. With limited space in this document, one can read into the example above and begin to notice that by this point in the semester, students had begun to follow the instructor's lead in questioning each other's suggestions (Terry's question to the instructor/class after Cynthia described an alternate solution), in building on the ideas of others, and in finding ways to make sense of the suggestions offered (trying a simpler example). More examples and analyses will be provided in a paper accompanying the PME-NA research report.

Discussion

What was most interesting in the data were the patterns of discussion that evolved over the course of the semester as students in this class learned how to present a valid argument for justifying a problem solution. This development appeared to be tied less to the mathematical content of the problems and more to natural growth of a taken-as-shared understanding about what it means to reason algebraically and about what counts as a mathematical justification.

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THE NATURE OF MATHEMATICAL DISCOURSE IN TWO UNEVENLY SUCCESSFUL STUDENT-CENTERED ELEMENTARY CLASSROOMS

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This poster is based on the part of a cross-cultural investigation of how teachers grapple with their own values and priorities relative to the similar reform ideals promoted for the profession in different national contexts. The purpose of this poster is to compare and contrast the classroom social norms of two U.S. second grade teachers who are implementing student-centered approaches.

After preliminary observations of 17 classrooms recommended, two classes were selected because of their inequality in the extent to which students' ideas were solicited and became the center of mathematical discourse. A total of 15 math lessons were video- and audio-taped. A total of 8 hour interview with each teacher was conducted as an attempt to explore mainly how she has been developing her own teaching approaches. Additional data included students' papers and projects.

The video-and audio-taped lessons were transcribed and analyzed using the grounded theory approach based on constant comparative methods. A preliminary analysis shows that the two classrooms constitute similar microculture wherein the teacher promotes communications about math and the students explain and justify their ideas. Despite the similarity of social practices, a close examination of mathematical discourse in the two classrooms shows dramatic differences with regard to the teachers' use of students' ideas. One teacher focuses on a pre-given formal algorithm after eliciting students' ideas, whereas the other teacher carefully orchestrates the path of discourse towards conceptual understanding. The successes and difficulties of the two teachers enable us to explore the challenges and issues in reply to: implementing current math education reform ideals.

ACCOUNTABLE ARGUMENTATION AS A PARTICIPANT STRUCTURE TO SUPPORT MATHEMATICAL LEARNING THROUGH DISAGREEMENT

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Recent emphasis on discourse in mathematics classrooms (e.g. NCTM 1991) has spurred a line of inquiry about different forms of talk in these settings. If mathematical thinking is understood to be a set of practices which include mathematical discourse, argumentation and the use of representations in argumentation require our analytic attention.

This paper investigates the interactional organization of public disagreements in Deborah Ball's third grade classroom by describing a participant structure called *accountable argumentation*. The norms, expectations, interactional roles, and uses of representations employed during accountable argumentation are explicated and then applied to the analysis of two public peer disagreement episodes that take place during the same whole-class discussion.

During the discussion, the evenness or oddness of different integers are called into question. The first disagreement episode illustrates some of the ways in which students employ representations in argumentation to position themselves and explicate their thinking during a controversy about the evenness or oddness of zero. The second disagreement episode illustrates how students use representations in argumentation to challenge others' positions and create mathematical generalizations during a controversy about the evenness or oddness of the number six.

The analysis of these two episodes illustrates the ways in which accountable argumentation supports mathematical learning through disagreement by providing students with the interactional resources to (a) manage the potentially personal feelings of disagreement; (b) articulate their mathematical reasoning through talk and over representations; and (c) produce mathematical generalizations.

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Functions and Graphs

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INVESTIGATING THE RELATIONSHIPS BETWEEN SUBJECT MATTER AND PEDAGOGICAL CONTENT KNOWLEDGE OF FUNCTIONS: CASE STUDIES OF TWO PRESERVICE SECONDARY TEACHERS

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This study investigated the relationship between preservice secondary mathematics teachers' subject matter knowledge of functions and their use of this knowledge in instructional planning. Significant factors included the critical role played by missing content knowledge, the effect of individual competencies and preferences for specific representational forms, and the ability to create illustrative examples of functions. These results are relevant to an overall characterization of preservice secondary mathematics teachers' knowledge of functions and have implications for preparing teachers to implement current reforms.

The purpose of this study was to examine the relationship between preservice secondary teachers' subject matter and pedagogical content knowledge of functions. The study characterizes several factors that played a significant role in this relationship, and subsequently, in the development of the in-depth knowledge of functions necessary for preservice secondary teachers to implement current recommendations for teaching mathematics (NCTM, 1989).

Research related to teacher knowledge, knowledge of functions, the structure of knowledge, and the interaction between knowledge and teachers' actions was incorporated into the framework of this study. Shulman's (1986) model of teacher knowledge, in which the organization of subject matter knowledge and the relationship between subject matter and pedagogical content knowledge play essential roles, served as the primary basis for examining subjects' knowledge of functions. In regard to specific knowledge of functions, results indicate many prospective teachers lack a deep understanding of the concept, and differences exist between prospective teachers' conceptions of functions and their thinking about teaching functions. (Even, 1993; Wilson, 1994). The use of concept maps as one means of assessing the organization of subject matter knowledge was based on a model for analyzing issues of learning with understanding and teaching for understanding (Hiebert & Carpenter, 1992). Within this model, internal representations of knowledge are viewed as vertical hierarchies in which some representations subsume other representations, or as web-like arrangements where pieces of information and corresponding relationships between the items of information form simple linear chains or complex networks. The final aspect of the study was based on the assumption "that an individual's knowledge structures and mental representations of the world play a central role in that individual's perceptions, thoughts, and actions" (Brown & Borko, 1992, p. 211). This idea is consistent with Sfard's (1991) assertions regarding the dual nature of functions and is supported by Fennema and Franke's (1992) work on the conceptual nature of teachers' knowledge.

Methodology

This study was designed to provide a view of preservice teachers' content knowledge of functions, resulting in two in-depth case studies drawn from the interviews of eight secondary mathematics education majors. Assessment instruments included concept maps, a written survey of function knowledge, individual interviews, and card sort activities. Data interpretation was qualitative in nature in order to provide an in-depth description of subjects' knowledge. Initial interpretation relied on essential aspects of functional knowledge, as well as additional features that emerged during data collection and analysis. Analysis focused on distinguishing general trends within the subjects' responses, individually and as a group, and detecting evidence of two dimensions of the function concept, concept image and connections. Data from each instrument were examined with respect to these dimensions, facilitating the comparison between results obtained on different instruments.

Results

Three aspects of content knowledge emerged as critical components in the characterization of the relationship between subject matter knowledge and the use of this knowledge in instructional planning: (1) missing content knowledge; (2) individual competencies and preferences for specific representational forms; and (3) familiarity with appropriate, illustrative examples of functions. The abbreviated case studies that follow provide valuable insights into these factors and the potential effectiveness of preservice secondary teachers' implementation of current recommendations for teaching mathematics. Similar trends were also evident in the responses of the remaining subjects.

Jill: Subject Matter Knowledge

Jill viewed functions as rules, or special relations, determined by a univalent correspondence between two sets. She was the only subject who asserted that all functions are not equations. Her concept maps incorporated a large number of terms that were moderately to extensively interconnected. The map using self-generated terms was organized around five main clusters: different representations of functions, the definition of a function, graphical properties of some functions, families of functions, and operations performed on functions. The directed map, based on predetermined terms, was composed of four similar clusters. Jill demonstrated a high level of competency with all four representational forms used in the card sort activities, easily sorting the functions according to underlying features as well as with respect to representational form.

Jill: Instructional Planning

A high degree of consistency was demonstrated between Jill's conception of functions and the sequencing of topics in an introductory lesson on functions. She indicated she would use her previously stated definitions of a function, "a relation in which each element of the domain is paired up with *exactly* one element of the range" and "a relation in which the *x*-values (domain) cannot 'repeat' at any point in the relation," with students. This perspective was reflected when she proposed introducing students to functions informally, and then immediately emphasizing the connection between the rule or the mapping and the formal definition. Jill introduced several topics prior to direct questioning— the vertical and horizontal line test, one-to-one, the relationship of the graph of a function and its inverse, and transforming graphs globally. She consistently returned to the underlying meaning of functions and attempted to connect related topics.

Missing knowledge. There were several instances when the nature of Jill's content knowledge appeared to negatively influence the proposed lessons. For example, a description of equations having "one variable" and functions having "two variables" appeared to hinder her ability to relate solving a quadratic equation to the corresponding function. However, when shown a quadratic equation, Jill was able to discuss the relationship between the zeros of the graph and the roots of the equation, and subsequently outlined several key points in a lesson emphasizing the relationship between a function and its graph. An ability to self-correct was also evident when discussing the properties of univalence and one-to-one. Although Jill had a solid conceptual understanding of these properties, at times she had to rephrase responses to clarify her intended meaning.

Individual competencies and preferences for representational forms. A variety of models and representations of functions were utilized when describing how to teach different topics. Jill consistently strove to establish the interrelatedness of these various interpretations as she moved from one representation to another. This ability to view functions in a variety of representational forms allowed her to present students with several methods of identifying functions.

Jill was aware of a tendency to approach functions graphically. However, when asked how one decides which representational form to use when solving a problem, she stated that it depends on "what I want to show. For inverses, I would choose a mapping or a graph. To show function operations, I would choose the equations. To test if the function is one-to-one, I would choose a graph, and so on." In several instances, her first approach was graphical, but this was followed by an alternate explanation that utilized a table of values, ordered pairs, or an equation.

Use of examples. Although cognizant of the benefits of generating real world examples and generally successful when identifying the type of function represented in a real world situation, Jill was initially unable to generate real world examples. This inability bothered her. As a result during the final interview session, she eagerly reported she had thought of several real world examples. Jill did use her integrated knowledge-base to consistently generate appropriate algebraic and graphical examples for the proposed lessons and was sufficiently familiar with the functions listed on her self-generated concept map to incorporate them into the proposed lessons appropriately. Other aspects of an ability to generate examples of functions were evident as she described an introductory lesson on functions in which a table with real data and counterexamples were effectively used to illustrate the defining characteristics of a function.

Marie: Subject Matter Knowledge

Marie's conception a function was a machine that carried out a certain predetermined rule or formula. Although she stated that functions are univalent, this property was not always used when identifying functions. Her self-generated map consisted of four interconnected clusters of 12 terms; *function* was surrounded by a line labeled "can be thought of in many ways." The closed chains "can be a machine that for every input value [it] spits out exactly one output", "can be a correspondence between sets", "can be a mapping [which] can be one-to-one [or] onto", and "can be a rule that assigns each element in the domain exactly one element in the range" emanated from the central term. The directed map was less integrated, as evidenced by the networks of relatively short chains and the lack of cross-links. The groupings on this map appeared to indicate that she was familiar with the terms, but did not have a solid conceptual understanding of their meaning or how they were related. A moderate amount of difficulty was experienced during the card sort activities. Although several groupings were considered (e.g., easy versus difficult, linear versus nonlinear, and representational form), the final groups included linear, geometric, exponential, simple problems, messy problems, and a table that represented a cubic function.

Marie: Instructional Planning

Marie consistently used the rule-based definition, "a machine which spits out one output value for every input value" and repeatedly used the terms "function" and "equation" interchangeably. This analogy was heavily relied on when applying her knowledge within the classroom scenarios. Due to the erratic and somewhat unconnected nature of her knowledge, Marie frequently became confused while describing how she would respond to the situations depicted in the classroom scenarios. Frequently she had to rephrase explanations in order to clarify or correct her intended meanings, and she tended to speak in generalities rather than citing specific examples.

Missing knowledge. The proposed lesson on quadratic functions was characterized by a lack of connectedness between algebraic and graphical representations; Marie did not mention connecting the roots of a quadratic equation with the zeros of the function. When questioned further about this, she saw no relationship between these ideas. She subsequently had difficulty outlining key points for a lesson emphasizing the relationship between a function and its graph. While Marie stressed the importance of exposing students to completing the square and analyzing how the parameters of a quadratic function affects the graph, no specific details how this would be done could be given. This lack of specificity was consistent with her own lack of proficiency with these aspects of working with functions.

The effect of a somewhat unintegrated knowledge-base was also evident in responses dealing with inverse functions. Although she could find the inverse of an exponential function algebraically and indicated the graph of an inverse was a reflection over the line $y = x$, this information was not used to determine the general shape of the graph of a logarithmic function. This lack of integration was reflected in the proposed lessons on inverses and the relationship between the graph of a function and its inverse.

Individual competencies and preferences for representational forms. Marie continually stated she was comfortable and proficient with both algebraic and graphical representations of functions, but preferred the graphical. This tendency was evident in many responses related to instructional planning where she relied primarily on graphical examples to illustrate different concepts. Despite repeated statements of proficiency, she did not always integrate these two representational forms.

A graphing calculator was the preferred method for approaching graphs; Marie consistently and strongly recommended its use in the classroom. She indicated a lesson would ideally begin with a graphical example, and then move to an algebraic representation; limited use was made of tables or real world situations. Although she discussed the importance of transforming graphs in the proposed lessons, a high degree of facility when working with graphs was not demonstrated.

Use of examples. Marie's limited use of specific functions to illustrate key points in the proposed lessons was consistent with her own difficulties identifying functions, sorting families of functions, and translating between representational forms. She had a vague graphical image of routine functions, but lacked the ability to consistently generate illustrative examples in the lessons. Based on the real world problem suggested, "think[ing] about just adding 3 areas or something together...like one has an area of ax^2, one has bx, and one has c....," it appeared Marie was confusing the concept of area with the form of a quadratic equation. This clearly indicated a lack of in-depth conceptual understanding.

Summary

Varying degrees of functional knowledge were evident throughout the study; there was a tendency to view many related topics as relatively isolated pieces of information. Formally stated definitions, some of which subjects contended they would use with students, were not always consistent with the justifications they provided when identifying functions. While sufficient knowledge of the procedural aspects of working with functions was usually demonstrated, this knowledge was not always onnected to the underlying concept. This lack of connectedness appeared to affect subjects' ability to plan lessons that focus on the central role of functions in mathematics.

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REVEALING PRE-SERVICE TEACHERS' THINKING ABOUT FUNCTIONS THROUGH CONCEPT MAPPING

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In this study, concept maps were used as an expressive representational system to reveal pre-service teachers' thinking about functions. We do not make any assumptions that the concept map contains a direct, one-to-one relationship to any internal representations that an individual might have. Rather, we see the map as a means of communicating a way of looking at the structure and patterns of relationships. We found that the teachers in this study showed a preference for non-hierarchical maps for the function concept and that they shifted from maps that had the multiple representations of functions as disconnected to linked. Few teachers showed ideas about change and variation or the use of technology in their initial concept maps. Most teachers integrated these concepts into their final maps on the concept of function. A significant shift occurred with the inclusion of nodes related to pedagogical strategies and student understandings in the teachers final concept maps.

Since the seminal work of Novak in the early seventies, concept mapping has been widely used in science education, with particular emphasis in the biological sciences (Horton et al., 1993; Novak & Gowin, 1984). Numerous researchers have studied the use of concept mapping as an instructional tool with learners at age levels from early elementary to post secondary (Al-Kunifed & Wandersee, 1990). Some researchers have looked at concept mapping as a research tool or an assessment instrument for measuring conceptual understanding (Markham, Mintzes & Jones, 1994; Williams, 1998). Others have explored the effectiveness of concept mapping in the preparation of pre-service teachers (Beyerbach & Smith, 1990; Roth & Roychoudhury, 1993). However, there is a somewhat surprising paucity of research on these uses of concept maps within the traditions of mathematics education research. In this study, we investigate and describe how concept maps were used in pre-service mathematics teacher education to support the development of pedagogical and mathematical insights into the concept of function.

Theoretical Framework

Within the theory of meaningful learning (Novak & Gowin, 1984), concept maps have served as a tool to support learners as they form meaningful connections between concepts, subsume detailed experiences or examples into more structured hierarchies, and differentiate between vague or weakly formed concepts. A key characteristic of concept maps is that they express a particular learner's ideas and hence they are to some extent idiosyncratic to the individual. Nonetheless, within the tradition of science education, there has been a trend towards using maps to evaluate an individual's understanding by various systems of scoring concept maps. Such scoring systems are most often a variation of a method which judges a map according to the number of central concepts, usually linked in a hierarchical way to an overriding concept. In addition, the number of scientifically valid links to sub-concepts or examples and the number of cross links connecting concepts are usually tallied. Those maps that are more highly linked (especially cross linked between more major concepts) are considered to reflect a higher level of conceptual understanding (Novak & Gowin, 1984). In addition to investigating the use of hierarchically organized maps, researchers have focused on maps that are organized as webs or networks. These maps are often characterized by several clusters of main ideas which are linked to each other and to their sub-concepts. The nodes on these webbed maps can be highly interconnected with uni-directional and bi-directional links among sub-concepts. Both hierarchical and webbed maps tend to reflect a view of knowledge structures that emphasizes the importance of the interconnections within a learner's conceptual understanding.

The appearance of hierarchical and webbed maps as static representations is in stark contrast to the dynamic processes by which they are created. Learners tend to want to re-organize their maps as they are being drawn. Ideas and relationships which may be fuzzy, unclear or unnamed need to be explicitly located and linked within the representation. This suggests that learners are actively adding new links, deleting or reconstructing sections of a map, or discarding the current map and re-starting with a new structure. This dynamic interaction between the learner and the map is particularly evident as individuals are observed drawing maps, as groups create maps to reflect a shared understanding, and as learners give oral or written reflections about their maps. We have focused on using the concept map as an expressive representational system for supporting the development of teachers' conceptual understanding of function. The creation of a concept map is seen as a means for supporting teachers' efforts to express and refine their mathematical thinking and to see new patterns and relationships among mathematical ideas. The construction of the map pushes one to reveal one's thinking to oneself and to others and potentially to revise one's views about the concept being mapped.

Methods and Data Sources

The subjects for this study were pre-service secondary mathematics teachers enrolled in one of two technology-based courses taught by each of the authors of this paper. The courses were taught at two different universities with the goal of supporting the development of teachers' conceptual understanding of functions and rate of change. The course material was designed to engage pre-service teachers in several instructional sequences designed to challenge their existing, secure knowledge about the concept of function and the mathematics of change through experiential and graphical ways and to evoke models of how others might learn these ideas. The teachers were asked to create (a) pre- and post- course concept maps and accompanying interpretative essays on their concept of function; (b) group concept maps on the concept of function; and (c) individual and group concept maps about a learning activity on the relationship between position and velocity. In this paper, we focus on the individual pre- and post-course concept maps and interpretative essays that were completed by 11 subjects enrolled in one of the courses. All but three teachers were familiar with concept mapping from previous course work or learning experiences. The task of map construction was deliberately posed in an open-ended way, emphasizing that the teachers were free to structure their maps in any way that seemed reasonable to them.

The data analysis involved three phases to ensure that the descriptions and features we identified were empirically grounded in the data corpus and could be substantiated by back-tracking to the teachers' concept maps and essays. In the first phase, we identified the overall structure and most central features in the teachers' pre-course concept maps by analyzing copies of all maps. The second phase involved comparing, contrasting, and elaborating each of these features in light of the teachers' essays. This served to differentiate concepts that could be used to describe, at least in part, the teachers' thinking as expressed in their concept maps and to elucidate the relationship between their written descriptions and the more structural, connectedness of the concept map. The third phase of the analysis focused explicitly on the differences between individuals pre- and post-course concept maps. In this phase, we sought to discern any structural changes that may have occurred within the individual teacher's concept maps and to identify changes in the nodes that were included in the maps and in the connections among nodes.

Results

The pre-course concept maps created by the pre-service teachers were structurally of two types. The first type of map was characterized by a central idea (usually simply the word "function") with a relatively large number of subconcepts directly related to function, but with few details in the subconcepts. These maps were often structurally similar to the central hub of a wheel with subconcepts at the ends of the spokes. There was limited interconnectivity among the subconcepts. Three of the maps were of this type. The second type of map was more of an amorphous web with several clusters of subconcepts linked to the concept of function. These maps exhibited a higher degree of interconnectivity between the clusters. Eight maps were of this type. We observed that no teachers constructed hierarchically organized maps flowing from top to bottom. This could be in part an artifact of the web-based illustrations that were given in class. However, since most of the teachers had prior experience with concept maps, we consider the preference for web-based maps at least in part a reflection that their thinking about the concept of function is non-hierarchical and non-linear.

The post-course concept maps created by the pre-service teachers maps showed two kinds of shifts from their initials maps. Of the three teachers who created spoke-like maps, two of them dramatically changed their final maps to more web-like structures. While still keeping most of their subconcepts directly linked to the main concept, these new maps showed a higher degree of interconnectivity, a larger number of subconcepts, and more structure in the relationships among the subconcepts. Of those teachers who began with more web-like structures, we found that the significant changes were not in the overall structure of the maps, but rather in the new and changed relationships that were represented in the final map.

On their initial maps, all but one pre-service teacher included some ideas about multiple representations (tables, graphs, symbols, words) of functions. For eight of the teachers, however, these representations were not linked to each other, but only to the concept of "representations" or "function." On their final map, half of these teachers now expressed a linked relationship between various representations of a function. Only two teachers showed ideas about rate of change or variation in their initial maps. On the final maps, six teachers had integrated these concepts into their overall conception of function, as expressed in their concept map. None of the pre-service teachers had seen technology as an initial aspect of their function concept; four of the teachers had technology nodes in their final maps.

A major shift in the concepts represented from the initial to the final maps was in relationship to ideas about teaching and learning about functions. Only one teacher had two single nodes in his initial map that were related to pedagogical strategies or student understandings. However, on the final maps, nine of the eleven teachers had added multiple new nodes explicitly related to teaching about functions and learning issues for students. This suggests that prior to this particular course, pre-service teachers' views about the concept of function were largely disconnected from any pedagogical strategies or learning paths or obstacles that students might encounter. The teachers' final concept maps indicated that teaching and learning issues were now linked to their mathematical understanding of the function concept.

Conclusions

The concept map as used in this study was seen primarily as a tool for expressing pre-service teachers' ideas about the concept of function. The concept map has the advantage of revealing concepts and their interrelationships both to the person who makes the map and to others who might read, interpret, share and discuss the map. We do not make any assumptions that the concept map contains a direct, one-to-one relationship to any internal representations that an individual might have. Rather, we see the map as a means of communicating a way of looking at the structure and patterns of relationships. In this study, we found that pre-service teachers showed a strong preference for non-hierarchical, web-like maps for expressing their understandings. In addition, simpler spoke-like maps later became modified into more web-like clusters. Significant numbers of teachers changed their concept maps to include linked nodes for multiple representations, for the concept of variation, and for the use of technology. Nearly all the pre-service teachers in this study integrated ideas of teaching and learning into their concept maps of function during a one-semester course of study.

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CONNECTIONS BETWEN DIFFERENT MATHEMATICAL DOMAINS USING TECHNOLOGICAL TOOLS: THE ANALYTICAL CHARACTER OF THE ALGEBRAIC TASK-RESOLUTION

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Abstract. This paper presents a teaching experiment with 10th grade students in a French Licée. The mathematical content approached was the resolution of rational algebraic inequalities by designing and to set up an algebra-learning scenario using the Cabri-II microworld. The activities included were the modeling of curves, as well as of graphical representation of algebraic expressions. We obtained results that point out the scenario was useful to analitical resolution of inequalities since a comparison and revision by the students of the every resolution procedures was observed. However, perhaps the most important result in this experimentation was to have sign that this scenario was "*supporting the forging of connections across domains*" (Noss & Hoyles, 1996), by promoting the understandig of dependence between variables —a essential character of the function— probably to connect the others more abstract or different contexts than algebraic inequalities.

Introduction

Abstraction, as a fundamental characteristic of mathematics, is a factor which is present in the school mathematics contents, and which motivates the didactical search. In the words of presentation of the EIAH[1] team, data processing opens an original access-way to artificial realities which have the peculiarity of materializing real phenomena by means of mathematics itself. In the Mexican basic education system, algebra is perhaps the subject to which such an abstraction feature is more extensively applied. In particular, research findings[2] (Hoyos, V., 1996) reveal that, in general, upon finishing the basic Mexican mathematical education on this subject -and included here are the two first years of the Mexican high-school-, the students do not possess the mathematical resources permitting them to control the algebraic executions which would be usual at this educational level, such as the solving of equations, the algebraic transformations by factoring or by reduction to similar terms, etc.

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[1] EIAH stands for the French research team *Environnements Informatiques pour l'Apprentissage Humain*, which is located at the Leibniz Laboratory of the Applied Mathematics Institute, Grenoble (IMAG), France; within EIAH, this team has developped the *Cabri-II* microworld , a data-processing environment of interactive learning.

[2] For instance, in one of Hoyos' (1996) case studies it becomes apparent that in spite of the subject being a high-stratum student who is taking analytical geometry in the second year of the Mexican baccalaureate, he fails when trying to effect an algebraic reduction which is, however, usual at this school level. One tries to

The large number of findings that have been reported about the difficulties on the learning of algebra, on the construction of meaning around symbols, and on the development of an algebraic thinking attest to an interest on the part of the international community of researchers in mathematical education, in this respect. In this paper, particularly, an attempt is made to contribute to this reserch current by presenting some of the results from a teaching experiment which was carried out at a French Licée -with 10th grade students-, with the set up, and the experimentation with an algebra-learning scenario using the Cabri-II microworld. This scenario was structured around the resolution of rational algebraic inequalities of the type $1 \div x < 3 \div (x+1)$, on the basis of two global, meta-level, mathematical activities -in the sense introduced by Kieran (1996)-: (a) modeling a curve obtained by the simulation of Descartes' machine as a Cabri-II diagram (Hoyos, Capponi, & Geneves 1998); and (b) the graphical representation of algebraic expressions within Cabri-II microworld, too.

The resolution of algebraic inequalities is a usual school task among 10th graders at the French Licée level, and it demands something more than a blind manipulation of symbols and/or algebraic literals.

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show with this, that students do not achieve forms of control concerning algebraic manipulation:

EP: Look, I'm going to draw this straight line... (EP points to equation 2x + y = 16)... This then would be... (EP writes first 2x + y = 16, and right away he obtains: 2x - 16 = y)...

Let's see... If x equals 1 (EP constructs a two-colum table, one for *x* and another for *y*, and in the *x*'s column he writes number 1)... When *x* is one... Four minus... All right, if this is two...

(EP is referring to the *x* column, and he writes)... four minus sixteen, that is, well... here it would be minus fourteen...

If *x* equals 2, four, and... that would be minus sixteen...

(EP writes -12 under the *y*-column)... this would be... *x* and *y* over here

(EP also marks point (2, -12), and in this way he draws the straight line passing through (1, -14) and (2, -12), thus drawing a straight line which does not correspond to the given equation 2x + y = 16, which was initially mentioned by EP).

Objectives of the Research

The main assumptions which guided the design of the learning scenario, were: (a) that the generation of expressions and equations by the modeling of the curves issued from the simulation of drawing machines on Cabri-II microworld would be a kind of mathematical experience which will allow the student to establish connections between the systems of representation involved; and (b) that the graphical representation of rational algebraic expressions Q(x) within Cabri-II would permit the dynamic perception of the functional dependence between x and Q(x). Cabri-II diagrams provide the visualization of general relations between the independent and dependent variables. This general relation is a fundamental characteristic for functional comprehension, and a mathematical element which is essential for the resolution of algebraic inequalities, that what is the subjet matter of this study.

It was expected, particularly, that the performing of activities with Cabri-II would promote an analytical reflexion[3] during the resolution of algebraic tasks which are usual at this school level; in this way, students acquire the control means that would particularly allow them to improve their algebraic executions in this respect, considering that such executions are usually a failure at this school level.

Elements of the Theoretical Framework

The analysis of algebraic activity unfolded in the context of school algebra (Kieran, 1996) leads to three types of activities based on the use of algebra as a tool: activities in the domain of the generation of expressions and equations; transformation activities, based on the rules of algebra such as the reduction to similar terms, factorization, equation solving, etc.; and global mathematical activities of a meta-level, such as problem solving, modeling, finding a structure, justifying, proving, and predicting. Kieran recognizes that the latter activities are not exclusive of algebra, and that they could even be approached without the use of algebra: *they are, in fact, no part of algebra. But attempting to divorce these meta-level activities from algebra removes any context or need that one might have for using algebra* (Kieran, 1996, p. 2). From our viewpoint, all seems to indicate that the activity of the resolution of an algebraic rational inequality of the type we chose for our teaching experiment, pertains to the meta-level kind of global mathematical activities.

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[3] Evidence of the accomplishment of the aforementioned analytical refelexion, would be given by "the use of any of a variety of representations in order to handle quantitative situations in a relational way" (Kieran, 1996), in the way to solve the rational algebraic inequalities. That would identific this task as proper of "algebraic thinking" as defined by Kieran (1996, p. 3).

On the other hand, the reasons for choosing the set up of learning scenarios in Cabri-II were the characteristics of Cabri-II microworld of direct manipulation, which, when interacting with the student, promote the perception of invariant geometrical properties: *nous faisons l'hypothèse que l'ordinateur peut guider l'élève dans l'exploration de dessins en l'incitant à repérer des invariants entre plusieurs dessins, invariants pouvant déboucher sur la mise en évidence de notions et propriétés géométriques générales. En effet,... il peut proposer plusieurs dessins répresentant un même ensemble de donnés théoriques*[4] (Bellemain, 1992).

Here, we are conceiving the computer (Noss & Hoyles, 1996) as integrating a system of educational support in which knowledge, the student(s), the teacher, and the learning environment participate. Elements of this system are: the fluidity and flexibility of the computational environment; the possibility to indicate available use-trajectories, rather than to point to one single proposed goal; there are a local and a global support stuctures. The idea of constructing meaning (Noss & Hoyles, 1996) under a system of this nature, appeals to the presence of a structure on which students can build and rebuild, with a (real and virtual) supporting basis, in any manners they choose as appropriate to their effort in the construction of the meaning of the mathematical points in question.

Finally, it is worth while to mention an element which we also believe played an important role in the design of the generational activity[5] of our teachng experiment: we introduced a 'voice' of history[6] in the classroom - Descartes' voice in this case, as an additional factor that would make it possible for the student to establish links between the voice and his/her own interpretations.

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[4] "We make the assumption that the computer can guide the student in the exploration of designs while inciting him/her to discover invariants among several designs, invariants which may lead to the unveiling of geometrical notions and properties of a general nature. As a matter of fact,... it can propose several designs representing one and the same set of theoretical data"

[5] This is activity consisted in obtaining the equation for the curve drawn by the Descartes machine.

[6] "*Each of these expressions conveys a content, an organization of the discourse and the cultural horizon of the historical leap. Referring to Bachtin (1968) and Wertsch (1991), we called these expressions 'voices'. Performing suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own interpretations, conceptions, experiences and personal senses (Leontiev, 1978), and produce an echo, i.e., a link with the voice made explicit through a discourse.*" (Boero *et al.*, 1997-98).

Our Teaching Experiment: Methodology and Development

We carried out our experiment during the 1997-1998 school year, with a class of 10th graders at a French Licée —students approximately 16 years old—. The work sessions we held with them were "practical work" (PW), and they were complementary to the regular mathematics course. As a part of this course, students had been following the traditional instruction regarding the resolution of algebraic inequalities, to wit, the one consisting in practicing a procedure which permits erecting a table of signs around the zeroes of the algebraic expressions in question, and finally around the quotient, if that is the case

In the classroom where the PW sessions were held, each student was able to use a computer with the Cabri-II microworld, but students were at liberty to communicate among them and to request the Professor's help when needed. PW sessions were structured in the following manner: (a) for the modeling of the curve, we set it up by constituting three sessions of practical work of one hour each, and all the students in the class participated in them; (b) as to the sequences about the graphical representation, these were carried out during two extra-class practical work sessions, which amounted to two hours of directed work with the pupils, only 7 volunteer pupils participated during this first session, and two during the second one -because that was the end of the school year.

The action of the students in the scenario was developed around two basical algebraic activities: (a) the obtention of the equation for the hyperbola (Hoyos, Capponi & Geneves, 1998); and (b) the graphical representation of algebraic expressions (see Hoyos & Capponi, 1999). It is important to mention that immediately after instrumenting the simulation scenario of Descartes' machine, we applied a test concerning the resolution of algebraic equations and inequalities -previously, in class, students had already revised a resolution procedure, that of the table of change of signs; also, in order to solve one of the exercises the students were asked to approach it by means of their calculator-graphicator (a type of calculators very frequently used at this school level. We were obtaining with the application of this instrument a proof that the resolution of rational inequalities constituted, indeed, a catalyzer of difficulties in the understanding of the notion of function.(Hoyos & Capponi, 1999)

During the activity of the graphical representation within Cabri-II, we could observe in the first session, for instance, student AM, who had already placed the variable point M on the axis of the abscissas, could not mark the point corresponding to f(x) on the axis of the ordinates, a matter which at first she attributes to an anomaly in the software. In order to be able to escape this false impasse, she must have point M, the point of variable abscissa x, move. Let us now see this protocol summary between student AM and Professor P:

AM: (after appropriately following the given instructions) But... this leads to nothing, right? Sir, this is a measuring relation!

P: Is it, Audrai?

AM: This does not place a point f... Why, is it that I'm wrong or not?

P: What do you say? Yes, what is it you're saying? 13, 62? Yes.

AM: I have indeed taken a measure relationship... The 13,62...

P: Yes, but wait. Before you go on, according to you, where is the 13,62, if you have marked it on this axis?

AM: Oh, but how stupid I am!

P: Well, then, what should we do?... So, you escape, you simply escape. There, and try to see... There you are!

Do you understand what she is doing? (The Porfessor is now talking to student CA, who has been working with AM for the entire session).

CA: No.

P: Explain to her what you have done. She doesn't understand why you've done this.

AM: The fact is tht there was a number, not the function. This is 13,62; and you mark 13,62. This was over there (she signals towards the top of the screen). Well, then, I move M in order to have f(x) smaller...

P: She decreases x in order to have a smaller f(x). Do you agree? There you are.

There are then, two main actions in Cabri-II during the graphical representation activity, where from the student gets a challenge to escape from an impasse:

(1) that of making point M move in a dynamic manner; and

(2) the parallel visualisation of the effect of moving M on to the ordinate f(x) point of the y axis.

Also, during the last of these two PW sequences on graphical representation, we were able to observe the following behavior in one of the volunteer students who heeded our call to work in, student NS. Already in a situation of solving algebraic inequalities with Cabri-II, NS spontaneously resorted to the revision of his own resolution procedures: in the middle of the graphication activity of the algebraic expressions in question, NS decides to solve the inequality $(1 \div x) < (3 \div (x+1)$ through the usual procedure of setting up a table of signs, and to compare the results derived from the two resolution procedures applied. NS, at this moment, had at his disposal the visualization of the graphical representation of the algebraic expressions which he had just performed in Cabri-II (Hoyos & Capponi, 1999). To N's astonishment, the different procedures employed showed different results. NS demands the immediate help of the Professor, who revises what NS has done, and points out to him the errors of his decisions made only on the basis of a crossed product, followed by the construction of a table of changes of signs.

Some of the Results, and Conclusions

Once passed the moment of the revision of his own resolution procedures, which to us is crucial for student's understanding, the student was able to succesfully solve, by himself, the rest of the rational inequalities programmed for the aforesaid session. There is every indication that Cabri-II turned out to be a significant technological tool for the learning of algebra, in the sense that (Confrey, 1993): "...technologies -that is, any significant tool- necessarily alter the character of knowledge".

It is probably that the actions executed by the students by means of Cabri-II become fundamental building-blocks for an understanding of the functional dependence of values x an f(x), and in this way, the experimented scenario was useful to promote connections to the others more abstract or different contexts than algebraic inequalities.

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VARIATION AND ITS RATE OF CHANGE: A STUDY WITH SECONDARY SCHOOL STUDENTS

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This paper presents the results of an exploratory research study within a more comprehensive project that has the objective of investigating students' transformations between tables and graphs, where the underlying situation provides the meaning. In particular in this article we describe the responses of a group of thirty secondary school students to tasks, converting the variation of a quantity into its rate of change and viceversa, both in graphical and tabular formats. The analysis shows that some of the students' difficulties could be traced to the "mathematical non-equivalence" of the two representations.

Introduction and Theoretical Framework

A theme that has been given widespread attention in the mathematical education research is the one of representations. As suggested by Duval (1993), the coordination of registers in different representations is crucial for the understanding of the concept related and is by no means natural. Embedded in this conversion is the issue of congruence. The general problem we are interested in studying is the "congruence" between the tabular and graphic representations. Since a table is inherently discrete and contains, in most cases, only partial information of a relationship, it can not be equivalent with a graphical representation, so we can expect this to be a source of some of the students' difficulties. For example, if I have a table of values of a function, what is the proper graphical representation of this relationship? A set of isolated points, a step graph, a polygonal, a smooth curved or something else? Actually, unless a real context provides the missing information through its meaning (or an algebraic representation is given), it is not possible to answer the previous questions.

There is considerable evidence on students' conceptualizations of functional relationships and their difficulties in interpreting graphs (for example: Dugdale, 1993; Kieran, 1993; Kaput, 1993; Chazan, 1994; Yerushalmy, 1994; Schwartz and Yerushalmy, 1995). Some of these authors have studied the graphical representation within mathematizing activities and suggested it as a good starting point for instruction. Focusing on the qualitative aspects of graphs, students can describe quite complicated situations without the use of symbolics, which later on can be constructed through the analysis of graphs.

Another important aspect of mathematics is the analysis of real world situations through the construction and exploration of mathematical models (Mellar, et al; 1994). Within this activity, the relationship between the variation of a quantity over time and its rate of change is of fundamental importance. Calculus deals with continuous quantities in which the rate of change is instantaneous and therefore its understanding requires from the limit concept. However, these ideas have a "discrete analog" which is more suitable for younger students: the variation (accumulation) of a discrete quantity and of its change per unit time. One of the aims of this research is to investigate the conceptions of secondary students on the transformations of these discrete quantities.

The SimCalc project, directed by Dr. James Kaput at the University of Massachusetts in Dartmouth, puts together both themes; variation and its rate of change and a graphical approach, through an educational software package called MathWorlds. The basic elements of this software are piece-wise constant and piece-wise linear graphs. Since, there is currently a national project in Mexico to incorporate this technology into the classroom, there is a special interest in this study to survey students' ideas on this type of functions.

Methodology

For the purposes mentioned above, as a first step, we elaborated a questionnaire with ten problems. Some questions asked to produce the variation of the rate of a quantity given the quantity in graphical or numerical form. Others requested the quantity itself given its rate data (there were also some complementary questions about reading and interpreting graphs). This questionnaire was applied as a pilot study to eighteen students of different grades at a junior-high school. We then selected 5 problems for a new questionnaire (described in detail in the appendix) which was applied to thirty students (fifteen and sixteen year olds) from the last grade of a secondary school in Mexico City. After the answers to the questionnaire were analyzed, six subjects were selected (on the basis of representative and interesting work) for an interview to probed their responses further.

The analysis of the questionnaire was focused on exploring what Mexican students can do with the type of tasks described above, keeping in mind Duval's (1993) theoretical framework on the ideas on representations. In particular we looked at the difficulties they might have due to the internal structure of a given representation and also at translating from one to the other. On the interviews we were interested in observing how much further the students could answer with the assistance of the interviewer, from a vygotskian perspective.

Results and Conclusions

The first problem of the questionnaire had the purpose of finding out the students' performance (as a background check) on some basic ideas of change and graphing from data given in tabular format. In general, students responded well on all three questions of the change in temperature (positive, negative and zero). In plotting the graph, 16 students connected the points with a polygonal and 8 left the points without connecting (the rest, connected only parts of the points). This diversity is probably due to the fact that a table representation doesn't contain enough information to complete the corresponding graph (not equivalent representations).

The objective of problem 2 was to observe students' responses transforming rate data into "totals" of a quantity. Their answers fell into four categories: a) ignore the initial value and take only the current change (10 students); b) ignore the initial value but add the changes (6 students); c) take into account the initial value but add to it only the current change (one student) and d) take into account the initial value and add to it the changes (12 students). Thus, in most cases, the students didn't use the complete information given, suggesting that basic tasks as this one is not natural for these students and have to be learned.

In problem 3 we wanted to find out how students draw the rate (velocity) from a simple linear distance-versus-time graph. Most of the students could find the distances traveled in different intervals, find the value of the velocity and state that it is constant (24 students out of the 30). However, we detected a consistent difficulty of the students to draw the corresponding constant velocity graph. Most of them (25 students) draw for the velocity the same linear position graph (either point by point or a continuous line). The other five did not draw any graph. This shows that they can make the transformation to a table (discrete) but not to a graphical one. As in problem 1, this difficulty could be due to a non-equivalence of representations.

Problem 4 used a polygonal graph taken from a textbook of mathematics at the secondary level, which represents the sales per year of a drink in millions of pesos. This is an example of a discrete data (average sales per year) represented in continuous (polygonal) graphical form. When the students were asked to get the total sales through some interval, they added the rate values in very different ways: A) adding correctly the values of each year (11 students); B) adding the values, but omit one of the extreme points of the interval (3 students); C) adding only the values of the extreme points (3 students); D) adding not only the values of each year but also including the values at the middle of the years (4 students) and E) taking only the value of the last point (3 students). In this problem, the graph representation gives "distracting data" (since the graph has a grid every half a year) which provokes responses like option (D). This type of response is probably due to the fact that the polygonal graph is not the proper representation of this discrete data. Does it make sense for non-integer year values in the graph? This is an example of a misleading graphical form (Why is it a polygonal?)

The objective of problem 5 was to observe the students' responses in extracting the changes of a quantity (value of a house) given in graphical form (here we can ask the same question as before: why is it a polygonal? Although in this case of totals doesn't produce the same confusions). About 70% of the students answered correctly the changes in value requested. The rest gave only as answer a value read from the graph. It seems that these students confused the changes with the total values (this error of "ignoring" the labels on the axis is frequently reported in the literature. However, it probably goes deeper on how students conceptualize functional relationships).

On the interviews, where the interviewer tried to probed the students' ideas further, the students were able to answer correctly many questions that they found difficult in the questionnaire. For example, they could describe now the meaning of horizontal sections in the graphs. They were also able to obtain the changes from piece-wise linear functions, or tell in which section there was a faster rate of growth. Even two of the students, who couldn't draw before the velocity graph from the linear position graph, with the guidance of the interviewer, were able to draw an horizontal line. This shows the importance of the interaction with an "expert peer" to create a zone of proximal development.

This work describes how students transform the variation of a quantity into rates and back in different representations. We discovered incongruencies between the table and graph representations that were sources of difficulties for the students and have to be studied further. This is a relatively new area of research, so this article hopes to stimulate more research along these lines.

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Appendix

In this appendix, we describe the five problems used in the second questionnaire and the interviews that followed:

1. "A nurse keeps the following record of the temperature of a patient:"

Hour:	15:00	16:00	17:00	18:00	19:00	20:00	21:00	22:00	23:00
Temperature:	36	37	38	39	39.5	40	40	40	38

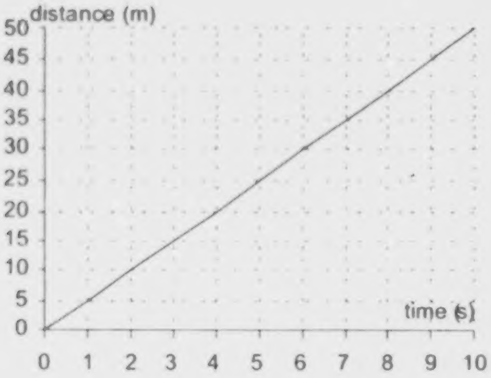
The students were asked for the change in temperature in several intervals, to graph the data and to interpret it.

2. "In a greenhouse a register is kept about the amounts two plants (A and B) grow per week. Initially both plants had a height of 20 centimeters. The data collected is given below:"

	Week:	First	Second	Third	Forth	Fifth
Centimeters grown in that week:	*Plant A:*	*3*	*3*	*3*	*3*	*3*
	Plant B:	*5*	*4*	*3*	*2*	*2*

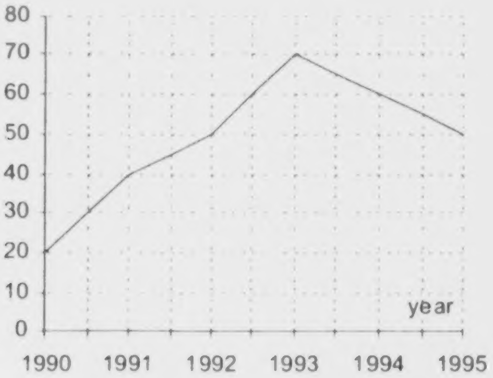
After this table, the students were asked for the height of each plant after one, two and three weeks and to complete a table of the height of the plants. After this, there were some questions about which plant grows faster at the first, third and fifth weeks.

3. "The next distance-time graph shows the distance traveled by a runner:"



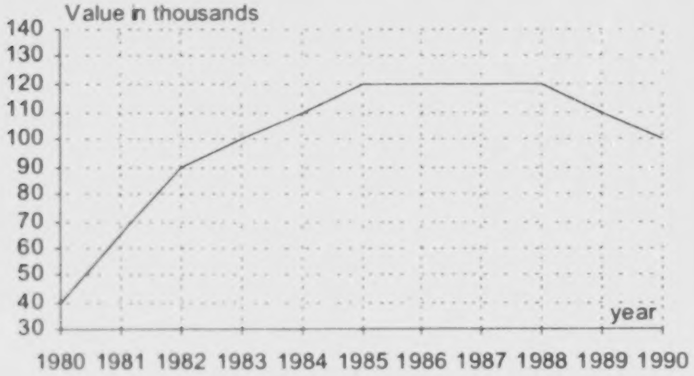
The students were asked to get the distance traveled in some intervals of time, to obtain the velocity of the runner and to draw his velocity graph.

4. "The following graph represents the sales per year of a drink in millions of pesos:" (this graph was taken from a textbook of mathematics at the secondary level).



The students were asked to give the sales for some of the years and the total sales since 1990 up to a given year.

5. “The following graph shows the value of a house registered per year, during ten years:”



The students were asked for the changes in the value of the house in several intervals of time and to make a table of the change in value per year.

PROFESSIONALS READ GRAPHS (IMPERFECTLY?)

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One research question that has not received much attention is, "Are scientists generally competent readers of graphs, or are graphs indissolubly tied to practices and understandings of their everyday workplace?" From an extensive database on graphing involving university students to professionals, I selected two case studies and interpret them within a theoretical framework grounded in semiotics and hermeneutic phenomenology. The first case study provides a detailed analysis in which a scientist wrestles and in part inappropriately interprets an unfamiliar graph used in undergraduate ecology courses; the difficulties of the scientist include some of those identified among students in the graphing literature. The second case study provides an example of the transparent use of graphs in the work of a water technician who is not only familiar with her graphs, but who has an intimate, embodied knowledge of the world to which the graph refers. These results therefore have considerable implications to mathematics education.

Graph-related practices are central to scientific endeavors and graphing has long been hailed as one of the core "general process skills" that set scientists apart (Leinhardt et al., 1990). One research question that has not received much attention is, "Are scientists generally competent readers of graphs, or are graphs indissolubly tied to practices and understandings of their everyday workplace?" This study, which inscribes itself in an interdisciplinary research program of mathematical practices from middle school to professional practice (e.g., Roth, 1996; Roth & Bowen, 1999) was designed to better understand the reading and interpretation of familiar and unfamiliar graphs.

Theory and Problematic

Cartesian graphs are central to the representation of the world in the natural sciences. A recent analysis showed that there were more than 420 Cartesian graphs in 2,500 pages from 5 top-ranked ecology research journals (Roth, Bowen, & McGinn, in press). Much of the ethnographic work in scientific laboratories conducted over the past two decades suggested that not only are graphs used to construct phenomena, but they also serve as proof for the existence of the phenomena and, as thus, are employed as rhetorical means in scientific publications (Latour, 1987). Graphs have been shown to be central to the constructive effort of collectives in establishing just what is seen in and evidenced by the unfolding pattern of a graph.

Recent theoretical work on graphing suggests that a sociocultural orientation toward graphing as practice avoids a deficit approach to students' graphing-related actions (e.g., Roth & McGinn, 1998). This approach has become even more convincing and necessary as a recent study among scientists shows. When scientists are not familiar with a graph, even if this graph is from introductory textbooks to their own discipline (ecology), they frequently do not arrive at the standard interpretations (Roth et al., 1998). Some of the breakdowns they encounter while reading graphs (and captions) are of the same type that have been identified among middle and high school students. However, because of their extensive training (M.Sc., Ph.D.), experience (minimum of 5 years independent research), and career-related success (many received national and international fellowships, grants, and awards), it would be difficult to argue that these scientists hold "misconceptions," have "cognitive deficits," or other "deficiencies." Here, I take the notion of graphing as practice further. By fusing semiotics and hermeneutic phenomenology, I account both for individual aspects of reading graphs and for the social matrix within which each individual operates. That is, my framework allows me to link mathematical experience with experience in the world.

Research Design

Sixteen practicing scientists who had obtained an M.Sc. or Ph.D. and who had a minimum of 5 years of experience in independent research were asked to interpret the same set of three graphs. In addition, they read one or more graphs from their own publications. To better understand the reading practices related to familiar graphs, a water technician was videotaped on four occasions reading pen chart-recorded graphs from her work (a) in real time at the work site (twice) and (b) during a public exhibition of her work as part of an open house organized by an environmental activist group.

My analyses, grounded in semiotics and hermeneutic phenomenology, are based on the assumption that reasoning is observable in the form of socially-structured and embodied activity (Livingston, 1986). In my analyses, videotapes, transcripts, and artifacts produced by the observed individuals are natural protocols of their efforts in making sense of, and imposing structure on, their activities. These protocols constituted our texts that we structured and elaborated in our analyses. I followed the advice of hermeneutic phenomenology to produce alternate readings of the data and thereby engage in a dialectic of understanding and explanation (Ricœur, 1991). The ecologist read the transcripts in the light of his own graph-related learning process during his undergraduate and graduate training; the physicist engaged in hermeneutic and mathematical structural analyses.

Professionals Read Graphs

To better understand how professionals read familiar and unfamiliar graphs, I produced two detailed case studies. These case studies show how graph reading unfolds in real time, and therefore has all the mumbles, stumbles, malapropisms, metaphors, tics, seizures, psychotic symptoms, egregious stupidity, strokes of genius characteristic of cognition in real time. My first case study illustrates problems in the reading of one scientist, Ted (6 of the 16 scientists who struggled even more than Ted); the second case study exemplifies transparent reading and the kind of knowledge and understanding that accompanied competent graph reading.

Reading an Unfamiliar Graph

In summary, the episodes provide glimpses of a scientist attempting to read graphs with the purpose to leap beyond to relate it to the world he knows. But this reading is not effortless and in some instances non-standard because (a) standard conventions were not observed, (b) structural analysis produced individual and ensembles of signs which did not or were not used to constrain their respective interpretants, and (c) visual features of the graph were used to make inappropriate structural arguments. I also saw an interaction between structural analysis and understanding of how the world works, and dialectic relations between sign and referent dimensions in each of the two processes at work. The video in my data base show that (a) 6 of 16 experienced scientists experienced even more trouble than Ted with this graph and (b) scientists generally connected their reading of this graph much more than non-scientists to their understanding of the world or mathematics. The difficulties faced by scientists starkly contrast with their reading and use of graphs from their own work. The following is a detailed excerpt from my paper relating to Ted. In the process of reading and interpreting, he scribbles and draws on the graph presented to him as shown in Figure 1.

As his interpretation unfolds, we can see Ted engage in the processes outlined in my semiotic model of graph interpretation, oscillating between graph (sign) and possible referents. Furthermore, when encountering difficulties, he enacts readings that are analogous to the difficulties identified among high school students. (Roman numerals are used to number utterances.)

> *[Ted:] [i] N would be the number like the individual in a population, rate would be a number differentiated by time, so this would be a measure of change. [ii] They are obviously plotting some population where we take, find the number of individuals and we see that as the number of individuals goes up the rate, the death rate increases.*

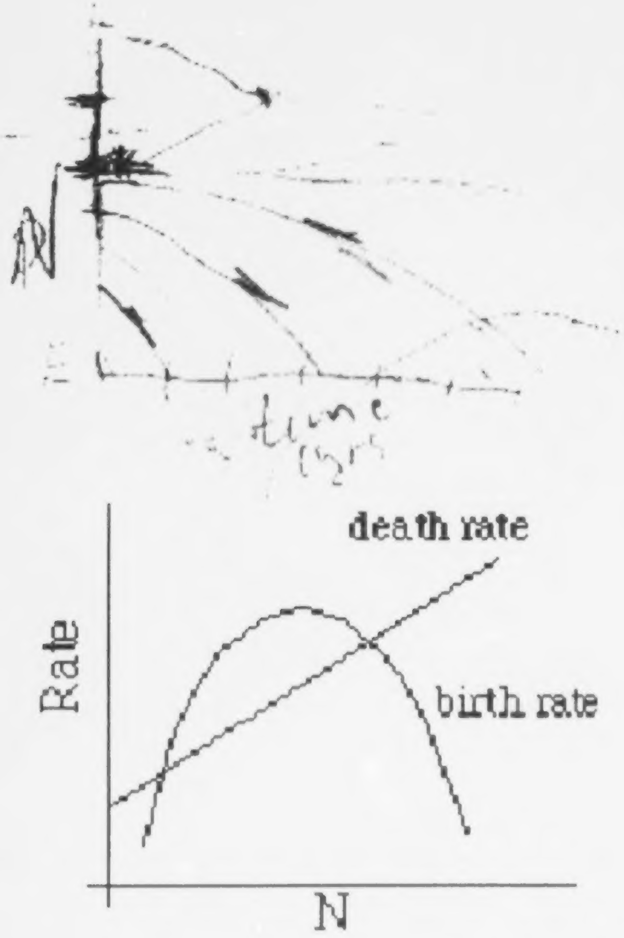

Figure 1. Graph in problem set marked in the course of his reading by one scientist. Notably, he produced his own graph representing the development of population size over time in the different regions he had identified.

> *. . [iii] There are more individuals dying per unit time as the number goes up and birth rate increases as the. . . [iv] It probably should go to zero if there are no, well it should go to 2 probably; if there are no parents, there will be no births. [v] The birth rate increases to a maximum, at some optimum number and then the birth rate falls of as the number of individuals increases, [vi] probably because of limits in the environment or competition or disease or overcrowding or social problems within that population. [vii] Now, the logistic model is a mathematical formulation and it's very common I know in the ecological literature and this particular curve*

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POINTS[birth rate], logistic curve, has been shown, it has been shown or has gained a lot of notoriety because it demonstrates chaotic behavior in non-linear. . .

In this opening reading, and without reference to the caption, Ted immediately elaborates the signs "N" and "rate" with the interpretants "number of individuals in a population" and "number differentiated by time," respectively [i]. Both interpretants are not immediately available from the task, but arise from common conventions with which more theoretically-oriented scientists are familiar. Ted then enacts a literal reading, and produces an interpretant which translates the trajectory of lines with respect to the coordinate grid into a verbal representation [ii], which he then further elaborates in [iii] with a second interpretant. All of a sudden, Ted engages in a reverse movement where he takes everyday understandings of reproduction (no organisms no births, it takes two parents to procreate) to project where the birth rate curve should intersect with the abscissa [iv]. In this, we observe a movement from the referent to the sign domains. Ted continues to provide interpretations that describe the shape of the birth rate curve [v], and then again uses familiar understanding of ecology (referent) that legitimize the dropping of the birth rate (sign) with increasing N [vi]. Here, the situation descriptions "limits in the environment", "competition", "over crowding", and "social problem" are not available in the graph or in the caption. Yet here, Ted's understanding of ecology is consistent with his interpretant of the birth rate curve for larger N. Therefore, the ecological understanding (and prior experiences) and his reading of the graph reified each other. Ted then turns to the text and reads the term "logistic model," leading him to relate it to mathematical formulations and chaotic behavior in non-linear systems [vii].

Reading Transparently

In summary, Karen's reading is in many ways representative of the processes of reading graphs transparently which we recorded when scientists engaged with graphs from their own work. Each time, extensive situation-specific knowledge of the setting including tools and instruments used to collect data was used as resource to explain how their graphs should be read. Our own, two-year ethnography in which we followed an ecologist around observing all activities that ultimately led to conference presentations showed how much scientists use implicit understandings ("anecdotal knowledge") in their reading (Roth et al., 1998).

Conclusion

Based on these research results and our theoretical framing, I do not expect students inexperienced in the conventional practices of graphing, nor in the possible nature of signs or the phenomena the signs are about, to infer the structure of the translations and the unfamiliar content domain. This is because all representation is at the same time interpretation and translation. If we do not recognize it as such, this is because of our familiarity with the representations that have become transparent in our activities. Even the apparent visual similitude between the outline drawing of a horse and some living horse is produced, consistent with some cultural decision, and therefore has to be learned (Eco, 1976). Thus, geometric similitude and topological isomorphism both are transformations which require rules and conventions. These rules and conventions specify how a point in effective space of the expression is translated and therefore corresponds to a point in the virtual space of the content model. Transformations may differ by the mode of correspondence and the class of elements made salient (and therefore pertinent) by the conventionalizing procedure; some elements are retained as invariant whereas others are varied. Some transformations aim to preserve geometric properties, others metric properties, still others topological properties, and so on. It must be retained that transformations are not natural correspondences, but require knowledge of the rules and conventions by which an expression relates to some content. They also require that the reader attends to and perceives those features in both domains that allow such a translation to occur. Neurophysiological research shows that both attention and perception are functions of prior knowledge and experience (Jarvilehto, 1998). Thus, without knowledge and experience in both graph and referent domains, I expect interpretive processes to be stymied. Through interpretation, a reader's understanding does not change into something else or reveal new information. Rather, through the interpretative process, the reader works out possibilities already projected in the understanding she brought to the graphing task.

Relationship to Conference Theme

Graphical representations are central to the efforts of mathematicians and scientists. My research shows that, contrary to commonly held assumptions, scientists do not easily (or at all) interpret unfamiliar graphs although these came from undergraduate courses in their own domain (ecology). My paper provides detailed exemplary case studies of interpretations when graphs are familiar and unfamiliar. In general, "expertise" was tied to familiarity with graphs and the contexts these represented. The results of my paper, which question graphing as a general process skill have considerable implications for mathematics education.

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PRESERVICE SECONDARY TEACHER UNDERSTANDING OF THE CONCEPT DEFINITION OF FUNCTION

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In this study, the concept definition (Vinner & Dreyfus, 1989) of function was examined in preservice secondary teachers who were in traditional or alternative certification programs. Concept maps and a written assessment were analyzed qualitatively. Findings included that most students did not hold a modern conception of function and that more similarities than differences existed in the concept definition of function in the students in these two types of certification programs. The goal of this study was to add to our understanding of teacher content knowledge on functions.

Functions are "an important unifying idea in mathematics," (NCTM, 1989) which "grow in importance as one progresses in the depth and breadth of one's understanding" (Yerushalmy & Schwartz, 1993, p. 41). Dreyfus (1990) summarized the three major problem areas in the teaching and learning of the function concept as the "discrepancies between the mathematical definition, . . . the concept definition that the student knows, and the mental concept image that the student actually uses," (p. 122) difficulties moving from one representation to another, and difficulty of moving beyond the notion of function as a "procedural rule and conceiving it as a single entity, a mathematical object" (p. 122). This study used data from concept maps and written assessments to examine the nature of preservice secondary teacher knowledge in those problem areas.

Cooney and Wilson (1993) concluded that although work had been done on student understanding of functions, not enough research had been done on secondary teacher knowledge of specific areas of mathematics, including the concept of function. They traced the history of the concept of function and showed how the definition of function developed as "dependence relations describing real world phenomena then as algebraic expressions, then as arbitrary correspondences, and finally as sets of ordered pairs" (p. 146). They suggested that the historical development might be an effective pedagogical sequence. The historical development of the definition of function correlates with the following categorization of function concept definition developed by Vinner (1983) and refined by Vinner and Dreyfus

(1989): correspondence, dependence relation, rule, operation, formula, and representation. The present study examined the types of concept definitions held by preservice secondary teachers.

Student data was also analyzed for evidence that each student understands the univalent and arbitrary nature of function, what Freudenthal (1983) called its essential features, in his phenomenological analysis of the mathematical structure of function. *Univalence* is the condition that each element in the domain is paired with only one value in the range. The *arbitrariness* of functions is the idea that the correspondence between the two sets or the sets themselves may be arbitrary. The correspondence does not have to follow a pattern or have a "nice" graph. The sets themselves may be any type of objects including graphs, rotations, or children on a playground.

As part of a larger study, twelve preservice teachers (six in an traditional preparation program [TPP]and six in an alternative preparation program [APP]) created concept maps and completed a written assessment that included items designed by Even (1993) and Vinner and Dreyfus (1989). Williams' (1998) research supported the use of concept maps to "capture a representative sample of conceptual knowledge" (p. 420), in particular, function knowledge. The researcher (who was not their instructor) administered both assessments during a mathematics methods course the fall semester before these students were to student teach.

The TPP students were earning a degree in mathematics with a concentration in education. The APP students were Masters of Education students who were returning to school to earn certification after earning a Bachelor's degree in mathematics or a closely related major. Demographically these groups were similar.

Students were asked to define function on the written assessment, as well as to answer a variety of items designed to assess their definition of function. Concept map data were used to provide clarification and elaboration on each students' concept definition of function. Eight of the twelve students selected what Vinner and Dreyfus (1989) would categorize a correspondence definition. Of that eight, three APP and two TPP students included univalence in their definition. Albert's (TPP) definition is the best example of a univalent correspondence definition: "a mapping where one x-value is mapped to one and only one y-value." Barbara (APP) gave a correspondence definition without univalence: "a function is a mapping of points from one set to another."

Of the four students who did not give a correspondence definition, two mentioned function machines or devices. Charles (APP) called a function "a mathematical computation device"; Dana (TPP) wrote that a function is "a machine that takes an input value and gives an output value." Both might be categorized as representation definitions because they use a representation to define function, but Charles's definition may be better categorized as an operation definition because of his emphasis on operation or computation. Neither his concept map definition nor any of the items on the written assessment expressed univalence.

The remaining two students left blank the question asking for a definition of function. From her concept map and other answers, Elizabeth (TPP) seemed to hold an operation definition, and Fran (APP) seemed to hold a representation view of function. Elizabeth's concept map has "calculations" linked to "function" by the word "performs." Predominant in Fran's concept map was a mapping diagram.

It is questionable whether any of these twelve thoroughly understand the arbitrary nature of functions. Only one TPP student Nicole responded negatively to the question "Can all functions be represented by equations?" Although follow up is needed to determine further what affirmative answers mean, they seem to imply an expectation of pattern or orderliness. TPP student Albert and APP student Gena were not entirely comfortable answering "yes" or "no" to the question. Albert wrote "of some type, yes they might be piecemeal, or some other type." Gena hedged similarly in her answer. Both answers need additional probing, but suggest that each of those students may have some additional understanding of arbitrariness compared to their peers. Nicole did not consistently hold that understanding of functions as arbitrary; on another item she argued that {(1,4), (2,5), (3,9)} was not a function because it is "a set of points." Five students did correctly answer that that set of points was a function.

To learn more about preferred representations in their concept images of functions, students were asked to provide three examples of functions, then to tell which representation is clearest to them and which they think would be clearest to high school students. All except one of the nine students who answered that question provided examples in the form of algebraic expressions "f(x)="; the TPP student who did not use the f(x) notation provided equations in the very similar "y=" form. The examples provided most often were linear and quadratic functions, followed by trigonometric functions. The only other functions given as examples by any of the students were the constant function, cubic, absolute value, natural log, the general x^n, and the nth root of x (the only incorrect answer). The two most commonly given specific examples were $f(x) = x^2$ (6 students) and $f(x) = \sin x$ (4 students).

In response to the question about which representation was clearer for them to understand, a majority picked an algebraic one for themselves and a graphical one for their students. Dana (TPP) and Fran (APP) suggested the only answers besides algebraic and graphical. Dana mentioned a function machine. Fran mentioned the example of having all children on the playground point to their mothers.

Several of the items on the written assessment had students explore their definition of function by interpreting whether or not specific examples were functions. Eight of the students successfully applied the vertical line test to determine whether or not a specific example was a function. Only seven of them could state what the vertical line test is. Only four used their knowledge of the definition to check examples given to them in nongraphical representations.

These results agree with previous studies (Even, 1993; Vinner & Dreyfus, 1989; Wilson, 1994) that found that students do not hold a modern conception of function. It found, as Vinner and Dreyfus (1989) anticipated, that most of them had a correspondence concept of function. Vinner and Dreyfus found that the higher the level of mathematics training, the more likely students were to hold a correspondence definition. Less than half of these students stated a definition that included the univalence of functions and few gave sufficient evidence to show understanding of the arbitrary nature of functions.

Questions about unusual functions, such as the Dirichlet function, were left blank by most of the students. Vinner and Dreyfus (1989) discussed the problem of compartmentalization, "of holding two different potentially conflicting schemes in his or her cognitive structure" (p. 357), that at least four of these students indicated. These students used the vertical line test to see whether a specific relation was a function and could state the idea of univalence in their definition, but could not apply the idea of univalence to test to see whether a relation was a function. These students seemed to only procedurally understand the use of the vertical line test. Concept maps by these students confirm that they recall some of the basic terms related to function, but have not formed relational (Skemp, 1978) understandings to link these ideas together coherently.

Others (Norman, 1993; Vinner and Dreyfus, 1989; Even, 1990, 1993) have explained that most textbook instruction presents a set of common functions that can be represented by equations and have continuous graphs. That may be illustrated in the examples of functions given by these students. When asked to provide examples, every single example given was an algebraic representation. Although that may be the most compact representation, the similarities of the responses, down to the examples of two particular functions provided by a third and a half of the students respectively reflect the common approach taken by most textbooks in discussing functions.

Norman (1993) also argued that the topic of functions was "assumed to be already understood by mathematics majors" (p. 179). These findings would argue that instruction on function needs to include a greater variety of functions and more nonroutine problems. The role of the concept of function needs to be made more explicit so that preservice teachers will begin to appreciate function as "no doubt one of the most important [topics] in modern mathematics" (Dreyfus, 1990, p. 119).

Although all of these soon-to-graduate preservice teachers exhibited a function concept that is still developing, students from neither the TPP teacher education program nor the APP group were distinguishable from each other in function concept definition. No more than two students demonstrated a modern concept of function that included univalence and arbitrariness.

Further research will follow the function concept development of these teachers as they begin teaching mathematics. It also needs to explore whether having a more networked, more conceptual understanding of functions and an understanding of the critical role of function in mathematics influences pedagogy.

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THE ROLE OF REPRESENTATIONS IN THE CONSTRUCTION OF ALGEBRAIC EXPRESSIONS: THE CASE OF POLYNOMIALS

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The use of representations, in any mathematical activity, is basic for conceptualizing process since it integrates several cognitive activities such as formation, treatment and conversion (Duval, 1993). For instance, within construction of algebraic expressions, treatment from graphical, numerical and algebraic representations play the basic role for integration's process. The present study explors the treatment performed by students at each representation while constructing cuadratic functions and the related conversions used to establish connections between them.

Students from Geometry and Trigonometry courses (between 15 and 16 years old) completed (45 minutes long) written assessment comprised of nonstandard, open-ended involving conversions among graphics, tables and algebraic expressions. Participants' reponses were analized in a qualitative manner. As a result, two situations from development of algebraic expressions emerged from data analysis. First of all, the use of a particular representation and secondly, the typical isolation between representations. These results show that 38.5% of the participants made graphic's treatment throughout selecting individual points; this group identifies numerical sequence within a table, which characterizes the curve and they do not develop any algebraic expressions. Alternatively, 61.3% of participants have chosen algebraic representation manipulating algebraic procedures; after analyzing the numerical table it has been concluded that students identify numerical sequence for establishing the pattern that generalizes algebraic procedure's behaviour. Moreover, treatments applied at representations do not provide strong basis in order to make connections. The present research is focused on developing required treatment from graphical, numerical and algebraic representations, based upon global interpretation to strength conexions between representations. The consequence shall enhace construction of algebraic expressions based upon treatment and conversion of those representations.

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THE (GRAPHIC, NUMERICAL, ANALYTIC AND VERBAL) REPRESENTATIONS OF VARIATION IN THE FORMATION AND DEVELOPMENT OF THE FUNCTION AND DERIVATIVE CONCEPTS

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This report includes the results of an investigation to determine the role that the (graphic, numerical, analytic and verbal) representations of variation play in the formation and development of the concepts of function and derivative, and also how those representations are used in problem analysis and solving.

The investigation was performed essentially in three stages. The first consisted of a comprehensive examination of the extant literature in the area concerning the concept of function. It is noteworthy that nearly all the works surveyed deal with the notion of function as an *object*. In the investigation reported herein, the function and derivative concepts are understood as models of variation and speed of variation respectively, and thus treated as tools. Furthermore, at the outset, graphs, numerical records and analytical expressions are understood not as representations of functions but as representations of variation.

The goal of the second stage was to find out the difficulties encountered when analyzing, interpreting and solving problems about variation, especially those related with the use of representations. The inquiry was carried out with a sample ranging from children of fifth and sixth grade of Elementary School to college teachers of mathematics. A list of twenty three obstacles was obtained and classified in seven types. Those obstacles range from the difficulty in identifying and abstracting the varying magnitudes in a certain situation or phenomenon and establishing the dependency relationship among the involved variables, to obstacles related to the continuity of the variation, which manifest themselves when students interpret the expression " variation of magnitude" as equivalent to "how much the magnitude varied" or else, the expression "represents the variation" as equivalent to "represents iconically the variation". In both cases, the interpretation can be explained assuming that only discrete states are perceived in the variation process, that is, it seems clear that the obstacle is related to the continuity.

From the identified obstacles a number of conjectures were made about the role of the representations in the process of formations and development of the studied concepts. The central idea here is the assumption that as far as the graphs, the tables and analytical expressions become meaningful as representations of the variation, and also functions and derivatives are understood as models of variation, their significance will be enriched by the new experiences.

In the last stage, a didactic project consisting of a number of learning situations was designed to test the conjectures previously formulated. The obtained results show that the high school students who participated in this project overcame many of the obstacles detected in engineering students who have passed one, two and even three courses of college calculus. Consequently those results support the validity of the posed conjectures.

SPREADSHEET AND COMPOSITION OF FUNCTIONS

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Research reports that spreadsheet environment may support the development of algebraic concepts (Rojano and Ursini, 1997). It has been found that students have difficulties in making sense of the composition of functions (Ayers et al., 1988). Our concern now is to investigate the mediating role of the spreadsheet in making sense of the composition of functions and its algebraic symbolisation. Ten students14-15 years old worked in pairs during 8 sessions. They had no experience in composing functions and a short experience with spreadsheets. Students were asked to use the spreadsheet for calculating "the square of the number in the cell on the left" and to label each column using an analytic expression. All the students generated a column (A) of natural numbers, a column (B) by the formula =square(A3) and a column (C) by the formula =square(B3). They labelled column A with x and column B with square(x). The difficulties appeared when trying to label column C: two pairs of students tried to reproduce the numbers in column C by the formulae =square(A3/2) and =square(A3*2). After these failed attempts this monologue was recorded"... the square of the cell on the left .... but I already have it, so .... it is the square of the square ..." and she wrote the formula =square(square(A3)). Using this formula to generate a new column she verified that it was equivalent to the one she used to generate column C. She labelled then column C with square(square(x)).

In the spreadsheet environment it is very easy for students to compose functions without being fully aware of the complexity of the operations done. Having to label the columns produced with the spreadsheet using an analytic expression for it, leads to a reflection process. This helps students both, to get aware of the operations done with the spreadsheet when composing functions, and to give meaning to the analytic expression used for representing the composition of functions.

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STRATEGIES TO SOLVE MATHEMATICAL PROBLEMS WITH A GRAPHING CALCULATOR: ARE INSERVICE TEACHERS DIFFERENT FROM PRESERVICE TEACHERS?

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In current mathematics education reform, students are expected to become problem solvers. Teachers play a central role in this expectation for no one questions that what a teacher knows is one of the most important influences on what is done in classrooms and on what students learn (Fennema & Franke, 1992). This study investigates strategies used by both inservice and preservice secondary teachers to solve mathematical problems with graphing calculators included in a final exam in a course on technology in math ed during fall 98. The author taught one course for 10 inservice secondary and another course for 13 preservice secondary teachers. Author believes both courses were comparable in content and activities. Problems included global behavior of a function, inequalities, optimization problems and solving equations. Preliminary analysis for the following problems is provided.

(a) *Find the coordinates of the point or points on the curve* $y = x^3 - 6x - 2$, *which are closest to the point (1, -2). What is the minimum distance?*

(b) *Find the coordinates of all the points on the graph of* $y = x^3 6x - 2$ *that are at a distance, d = 1.2 from the point (1, -2).*

2/13 preservice teachers and 3/10 inservice teachers gave a correct answer to part a. None of the preservice teachers gave a correct response to part b; 4/10 inservice teachers responded correctly. Possibly most of these teachers did not learn mathematics with technology. However, the course emphasized this type of problems. One can also argue that 7/10 of the inservice do not teach these types of problems. While this is true, most of the errors were algebraic in nature. Some could not set an equation, or substitute to find the coordinates of a point. It is a major concern that inservice teachers missed one part of the problem. Preservice teachers relied on visual clues to solve the problem: 6/13 used the point (0, -2) as the solution of the problem (which appears to be the closest point if one graphs the equation of the function given). In both cases, even when teachers had an idea of how to solve part a, they did not use the distance equation generated there to solve part b. So even teachers with relatively advanced mathematical background and experience can be expected to ignore their own equations (produced immediately before a problem where they can be used). Data suggests that preservice and inservice teachers used the same weak strategies (with some minor differences) and also made the same mistakes (with some differences and to a lesser extent). This suggests that teachers need strong help to implement the ambitious recommendations of the NCTM standards to help students to become better problem solvers.

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GRAPHIC AND ALGEBRAIC REPRESENTATIONS AT THE LEARNING OF RELATIONSHIP BETWEEN TANGENTS AND AREAS

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The way of teaching some mathematical concepts or objects at classroom has forgotten its own genesis as a consequence of privileging the treatment of algebraic representation (Duval, 1993) among drawings, diagrams, schemes, graphics, etc. These last ones are as old as mathematics itself and most of schoolwork leans strongly in elements of its type, just like in the case of the Fundamental Theorem of Calculus. Traditional version, shown at the classroom or provided by texts, has ignored graphic register representation of the *inverse relation* between tangents and areas, which is fundamental for conceptual apprehension of such an important and generic result. Common representation carried out in practice and with just algorithm's aspects exploited is not the best alternative for guiding students to identify independent cognitive variables that help on their understanding. This idea was verified during preliminary studies carried out with last high school and first college levels' students. The obtained results show that even while having enough formal tools the students are still not able to reconstruct or to identify such relationship in its graphic form if it has not been worked previously.

At graphic representation, two central elements in the development of Calculus are manifested, providing a better opportunity for students to end up conceiving the relationship as an object more than as a simple tool or an algorithm. One of these elements deals with the possibility of representing an area's magnitude by means of a segment's magnitude. The second one, with the restriction that may not involve an appropiate notation, allows students to progress, in certain moment, at the evolution of a concept (Struik, 1969).

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CABRI GÉOMÈTRE AS A TOOL TO IMPROVE THE INTERPRETATION OF LINEAR VARIATION GRAPHS

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The objectives of the investigation were: to design strategies for the learning of the linear variation, using the software Cabri Géomètre as a tool; to investigate the effects that the application of these strategies might produce on the student's competence in the use of graphs, and the effects such application on the conceptualization of the linear variation could have among the High School students. The difficulties of both, reading and interpretation of the graphic Cartesian representations resides, basically, in the ignorance of semiotic correspondence rules between the graphical representation and the symbolic expressions (Duval, 1988). The graph, the table of values, and the formula are different types of representations, which are used in order to manage a concept. The existence of several registers of representation are necessary for the construction of concepts (Duval, 1993). An integral understanding requires the coordination of at least two registers of representation. The articulation between representations implies that an individual is able to convert one representation to another (and vice versa) preserving the meaning (Hitt, 1995). The investigation was carried out with a group of the Preparatory Cuautla School. Two constructions with the Cabri were designed, the students were allowed to carry out certain type of experiments with the purpose of exploring and discovering the properties of the graphs of linear variation and their relationship with the algebraic expressions. A previous individual test was applied and a second stage of the investigation consisted of three sessions of work. After carrying out the sessions, two individual evaluations were applied. The results indicate a significant improvement in the acting of the students on deducing the corresponding algebraic expression to a given graph, and this improvement in the conversion from a graphic representation to an algebraic expression, also enhanced the students' level of conceptualization of the linear variation.

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UNDERSTANDING REPRESENTATIONS THROUGH DISCUSSION: A CASE STUDY OF A STUDENT'S INTERPRETATIONS OF INTERVALS IN GRAPHS AND TABLES OF MOTION PROBLEMS

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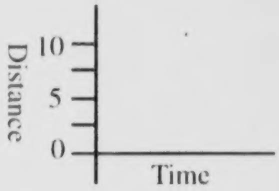
This poster summarizes an account of conceptual change and developing understanding of graphical and tabular representations of motion problems. It is a case study that examines the changes in an 8th grader's understanding of intervals in graphs and tables. The study was constructed from classroom observation data collected as part of a larger research project on mathematical discourse in bilingual settings. Two episodes focusing on one student's conceptions about intervals in graphs and tables were analyzed.

The analysis revealed the student's initial struggle with the graphing of intervals and the assignment of their values on a graph. The student used the phrase "going by five's" to describe the amount by which each two intervals on a graph had increased, rather than the distance between marks on the graphs, as he did in the Y axis of his graph:



This student also grappled with the difference between interval distance and cumulative distance as displayed in a table. The second episode analyzed shows the student using a different interpretation of intervals on the Y axis, and developing an understanding of the difference between interval distance and cumulative distance.

The poster also outlines how discussion with a peer and a teacher contributed to the change in this student's understanding and use of intervals.

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INVESTIGATING STUDENTS' DEVELOPING CONCEPT OF RATE

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Our goal in a recent 3-week whole-class teaching experiment was to develop and research a technology-intensive activity sequence that would support seventh-grade students' emerging views of rate. This instructional sequence 1) put the phenomenon (motion) at the referential center, and 2) provided an opportunity for students to connect the phenomenon with the graphical representation before encountering algebraic representations. Post-test analyses showed students made significant gains on many standardized test items. In order to investigate the nature of students' understandings of rate, we also interviewed six students pre- and post-instruction.

The theoretical framework that supported the development of the sequence and the subsequent analysis of students' understandings of rate is based on research conducted by Thompson (1994). In his work, Thompson conducted one-on-one teaching experiments to describe how students develop conceptions of speed and rate. His findings, which reflect a Piagetian perspective, suggest that, initially, students perceive speed as a length. After abstracting time as an extensive quantity, a student can develop an ability to see time and distance as covarying. Time and distance are seen in ratio, as the result of comparing two quantities multiplicatively. Students may express this in alternative ways (e.g., 35 miles in one hour or 70 miles in 2 hours); critically, the comparison is seen as between two specific nonvarying quantities. The most sophisticated conception of rate necessitates understanding the constancy of the result of the multiplicative comparison. Implicit in the concept of rate is that values of the compared quantities vary in constant ratio.

This poster reports on the generalizability and usefulness of this framework in our analysis of student learning in this context.

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USING COMPUTERS TO FACILITATE VISUALIZATION IN MULTIVARIABLE CALCULUS: A VARIETY OF OPTIONS

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At our university, we wanted to use computers to enhance the visualization of the three-dimensional objects studied in multivariable calculus. We also wanted to make available resources to facilitate instructor's efforts to incorporate computers. To this end, we constructed sequences of graphics with comments about the calculus content and the *Mathematica* commands used to generate the graphics. We made these materials available as overhead transparencies and also at a web site (www.math.ou.edu/~rgoodman/calculus.html). These materials can be used in a variety of ways ranging from "advertising" the web site to students (low effort for instructor) to assigning computer-based problems for students to work on (high effort for instructor and students).

We administered a short student survey in Spring 1999 (147 respondents, 81 of them in sections designated to participate in the project). The survey included a computation item, a "visualization" item, and items about the usefulness of course components. Results included (1) class means on the computation item that were not significantly different indicated that students who used the computer algebra system did not lose computation skills; (2) significantly different class means on part of the visualization item indicated that use of computers may facilitate visualization in some way and that further research is needed to study this phenomenon; and (3) class means indicated that each component of the course helped respondents to learn the material and thus that students want all the help they can get. A revised survey will be given in Fall 1999.

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Twenty First Annual Meeting

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Psychology of Mathematics Education

Volume 2

Centro de Investigación y de Estudios Avanzados-IPN
Universidad Autónoma del Estado de Morelos
Cuernavaca, Morelos, México

October 23 – 26, 1999

Editors
Fernando Hitt
Manuel Santos

ERIC *Clearinghouse for Science, Mathematics, and Environmental Education*

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PME-NA XXI

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- A vita and a writing sample.

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Geometric Thinking

EXAMINING WHAT PROSPECTIVE SECONDARY TEACHERS BRING TO TEACHER EDUCATION: A PRELIMINARY ANALYSIS OF THEIR INITIAL PROBLEM-POSING ABILITIES WITHIN GEOMETRIC TASKS

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In this study, 17 prospective secondary mathematics teachers were asked to pose problems related to each of four given geometric situations. They generated a total of 225 responses (199 mathematical problems or questions, 4 nonmathematical problems or questions, and 22 statements). The 199 mathematical problems were categorized as well-posed problems (168) and ill-posed problems (31). The most common strategies used to pose problems, and number of problems, were: generalization (38), variation of knowns (25), variation of unknowns (21), pattern problems (19), and a combination of strategies (12). Even though students generated a diversity of problems, only 10 students posed general problems and only two students posed at least one general problem for each mathematical situation. In addition, the students rarely posed converse-type problems and proving problems. Given that most of the students were majoring in mathematics, the findings are not very encouraging. Thus, there appears to be a need for prospective teachers to learn how to pose those types of problems.

Mathematics has developed into an extensive body of knowledge because there is, and has been, a continuous search for finding solutions to *problems posed* by someone. Therefore, problem posing is a fundamental activity of doing mathematics (Brown & Walter, 1990). Yet, we do not know the extent to which preservice teachers, who are themselves students, are able to pose problems from given situations or problems. Even though some researchers (English, 1998; Silver et al., 1996; Silver & Cai, 1996) have provided us with insights about issues pertaining to this line of investigation, we know little about prospective secondary teachers' abilities and strategies to pose problems. The purpose of this paper is to describe and examine the strategies used and the problems posed by 17 prospective secondary teachers within four *geometric problem situations.*

Conceptual Framework

From a mathematical point of view, generalizing, proving general statements, and generating problems by considering converse-type problems are important mathematical activities. Generalizing mathematical patterns is one of the most important processes that contribute to the development of mathematics. According to Sawyer (1982), "generalization is probably the easiest and most obvious way of enlarging mathematical knowledge" (p. 55). Proving those general statements is one of the central activities of doing mathematics. When creating a theorem, it is often worthwhile to investigate whether the converse of the theorem holds or what additional conditions or restrictions must be added for having a converse-type problem. In this study, the problem situations were chosen so as to allow for posing problems about *generalizations* of geometric *patterns, proving general formulas, converse-type problems,* and the use of the strategies described below. We used Contreras' (1998) and Moses, Bjork, & Goldenberg's (1990) frameworks for analyzing the strategies that the students used to pose the problems. The framework includes mainly the following seven strategies to pose problems: *variation of unknowns* (VU), *variation of knowns* or *givens* (VK), *variation of restrictions* (VR), *reversing knowns* and *unknowns* (*converse-type problems*, CP), *generalizing* (G), *thinking of patterns* (PA), and *proving* (PF) (Contreras, 1998; Moses et al. 1990).

Methodology and Data Sources

The subjects were 17 prospective secondary mathematics teachers (8 males and 9 females) enrolled in a college geometry course in Spring 1999. Figure 1 provides an abbreviated description of the four mathematical situations given to the participants to generate problems. Each subject was given 30 minutes to complete each of the problem-posing tasks during regular class time but they spent less than 20 minutes on each problem-posing activity. The students were always encouraged to take more time to pose additional problems. The tasks were given sequentially and the students focused on one task at a time. They responded to each mathematical situations individually. The written responses are the basis of the analysis as described briefly below.

Preliminary Analysis and Results

Data coding and analysis. We adapted portions of Silver and Cai's (1996) coding schema. Students' responses were first categorized as nonmathematical questions or problems, mathematical questions or problems, or statements. Only the mathematical problems were considered for further analysis. The mathematical problems were then categorized as well-posed problems (a solution can be investigated) or ill-posed problems (the problems made no sense to us or were poorly stated). Next, each problem was analyzed to examine the strategy or strategies involved in its generation. The first task we did was to pose what we call a *base problem* for each

Mathematical situation 1: Diagonals of polygons (DPMS)

Observe the following diagrams [*A quadrilateral, a pentagon and a hexagon with all of their diagonals are shown*]

We notice that we can draw 2 diagonals on a quadrilateral, five diagonals on a pentagon, and nine diagonals on hexagon.

Look at the examples and think about other diagrams, and write down any mathematical questions or problems that occur to you. Write down as many different problems or questions as you can.

Mathematical situation 2: points of intersection of nonparallel lines (PILMS)

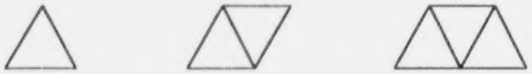
Notice that three lines, no two of which are parallel, have three points of intersection [*a diagram is shown*]

Look at the example and think about … [same directions as in mathematical situation 1]

Mathematical situation 3: Rows of polygons with toothpicks (RPMS)

Toothpicks can be arranged to form sequences of rows of polygons as shown in the following diagrams

(a) A sequence of rows of triangles



[*Similar sequences of rows of squares and rows of pentagons having one side in common are shown*]

Notice that the row with one triangle uses 3 toothpicks, the row with two triangles uses 5 toothpicks and the row with three triangles uses 7 toothpicks.

Look at the diagrams and also think about … [same directions as in mathematical situation 1]

Mathematical situation 4: Hexagonal numbers (HNMS)

The hexagonal numbers are associated with this sequence of dot patterns [*The dot representation of the first three hexagonal numbers is shown*]

The first hexagonal number is 1, the second hexagonal number is 6, the third hexagonal number is 15

Look at the examples and think about other sequences of dot patterns, and … [same directions as in mathematical situation 1]

Figure 1: Abbreviated versions of the mathematical situations (MS) presented to students

situation. The base problem was one of the most "natural" problems suggested by the geometric situation. The second task was to analyze each base problem in terms of knowns, unknowns, and restrictions. Table 1 displays the given mathematical situations, the corresponding generated base problems, and their knowns, unknowns, and restrictions.

Problem-posing responses. A total of 225 responses (199 mathematical problems or questions, 4 nonmathematical problems or questions, and 22 statements) were generated from the four mathematical situations. The 199 mathematical problems included 168 well-posed problems and 31 ill-posed problems. Students generated a total of 53 responses (49 mathematical problems or questions, 1 nonmathematical problem or question, and 3 statements) for the case of diagonals of polygons (DPMS). The 49 mathematical problems or questions included 44 well-posed mathematical problems and 5 ill-posed problems. Regarding the case of the number of points of intersection of nonparallel lines (PILMS), students provided a total of 48 responses (46 mathematical questions or problems and 2 statements). The 46 mathematical questions or problems included 31 well-posed mathematical problems or questions and 15 ill-posed mathematical problems or questions. With respect to the situation of forming rows of polygons with toothpicks (RPMS), a total of 70 responses (54 mathematical problems or questions and 16 statements) were generated. The 54 mathematical questions included 46 well-posed problems and 8 ill-posed problems. Finally, students generated a total of 54 responses (50 mathematical problems or questions, 3 nonmathematical questions or problems, and 1 statement) for the situation of hexagonal numbers (HNMS). The 50 mathematical problems or questions included 47 well-posed problems and 3 ill-posed problems.

Strategies used to generate problems. On average, each student generated about 3 problems for each mathematical situation. The most commonly used strategies were: generalization (38 problems), variation of knowns (25 problems), variation of unknowns (21 problems), thinking of patterns (19 problems) and a combination of strategies (12 problems). About 77 problems were generated using other strategies not included in the framework. It is worthwhile to notice that only one converse-type problem and only one proving problem were generated. The 38 problems dealing with generalizations were generated by 10 students. However, only two students (Brett and Peter) posed at least one general problem about each of the four mathematical situations. Mathematical situations RPMS (rows of polygons with toothpicks) and HNMN (hexagonal numbers) allow for double generalizations (e.g., what is the number of toothpicks needed to make a row consisting of n p-gons? and what is the nth p-gonal number?)

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Table 1: Knowns, unknowns, and restrictions for the base problems

Mathematical Situation	Base Problem	Knowns	Unknowns	Restrictions
1) DPMS	How many diagonals are there in a hexagon?	Hexagon	Number of diagonals	Convex polygon Two-dimensional figures
2) PILMS	How many points of intersection are determined by three lines, no two of which are parallel?	Three lines	Number of points of intersection	No two lines are parallel Lines are coplanar
3) RPMS	How many toothpicks are used in a row with three triangles?	Row with three triangles	The number of toothpicks	One type of polygon Two-dimensional figures
4) HNMS	What is the third hexagonal number, given its dot representation?	Dot representation of the third hexagonal number	Third hexagonal number	Two-dimensional figure numbers

However, only Brett posed a double-general problem for each of the two mathematical situations and Peter posed a double-general problem only for RPMS. Table 2 displays some examples of problems posed by students for each of the mathematical situations using the most comon strategies.

Since problem posing is a very open-ended activity, it is not surprising that students posed problems using not only a variety of strategies but also expressed their problems in a variety of ways. For example, the problems "find a formula that can tell us the *n*th hexagonal number" (Brett) and "find an equation to find out how many dots would be in our fourth figure?" (Jan) were judged as being essentially the same problem.

Discussion and Conclusion

Out of the 225 generated responses, 199 (88%) were categorized as mathematical problems and out of the 199 problems, 168 (84%) were classified as well-posed problems. Thus, about 75% of the 225 generated responses were categorized as well-posed problems. Most students generated not only some interesting problems but also some significant problems. This finding suggests that the preservice teachers had some initial abilities to pose problems. However, converse-type problems and proving problems were rarely posed, and general problems were not generated consistently by every student for every mathematical situation. Given that subjects in this study were prospective secondary mathematics teachers, most of which were majoring in mathematics, the findings are not very encouraging. Thus, there appears to be a need for prospective secondary teachers to learn how to pose those three types of mathematical problems in systematic ways, such as the ones described in our conceptual framework, but still be open to other creative ways to pose problems. They also need to learn how to pose a diversity of problems so that they become more confident and willing to engage their students in problem-posing activities. This is one of the objectives of our project "Problem Posing in Teaching and Teacher Education."

Table 2. Examples of problems generated by the most common strategies

Mathematical situation	Variation of knowns	Variation of unknowns	Generalization	Patterns
1) DPMS	How many diagonals would be on a shape with 23 sides?	How many new triangles are formed when all the diagonals are drawn in a pentagon?	Is there a formula to calculate the # of diagonals that would be formed in advance?	Is there a pattern of increase in the # of diagonals as the # of sides increase?
2) PILMS	What are the possible numbers of points of intersection of 4 lines in a plane, none of which are parallel?	Can the angle's made by these 3 intersecting lines be determined? and if so what are they?	Can we develop a formula that will tell us how many points of intersection are formed by n number of non-parallel lines?	(No responses within this category)
3) RPMS	How many toothpicks would 5 hexagons contain?	(No responses within this category)	Does a formula exist to find out how many toothpicks would you need to form a figure of r number of connected polygons and with n number of edges of the polygon?	Is there a correlation between a particular shape and the number of toothpicks? Such as increasing by 2's or 3's or 4's
4) PNMS	If the 1st decagon # is 1, the 2nd decagon # is ________	What is the next hexagonal number?	By looking at the hexagon, how many dots are in the nth hexagon number?	Is there a pattern to look for?

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ASSESSING STUDENTS' DEVELOPING CONNECTIONS BETWEEN EMPIRICAL WORK AND DEDUCTIVE THOUGHT

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In a geometry course requiring a strong connection between empirical activity and deductive work, students struggle because they have a gap between these levels of work. Geometry software addresses this gap. This study describes how the software relates to the student's development of deductive work based on his/her empirical thought in a small college geometry class. The investigator establishes base knowledge by assessing van Hiele levels, proof writing abilities and construction skills; the instruments are used again at midterm and end of term. Computer sketches and observations from the lab and in-class tests provide data for the description of student development. Data summaries for five students are reported. Both in the pre-test and at midterm, all students except one were using only spatial cues at an empirical level for constructions. The midterm experience forced them to face misconceptions and make initial efforts in bridging the empirical/deductive gap, but gains were modest. A key may be in how an instructor makes students deal with the struggle of discovery of constructions from the beginning.

In an ideal world, college mathematics students would have no trouble extending their traditional high school knowledge of Euclidean geometry to a college course. However, research as well as the experiences of instructors indicate that it is not an ideal world. The fly in the ointment is that learning geometry at a more rigorous level requires a strong connection between empirical activity and deductive work that is lacking in many college students.

Students who have learned proof in an almost algorithmic fashion fail to connect the deductive work to their own mathematical activity in work such as constructions. Schoenfeld (1986, 243) describes students as "naïve empiricists whose approach to straightedge-and-compass constructions is an empirical guess-and-test loop." In proof, the objects are hypothetical and theoretical; the standard of correctness for the student is logic. On the other hand, objects in constructions and diagrams have spatial-graphical properties and are real; the standard of correctness is accuracy of the drawing.

In part this gap is set up by a mismatch of learning and teaching. The van Hieles model (Fuys, 1988) describes the stages of learning geometry in these levels: (1) recognition, (2) analysis, (3) informal deduction, (4) formal deduction, and (5) rigor. The inherent properties of the van Hieles model (Usiskin, 1982) explain how the empirical/deductive gap can form as a result of teaching at a level higher than the student's actual development. A college student (level 3) may be able to order geometric objects concretely and network simple ordering propositions; however, the instructor (level 4) is using propositions as mathematical objects and networking the objects through proof. The instructor sees the theoretical aspects of a diagram while the student sees primarily the spatial-graphical properties. In short, the student and instructor are not communicating as well as they could for optimal learning.

Geometry software packages address the students' separation of deductive and empirical mathematics. Teles (1993), Laborde (1996), Clements (1995), and McGehee(1997) describe how the dynamic features of the software make it a philosophical tool that allows spatial-graphical diagrams to behave in a theoretical way. A sketch will react to manipulations of the user by following the laws of geometry if it has been constructed in a theoretical way, but it will fall apart if it has been drawn to approximate the spatial conditions for the user.

Objectives

The purpose of the study is to describe how the software relates to the student's development of deductive work based on his/her empirical thought. Specifically, I posed the following questions:

1. What is the distribution of van Hiele levels for entering college geometry students?
2. What are the proof-writing skills of entering college geometry students?
3. To what extent do entering college geometry students relate construction to proof?
4. How do GSP sketches demonstrate the development of the connection between deductive thought and empirical work of the college geometry student?
5. What changes take place after a term of study in van Hiele levels; proof-writing skills; connections between deductive thought and empirical work?

Methods

A three-hour college geometry course enhanced with construction software (GSP 3.0) is the context of the study; it is required for all preservice secondary mathematics teachers. During the spring semester of 1999, data from pretests, posttests, classroom tests, student interviews, and analysis of computer sketches were collected. The pretest/posttest instruments included the work from the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin, 1982) — specifically the van Hiele Geometry Test and the Proof Test. While these instruments were designed for secondary students, it seemed appropriate to assess the preservice teachers' incoming knowledge of high school geometry as well as the college course impact. Four items based on Shoenfeld's work (1986) were added to the Proof Test so that the connection between proof and construction could be assessed.

The van Hiele Geometry Test and the Proof + Four Item Test were administered on separate days in the first week of the term. The Proof Test was administered again in the ninth week following the study of congruence theorems and constructions. Students had the option of working on the computer or using a paper copy. The van Hiele Test items appeared again on the final, and some test items were correlated to the van Hiele levels so that student progress could be monitored by in-class assessments.

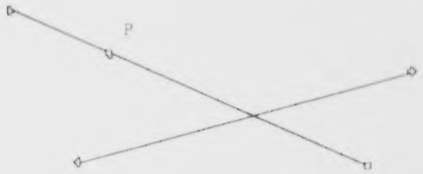
Computer sketches were helpful in studying the development of the empirical/deductive connections. The sketches were evaluated according to their "robustness" of geometric construction and the students' investigative text notes and measurements. The investigator took notes on observations and interviews as the students worked in the computer lab.

Results

The college geometry class for the spring 1999 term consisted of nine students who had never used geometry software. Four students were non-traditional (older than 25), and five students were in the traditional junior/senior age group. A summary of five students' work is reported.

Table 1 shows the student scores throughout the term. Because levels one and two were stable for all students, only levels three through five are reported. Each level is represented by five items on the van Hiele Test and is reported with Senk's (1989) scores: C4 denotes the strict classic criterion of four out of five and M3 is a modified criterion of three out of five. The six proof problems and four constructions are scored 1 point for being correct and _ point for good effort with incomplete or incorrect reasoning. Two of the construction items are given below (See Figure 1). It is also noted whether the student dealt with the construction spatially or theoretically. A revealing group of ten items on the final presents conditions for which students have to judge sufficiency for special classes of quadrilaterals. These items provide more insight into the student's understandings in levels three and four.

1.



You are given two intersecting straight lines and a point P marked on one of them. Tell how you would construct, using a compass and straightedge, a circle that is tangent to both lines and that has the point C as a point of tangency to one of the lines.

2. Find the locus of all points from two intersecting lines. Sketch your result. What does the word "locus" mean?

Figure 1. Construction Items

Aaron, a kinesiology major/mathematics minor, reported that he enjoyed the computer and making designs, but he did not value writing and proof in mathematics. Aaron was a student who improved in spite of himself. He used the computer advantageously on tests or just to investigate a complicated construction. However, I had to force him to write text about his explorations. It was surprising that at midterm someone who was so adept with the computer was still eyeballing the construction of the circle tangent to two intersecting lines. Eventually Aaron could talk through many of his proofs at the computer. For example, he explained to me the details to prove the existence of the nine point circle. I left him to type the text for this proof, but the final viewing of the sketch was disappointing. For him GSP was primarily a design maker.

Beth, a math minor/ English major, also was adept at learning the GSP program. She was especially talented in adopting the features of the program such as color and action buttons to enhance sketches for learning. While she was fearless on the computer, she would throw her hands up at proofs and problems. I had interpreted her nice sketches as work that was bridging the empirical/deductive gap, but her test performances indicated otherwise. When I suggested at midterm that proof could start as narrative of her creative sketches, I tapped into her creative writing skills that could develop into proof writing skills. She adopted a paragraph proof style and her work and confidence began to improve.

David, a post-graduate math major returning for certification, was very comfortable with technology. David could ask insightful questions whether

Table 1. Pre/Post Scoring for Five Students

		Van Hiele Level 3	Van Hiele Level 4	Van hiele Level 5	Proof test 6 items	Construction 4 Items	Nec./Suf. 10 Final Items
Aaron							
< 25 yrs.	Pre	C4	X	X	4	Spatial; 1	—
	Post	C4	C4	M3	6	Both; 3	8
Beth							
<25 yrs.	Pre	M3	X	M3	1	No attempt	—
	Post	C4	M3	M3	5.5	Both; 3	9
David							
> 25 yrs.	Pre	C5	C4	C4	4	Theoretical;2	—
	Post	C4	C4	C5	6	Theoretical;3.5	10
Jill							
> 25 yrs.	Pre	M3	C4	X	.1	Spatial; .5	—
	Post	M3	M3	C4	5	Spatial; 2	3
Susan							
> 25 yrs.	Pre	C5	M3	X	2.5	Spatial; 1	—
	Post	C4	M3	M3	4	Both; 3	8

he was in front of the computer screen or in a traditional class discussion. He started strong and used the computer to his advantage.

Jill, a post-graduate seeking alternative certification, was a good test-taker with chinks in her armor. While she showed some initial ability in the van Hiele levels and some improvement, she could not hold up to alternative questioning. She told me, "The computer is a toy. I don't see what you want us to get out of it." While the other four students usually tested their sketches with dragging, Jill often complained to me, "Look, it fell apart when I moved it." or "See, I put this perfectly in the middle, but the it doesn't work." Another complaint was, "I don't like this discovery learning. They [the book and instructor] need to teach [tell] me." Memorization was a tool that would work on an immediate test while failing her in other problem solving situations.

Susan, also a post-graduate seeking alternative certification, had maintained computers, managed a local swimming pool, and raised five children. In short, unlike Jill, she was a problem solver and viewed the computer as a tool. While she could help other students with GSP and produce good sketches, her work was not A quality. Susan missed opportunities to explore and record text. It could be that GSP's potential will be more fully realized when Susan teaches.

Conclusion

The entering college geometry students scored at least M3 at the third van Hiele level. In this small sample 5/9 of the class scored below M3 at levels four and five. The Proof Writing pretest indicated that students could recall several names from high school, but the application of theorems such as the Pythagorean theorem or "SAS," were usually inappropriate. Also students would assume properties such as parallel lines based on spatial-graphical cues. All attempts on the construction items demonstrated an eyeballing spatial method. . A typical response to #1 (Figure 1) was, "Find the midpoint between P and the other line—this is the center of the circle." They did not relate #1 to #2.

GSP sketches can both mask and reveal students' lack of developing connections between empirical work and deductive thought. The first round of sketches, Beth's group in particular, indicated good use of the computer as a learning tool. However, when the pretest construction items were put on the computer (week 9), David was the only student using theoretical properties. All the other students were still trying to center that circle between the lines. Question #2 helped them find the center, but most of them failed to use perpendicularity with the tangent point. Experiencing this pretest item on the computer was a turning point for students who had not made the move from the spatial-graphical properties of the sketches to the theoretical aspects. The struggle with this exercise made them face their misconceptions. The introduction of GSP involves the tricky balance of step by step instruction with the struggle of discovery activities. As frustrating as discovery of constructions may be to students new to geometry software, it may be the better approach for introduction of the program.

At the end of the term, all students scored at least C4 at level three; only one student scored below M3 at levels four and five. Students consistently improved scores for simple proofs. However, they were inconsistently engaged in harder problems with curiosity or persistence. The empirical/deductive connections were only at a beginning level for most students.

Future work will include better use of discovery construction items and an expansion to collect data from instructors who use either a traditional approach or an innovative/low tech approach. The empirical/deductive gap seems to create a glass ceiling for student work and progress. Comparing use of computer software to other instructional strategies may give a more complete picture of developing geometric knowledge and bridging this gap.

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PROOF SCHEMES DEVELOPED BY PROSPECTIVE ELEMENTARY SCHOOL TEACHERS ENROLLED IN INTUITIVE GEOMETRY

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Abstract:This paper examines proof schemes developed by students in intuitive geometry classes. Perceptual and inductive investigations which supported conjectures about and justification of relationships were processed with the aid Cabri, a dynamic geometry program. Student control of the dynamic environment was maximized. Through the triangulation of information gained from projects, journals, cognitive maps and exams, the proof schemes for four students whose work parallels work by their class members, 65 students, are examined relative to what they attend to and what constitutes justification. Instead of formulating proofs as "external convictions" each of the four students develop empirical proof schemes through creating objects and investigating patterns. Differences across students show that the empirical dimension supports many levels of justification from statements varying from individually constructed objects, to relationships determined between a small set of objects, to relationships underlying the development of operations.

Introduction

In a recent mathematics department faculty meeting, one faculty member questioned the name of a geometry course; the name of this course was "Intuitive Geometry." At the heart of her question was the issue that geometry should deal with proof. Her implicit challenge was the question of whether or not an intuitive geometry can encourage students to "prove" geometrical relationships.

The class in question is the second course in a sequence designed for prospective elementary school teachers. Certainly proof is difficult for these students. For example, Senk (1985) found that only 30% of the students in full-year geometry courses that teach proof reach a 75% mastery level in proof writing. Most of the students who take the intuitive geometry course are not in the successful 30% identified in Senk's study. Does this mean that these students, who could not write proofs in high school, can not build generalizations and justify their thinking in yet another geometry class, especially an intuitive geometry class?

In their paper, Harel and Sowder (1996), built a classifying process for proving. The proof schemes included three major categories; the external, the empirical, and the theoretical proof schemes. External conviction proof schemes are symbolic and ritual and are grounded in meaningless symbol manipulation or surface features. Empirical proof schemes are based on examples. While the theoretical proof schemes are certainly a major goal of mathematics. In their conclusions, p. 65 they state:

> Of greatest importance is instruction that promotes transformational proof schemes, since these are the foundations for the theoretical proof schemes.

The general characteristic of analytical proof schemes is that the students attend to the generality aspects of a conjecture and involve mental operations that are goal oriented.

The focus of this paper is, however, the empirical schemes. This intermediate step includes inductive and perceptual experiences. Could this be the "intuitive" step and could it be a stepping stone to the analytical transformations?

When one moves to develop a course which encourages empirical proof schemes, many issues arise. Two critical issues which arise include what students attend to during the intuitive investigations and what constitutes evidence in their eyes? A quote of Poincaré reported by Harel and Sowder (1996, p. 59) reflects the importance of these issues:

> Satisfaction of the teacher is not the unique goal of teaching; one has at first to take care of what is the mind of the student and what one wants it to become

Empirical investigations created in a dynamic geometry environment contain objects and transformations on the objects. Ideally, the investigations should encourage the students to see and describe a pattern and perceptually explore that pattern in order to determine a generalization. In creating explorations, it is possible to control the perceptual options by defining the object and limiting which components can be transformed. For example if students are asked to think about the relationships between the diagonals of a square, it is possible to create a square, in a dynamic environment, so that only the orientation of the square and the side lengths can vary. Then the students can examine what varies and what remains the same. But, the existence of teacher directed control may force students back to the external conviction proof schemes. When operating with teacher constructed objects and teacher directed variations, are the students developing their own patterns or are they being forced into the external conviction category where they are developing the teacher's pattern? For this study, students were given control to create objects and define variations on these objects. With the control residing in the students' hands, the key question becomes that of what the students are attending too in their environment and what constitutes evidence of relationships identified.

The purpose of this paper is to examine students' work (exams, projects) and reflections (journals, cognitive maps) in an attempt to gain an understanding of what they are attending to, relationships investigated and the proof schemes used in these investigations.

Method

During the Fall 1998 semester, the author of this paper taught two sections of a fifteen week, 45 class hour, "Intuitive" geometry course designed for prospective elementary school teachers. One class had 30 students and the other contained 35 students. Students ranged in maturity from freshmen to seniors. Content of the course focused on the development and justification of relationships in geometry derived through active experimentation. Students were encouraged to use manipulatives, dynamic geometry, and multiple representations in their pattern searching. During the semester, student thinking and reflections were collected every week, including four journals, four projects and three exams. In each journal the students were asked to reflect on key ideas and build concept maps connecting these ideas. Projects encouraged students to provide written explanations of patterns identified and justification of these patterns. Exams were cumulative with one subset of exam questions developed from student patterns reported in the projects. In these questions students were asked to decide which patterns were correct and why. One of the final reflections asked the students if geometry were an animal which animal would it be and why. Through the journals, projects, and exams it was possible to triangulate student justification schemes with their concepts of connections between key ideas and their personal beliefs. The data shared, however, will focus on information collected during the last three weeks of the course. Proof schemes will be examined relative to two problems, one from the last class project and one from the final exam. In each case proof scheme observations will be supported through beliefs reflected in journals and/or concept maps.

Results

Case 1 - Slopes of perpendicular lines

In the last project of the course, students explored the relationship between the slopes of lines. For this investigation, students had access to a computer and the Cabri geometry program. In the final journal they were asked to describe why the slope of perpendicular lines are negative reciprocals of each other. Most of the 65 students gave a table to illustrate the pattern with data collected from the measurement tool on Cabri. Only 5 students made an attempt to move back to the picture and produce a justification relative to the geometrical objects. Below is an example:

> S_1: *Perpendicular lines have a 90 degree rotation around the point of intersection . . . It is easy to see that the x and y coordinates are reversed. One slope will be negative and one positive unless they are the y and the x axis.*

Students using the tables for explanations are using empirical schemes. They are producing patterns from one representation. There is some question as to the role, if any, of the diagram.

On the other hand, in the case identified above, student S_1 is moving to a transformational scheme by using the table and the computer graphics to generate her pattern but then operating on the objects (lines and coordinates) to justify the relationship between the slopes of the perpendicular lines. It is true that she seems to be restricting her investigation to lines through the origin but she is not only working to move back to the picture she is also addressing the special case of the axes.

When one examines this student's beliefs through some of her reflections the role of transformational schemes is further supported. For example, in the final journal she reported that her favorite part of the course was *figuring out why the Pythagorean theorem works. Its amazing to have a visual of a theorem that was just a formula for so many years of math classes!*. Her concept map is simple and contains three columns entitled Solids, Measurement and Plane. The first row identifies basic objects and the second focuses on relationships and functions.

Case 2 - Transformations

In the second example, the statement of the question was based upon actual responses given by students to an earlier project investigation. The question was:

> In the figure below you will find triangle ABC inscribed in a circle with center O. In the exploration of transformations students found that it was possible to use a transformation on triangle ABC to form a rectangle ABCD with segment AB as the diagonal. To form the rectangle, one student reflected triangle ABC about segment AB while another rotated triangle ABC about point O through 180 degrees. Which student was correct and why? Sketch the resulting rectangle ABCD.

Many difference responses were given to this question. Each of following three responses to this question represents most of the various responses and were selected because they provide different justification

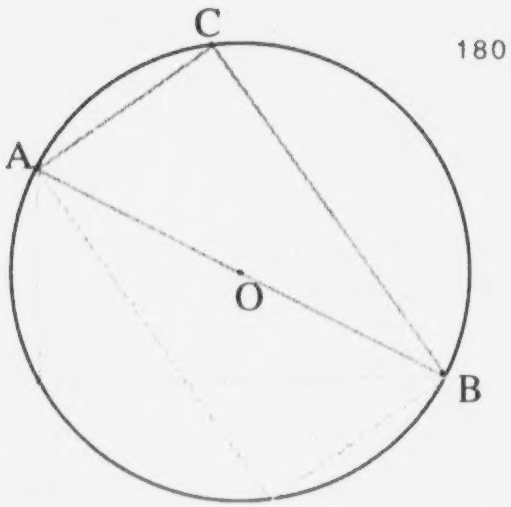

Figure 1: Transformations exam problem

levels, attend to different objects and use the diagram with different levels of generality.

> S_2: When I did a reflection of triangle ABC about segment AB, I had a new triangle ABC'. It is not a rectangle. Therefore, the first person who used reflection was wrong. When I rotated triangle ABC about point O and through 180 degrees, I got the red triangle. This triangle and triangle ABC gives rectangle ABCD

This student talks about when "I" did the reflection. Her empirical proof is based upon her perception of personally created objects, like the red triangle which is a specific triangle shown in her answer, and shows no attempt to generalize or mentally connect generalizations. In this student's end of the year reflections she reports that *I found the most challenging thing to do in this entire course was to relate one idea to the other. I have a difficult time drawing a cognitive map. It's easier for me to do calculations.* Her end-of-the-course cognitive map is very complex. It is organized by placing the word geometry in the middle with geometrical objects on the top and bottom, measurable quantities on the left and transformations on the right. This partitioning of discrete objects parallels her response to the question. When asked if geometry were an animal, which animal would it be she cites a bee and explains *because when a bee is constructing his house he needs to know well about the tesselations, angles, and shapes.*

As with the concept map this answer further supports her focus on discrete objects and primarily the measurement of these objects.

S_1: The student that rotated the triangle 180 about point O was correct. By reflecting the triangle, point A and B remain positioned along the original segment AB while C is reflected across the circle to a point which results in creating a kite rather than a rectangle. The dashed lines indicate the figure if done by reflection which is incorrect for creating a rectangle.

In this case the·student is generalizing to the properties of the diagonals of a kite which he has explored in the projects. He has used the empirically based question and moved mentally into the generalizations where the objects are the kite and the diagonals of the kite. He is moving through the inductive dimension of empirical schemes and is beginning to transform these patterns into generalizations. These generalizations, however, still lack structural connections. In his reflections he writes that *in geometry, working with shapes and figures, I can readily understand specific concepts and their respective relationships.* His cognitive map is simple and emphases the role of geometric pattern explorations.

S_4: The student that rotated triangle ABE was correct because in order to form a rectangle point D had to be on the other extreme of point C so it could form another diagonal.

In this final case, the student has actually created a new transformation. The idea of a symmetric point was never discussed in class. Symmetry was used by only one other person, S_1, to explain why. This is a case where student S_4 is creating her own objects and is developing a theoretical proof scheme. When asked if geometry were an animal, which animal would it be and why student S_4 states that *it would be a leopard because leopards can change their spots.* Her concept map has transformation as the key idea with three subcategorizes of translations, isometries and size transformations. Each column has a different number of entries which are clustered relative to connections but the maximum of seven entries appears in any one column.

Conclusions

To return to the challenge of one of my colleagues, she was very perceptive. The question of the role of intuitive geometry is simple. Intuitive geometry based upon dynamic geometry investigations does not preclude proof. On the contrary, as Harel and Sowder suggest, empirical proof schemes including both deductive and perceptual dimensions serve as a critical step leading from the external conviction derived from ritual, authoritarian and symbolism to the analytical proof schemes built on transformations and axioms. From the examples student S_2 is working from perceptual experiences with little generalization, but she is creating her own objects and characterizing them. For student S_3 pattern searching and inductively created generalizations are developing. His empirical/ intuitive proof schemes lack structure but the willingness to explore and confidence to see relationships is producing a bridge to the development of analytical proof. Students S_1 and S_4 continue to initiate their pattern searching based up induction and/or perception. They are able to generalize, connect ideas and even form new transformations through these connections. In each case the empirical level is documented and appears to be an important transition from the external conviction proof scheme to the analytical proof scheme.

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A CONCEPTUAL NETWORK FOR THE TEACHING-LEARNING PROCESSES OF THE CONCEPT OF VOLUME

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In this paper a conceptual network for the mathematical concept of volume is developed. The principal purpose of this work is to obtain a theoretical framework for planning, designing and analyzing didactical activities, problems, evaluations, didactical materials or didactical situations related to the concept of volume. The net is a schematized organizer to present this framework. The results presented in the net were obtained by applying both a phenomenological and a didactical-phenomenological analysis to the concept of volume. The net exhibits different paths that lead to different mental objects associated to the volume concept. In agreement with the didactical treatment chosen, the net forks at the beginning, pursuing a qualitative and a quantitative perspectives, which in term continues to open new branches according to the different aspects of the concept involved.

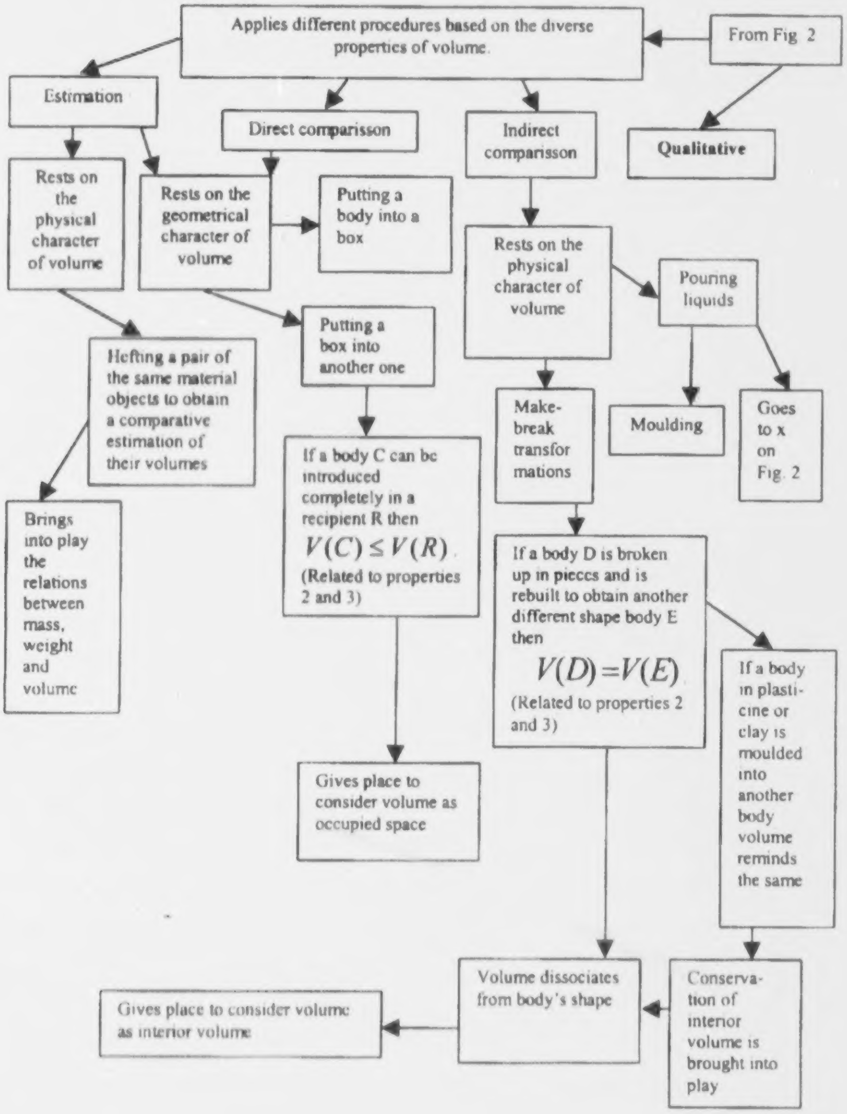
Volume is a meaningful mathematical concept that plays a considerable role in grade-school mathematics. In spite of the fact that some researchers have shown an interest for this topic, the concept of volume has been very little studied as compared to other notions. This fact has motivated the development of a research project on the concept of volume and its teaching at the grade-school level, which has been in progress for the last two years. The most important objective of this work has been the creation of a local framework, as per the theory developed by Filloy (see, for instance, Filloy, 1990), in which the experimental study that has to do with the teacher's ideas on, and knowledge about this concept may be inserted. Filloy's theory proposes the creation of a methodologico-theoretical framework through the development and integration of four components: models of formal competence, models of teaching, models of cognitive processes, and communication models. The findings included in this paper maintain a link with the inherent relationships existing among the first three of these components.

Methodology

As Freudenthal (1983) has said, before starting the design of plans and programs for the study of mathematics or any other kind of activity related with grade-school mathematics, it becomes necessary to apply a phenomenological analysis to the concepts that are of interest. On the other hand, he has stated that the acquisition of concepts is preceded by a solid constitution of mental objects. When taking these ideas as didactic principles and while setting as an objective the creation of a local theoretical framework for the study on volume, a starting point has been the phenomenological and didactical-phenomenological analysis that Freudenthal (1983) applied to this concept. His findings have been enriched by means of new applications of didactical-phenomenological analysis, and through a phenomenological analysis of the concept with which we are dealing, as well as by the perusal of specialized literature relating to the cognitive processes which take place in the learning-teaching processes of the concept of volume. The next section deals with the findings and the methodology, which has been followed.

Discussion and findings

a) On the application of a new didactical-phenomenological analysis. This analysis has been performed through the accomplishment of several tasks. One of these has been the perusal of primary-school textbooks in order to identify teaching models for volume as used in Mexico throughout the last one hundred years; as a result, a classification of the textbooks in seven stages has been obtained (Saiz, 1999). Another has been the contrasting of our characterized models with those recommended by the researchers (see for instance Inskeep 1976, Freudenthal 1983 and Del Olmo 1989), and the result has been a classification of didactical treatments related to the measurement and comparison of volumes in two categories: qualitative and quantitative (see Figures 1 and 2). The term *qualitative treatment* has been assigned to a teaching sequence where tasks and activities are privileged that bring qualitative aspects into play, without any recourse to numbers; whereas a treatment is a *quantitative* one when the aspects brought forth are numerical. For instance, a teaching sequence proposing activities consisting in the pouring of liquids with variously-sized, ungauged containers, is a *qualitative treatment*; on the other hand, a lesson or a series of lessons which start with the presentation of unity to then pass on to the use of formulas is a *quantitative treatment* (see Figures 1 and 2). Also, the mental objects to which the different treatments lead to, have been identified; for instance, the use of measuring units and formulas lead to the mental object of "volume as a number" (see Figure 2), and working with liquids leads to the mental object of "volume as capacity" (see Figure 1). It is important to mention here that the work outlined in this section is related, mainly, to the development of the second component, i.e., that of the teaching models.



Applies different procedures based on the diverse properties of volume.
From Fig. 2
Estimation
Direct comparisson
Indirect comparisson
Qualitative
Rests on the physical character of volume
Rests on the geometrical character of volume
Putting a body into a box
Rests on the physical character of volume
Pouring liquids
Putting a box into another one
Hefting a pair of the same material objects to obtain a comparative estimation of their volumes
Make-break transfor mations
Moulding
Goes to x on Fig. 2
Brings into play the relations between mass, weight and volume
If a body C can be introduced completely in a recipient R then $V(C) \leq V(R)$. (Related to properties 2 and 3)
If a body D is broken up in pieces and is rebuilt to obtain another different shape body E then $V(D) = V(E)$ (Related to properties 2 and 3)
If a body in plasti-cine or clay is moulded into another body volume reminds the same
Gives place to consider volume as occupied space
Volume dissociates from body's shape
Conserva-tion of interior volume is brought into play
Gives place to consider volume as interior volume

Figure 1. Fragment of a conceptual net for the mathematical concept volume (the qualitative approach)

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b) On the application of a new phenomenological analysis. This task is related to the shaping of the formal competence component and the core part of it has been to deepen the analysis pertaining to the physical characteristics of the volume concept The term "physical" is used as a qualifier for the relationships existing between the volume mathematical concept, and other concepts identified as being parts of physics, such weight, mass, and capacity. The main result obtained through this line of research has been the subclassification of didactic activities related to the concept of volume, in three subcategories: geometrical, numerical, and physical. On the one hand, within the qualitative treatment, geometrical and physical tasks are used or applied; and on the other hand, the quantitative treatment makes a more extensive use of numerical-type tasks, although it leans on procedures from the other two subcategories. Comparing heights or areas of the bases of solids, or looking for congruencies, for instance, are geometrical-type tasks, whereas pouring liquids, weighing or using the Archimedes principle are physical-type tasks. Numerical-type tasks are the use of formulas or the counting of units. Another task performed in this direction has been to carry out an historical follow-up of volume as a mathematical concept, and one result obtained is the identification of mental objects associated with volume by scientists of diverse epochs of humankind history. In this historical course, some of the methods employed to measure volumes have been recognized. Some of the facts and mental objects identified in this process are those of the Egyptians and Babilonians, who considered volume a number –exact or approximative–- useful to solve a real problem; Archimedes' discovery relating volume to the volume of displaced water (see Figure 1), and the mental object of volume as a measure in a three-dimensional measurable space. The latter example is derived from Lebesgue's theory, which considers volume to be a measure, i.e., a defined function in a collection of measurable sets of a three-dimensional space which adopts real values and fulfills four specific properties (see Figure 2). Although in the first grade-school levels it is a convenient idea that "the class of measurable solids must be narrowed down as far as possible" (Antonovskii, 1971) and that properties appear only implicitly, it is important to take into account that this mental object encompasses the other ones which relate to volume, and its constitution is favored by the solid constitution of other mental objects, which finds support in the use and application of properties 1-4 of Lebesgue's theory. For instance, if a solid is made up of small cubes whose edges are 1 unit, properties 2 and 4 are brought into play and support the constitution of the mental object "internal volume" (see Figure 2).

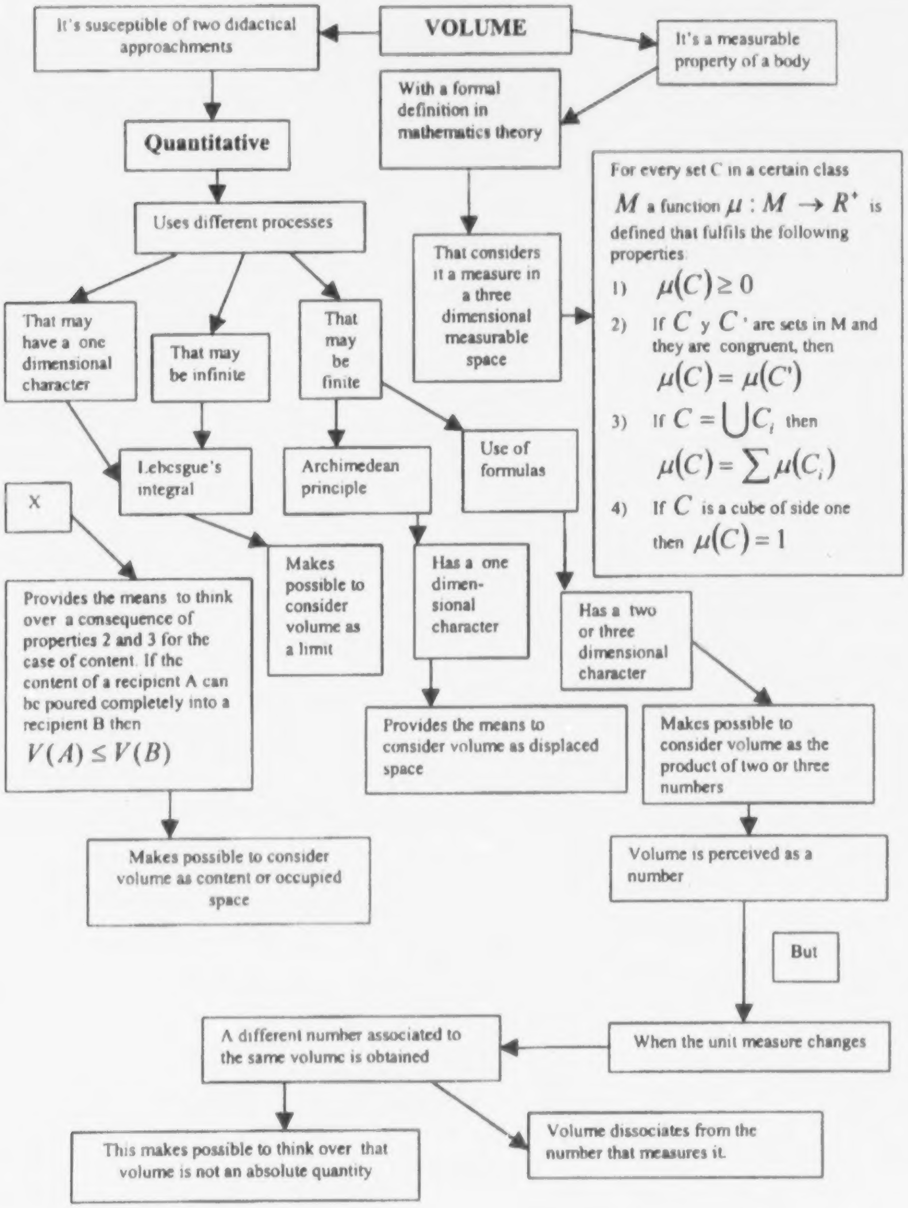
It's susceptible of two didactical approachments
VOLUME
It's a measurable property of a body
With a formal definition in mathematics theory
Quantitative
For every set C in a certain class M a function $\mu : M \to R^+$ is defined that fulfils the following properties:
1) $\mu(C) \geq 0$
2) If C y C' are sets in M and they are congruent, then $\mu(C) = \mu(C')$
3) If $C = \bigcup C_i$ then $\mu(C) = \sum \mu(C_i)$
4) If C is a cube of side one then $\mu(C) = 1$
Uses different processes
That considers it a measure in a three dimensional measurable space
That may have a one dimensional character
That may be infinite
That may be finite
Lebesgue's integral
Archimedean principle
Use of formulas
X
Provides the means to think over a consequence of properties 2 and 3 for the case of content. If the content of a recipient A can be poured completely into a recipient B then $V(A) \leq V(B)$
Makes possible to consider volume as a limit
Has a one dimensional character
Has a two or three dimensional character
Provides the means to consider volume as displaced space
Makes possible to consider volume as the product of two or three numbers
Makes possible to consider volume as content or occupied space
Volume is perceived as a number
But
A different number associated to the same volume is obtained
When the unit measure changes
This makes possible to think over that volume is not an absolute quantity
Volume dissociates from the number that measures it.

Figure 2. Fragment of the conceptual net for the mathematical concept volume (the quantitative approach)

c) On the perusal of specialized literature regarding cognitive processes. This task is related to the structuring of the component of cognitive processes, and one result has been to focus attention on the main errors and difficulties inherent in the teaching-learning process of volume,

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such as: the problems of conservation (Piaget, 1970), the difficulties which are associated with the concept of unity (Piaget *et al.*, 1970 and Steffe and Hirstein, 1976) and Vergnaud's (1983) remarks dealing with the conflict generated by a change of dimensionality, from an exercise of small cube-counting –which is a task of a unidimensional nature–, to the use of formulas –which is a tridimensional process. Also, mental objects in which researchers have shown interest have been identified; for instance, mental objects associated to the terms *inner space*, *occupied space*, and *displaced space* have been taken from Piaget (1970) (see Figures 1 and 2). The most common errors pointed out by experts include the belief that volume is doubled when the linear dimensions of a parallelepiped are doubled, or the problem arising when it is required to calculate volume with a certain unit, when that volume is already known with a different unit (Hart, 1984; and Figueras, 1984). Therefore, it is important to introduce tasks which permit overcoming or avoiding such errors (see Figures 1 and 2).

Contributions and conclusions

Comparisons of the findings obtained by means of the aforementioned analyses are integrated into a conceptual network for volume. Inspired by Corberan's (1996) work with respect to area, the network has been organized as shown in Figures 1 and 2. Also, the network constitutes an important part of the local theoretical framework into which research on the teacher's ideas and beliefs about volume and its teaching will have to be inserted. It can, furthermore, be a theoretical framework for other research works or for the design of materials and didactic activities related with the concept of volume.

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THE *CAN BE* RELATIONSHIPS BETWEEN QUADRILATERALS. A STUDY USING CONCEPT MAPS

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In Huerta (1995) we showed how we can use the technique of concept mapping to analyze students' relationships between quadrilaterals. Here we show how the relationships between quadrilaterals leveled as this A *can be* B, where both A and B are quadrilaterals, are difficult to understand by the students in almost all school level. This difficult is probably because the double mathematical meaning of the statement *can be*.

As we know, quadrilaterals can be organize from a partial relation of order defined on a set. This partial relation of order can be represented in a diagram form in which the concepts are related by segment lines in a top down structure. We also can read this diagram including the statement *can be*, if the reading is top-down, or *is* if the reading is down-top. So, we have the first approach to one of the meanings of the statement *can be*. But, this is not the only one, there is another meaning of the *can be* that is not derived directly from that partial relation of order. Of course, as we know, depending on the definitions Parallelogram and Right Trapezoid are related to. This relation can be represented by the statement: Parallelogram *can be* Right Trapezoid and Right Trapezoid *can be* Parallelogram. So, these seconds *can be* statements have not the same meaning as the first one because the last one derive from an no empty intersection of sets and not from a partial relation of order as mathematical inclusion. So, in diagram mentioned above, we can include this last meaning by a double directional arrow connecting those quadrilaterals that are related sharing this last meaning. The resulting diagram is called concept map of *the can be* relationship between quadrilaterals.

In this framework we investigated (Huerta, 1997) how the students organize the set of quadrilaterals using sets of words as *is* or *always is*, *can be* or *several times is* or similar, *is not* or *never is*, and what are the meanings that they seems to give to those sets of words. Here we only analyze the second group of words.

The students involved in our research (47 students) were students in three different levels. So, we investigate students at the end of primary school (11 years old), at the end of secondary school (16 years old) and in pre-college level (18 years old).

All groups of students answered the same set of items. They had to relate each quadrilateral to the resting quadrilateral from the set. If they supposed or were sure that the relationships existed then they had to include at the appropriated places what kind of the relation they found, putting there the set of words that explained or gave meaning to that relation.

From the students' answers to the items we build their concept maps following the framework we presented above. These concept maps were analyzed qualitatively in order to identify what quadrilaterals were related to, what groups of words were used to represent the relationships and what meanings seemed to give linked with the groups of words.

Classify we found is a task that is not seen equally by all students. While there are students that use the words *can be* in a mathematical sense of include in, depending on what kind of quadrilaterals they are relating to, other use they in a different sense that do not enhance them to do any classification of the set of quadrilaterals. By the other hand, the meaning of the no empty intersection for the words *can be* is not usually recognized by the students, of course seems that they change that meaning for this one: there exist a relationship but I do not know what.

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CONCEPTIONS OF HIGH SCHOOL STUDENTS ABOUT SMALLER ABSCISSA AND BIGGER ORDINATE BETWEEN POINTS ON THE CARTESIAN PLANE

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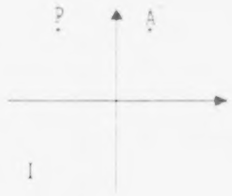
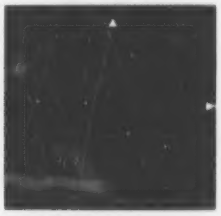
Our goal in this research is have a close explanation of know the high school students construct theirs concept image about the order relation between points on the Cartesian plane.

About the election of the smaller abscissa and bigger ordinate we know that other election would be different because the develop in each quadrant on the plane are different, but we have interest in the ways in which the students explain the order and this election does not change the observation. When we ask to a group (77,17 years old) of our students to plot a point with same ordinate but bigger abscissa than (-2,3) on the plane, they draw the point right only 2.5%. Some other explains rises in the solution of the questions some of them avoid they to grasp a successfully answer.The other option about smaller coordinate were asked in the same way, in those case the x-axis was the border. Follow we describe in graphing the more representatives' solutions and we propose some possible explanations.



I 16.8%



II 9%

III 6.5%

IV 20%

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And 13.7 % more aswer with bigger and smaller points, 34% others and abstentions.

I Here, the abscissa is use without sign, like absolute value but the solution is only plot in the place whit positive abscissa.

II Is possible that the students evoke the slogan "right is bigger and left is smaller" just like in the real line, this thought make sense if we work in the real line.

III Absolute value again but there are two possibilities in both sides of the y-axis

IV They plot the point in the first quadrant and the distance between of the point at the y-axis is bigger than the original point.

We met that some our students use the order between two different points on the plane like: a) they don't have any sing of positively in the sense of the absolute value and b) they don't think in points inner a bidimentional space only recognize the positively of the real line,

The y-axis is taking a border between two types of numbers, positives and negatives, in this case we think the y-axis is taking like an anchor (ancrage in France) where the students focus their attention and in this level avoid construct a right answer.

STUDENTS' UNDERSTANDING OF THE AREA CONCEPT OF PLANE SURFACES: A STUDY FROM PRIMARY SCHOOL THROUGH UNIVERSITY

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The study described in this communication is part of a research project (Corberán, 1996) whose principal objectives are: a) to carry out a didactical analysis of the plane surfaces areas concept using Freudenthal's work (1983) as a framework which was preceded by a study of the mathematical concept and its evolution throughout history; b) to investigate the level of understanding accomplished by students in primary school and to study if it evolves further in relation to secondary and higher education studies; and c) to design, experiment, and evaluate a teaching scheme for secondary education sustained by the results obtained in the preceding parts of the research. Results driven from the phase linked with the second objective will be discussed with particular emphasis on understanding the aforementioned concept.

A paper and pencil test was posed to 521 students in primary (13-14 years) and secondary (15-16 and 17-18 years) school as well as university students (20-22 and 22-24 years). The test contains twelve different aspects of the plane surface area concept and notions related to it. Qualitative and quantitative analyses of the students' answers were carried out. SPSS/PC + software was used to support the quantitative analysis.

The results indicated that general mathematics education acquired by students throughout their courses improves their knowledge of the measurement unit; area conservation; independence between area and surfaces' form; and formulas and numerical methods for calculating areas. However, no improvement was shown regarding independence between the area and the perimeter of a surface, the bidimensional character of area, the comprehension of formulas for area computation, and the knowledge and use of geometric procedures for the area study of plane surfaces.

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SOME IDEAS OF 12-YEAR-OLD STUDENTS ABOUT GEOMETRIC CONCEPTS REFERRED TO BODIES

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This study is part of an extensive research project (Guillén, 1997) that deals with the application of Van Hiele's model to geometry of bodies. The study explores the teaching-learning processes within the context of school courses for pre-service teachers and in clinical sessions with students aged 12 years. This paper will be focused on some ideas that students incorporated to mental objects they constructed fort prisms, antiprisms, pyramids and antipyramids (family of bodies) and for its elements (surfaces, vertices, and edges) when concepts were introduced through some tasks of construction and identification of examples and non-examples (we use the term *mental object* in the sense used by Freudenthal (1983)).

Results here reported were obtained through the analysis of information requested in working sessions carried out with two groups of students aged 12 years. It was confirmed that these tasks of constructing physic models for bodies lead students to construct and express "ideas" for the family of bodies, which allow them to identify those models and to explain the answer with the help of geometric attributes, or to associate these families either at a visual level or in geometric terms.

Thus, for example, ideas such as the following are expressed for antiprisms "*To these two* (alluding two polygons) *I put them triangles* (insert figure 1)*; I assemble them and now I have an antiprism* (insert figure 2*)*"; student adds (referring to pyramids), "*For pyramids I only need one of these stars, because I put all triangles together, I gather them. There are only triangles in a vertex*".

Erroneous *ideas* also deserve attention, for instance, "*the idea of base as surface in which objects rest*", or that "*each one of the pieces that forms a model corresponds to a face of a body*". Using the same color for bases and for pieces that remain plane (forming a face) facilitate the correct recognition of bases and faces.

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Probability and Statistics

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ADULT'S INTUITIVE ANSWERS TO PROBABILITY PROBLEMS: A METHODOLOGY

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ABSTRACT. Some of Piaget's experiments with two open urns and simple extraction were repeated with University students. A methodology was constructed in order to design different situations (i.e., combinations of favorable and unfavorable cases) using *combinations* and *locations*, and to detect different strategies used by the subjects, namely *centrations* and *relations* of different sorts, which they use in simple or composed forms. Subjects were interviewed using a variation of Piaget's clinical method. Situations were classified according to their difficulty in six levels, and according to their difficulty of verbal expression. Strategies were also analyzed, showing that the most frequently used ones are not correct strategies and that some tend to be used more in composed forms than others. Subjects were classified in one of the six levels; only 20% could solve the problems described by Piaget for the stage of formal operations and at most 66% those for the stage of concrete operations. A shelves model is proposed to account for the ways in which adults use their formal or intuitive knowledge.

PURPOSES

The main aim of this work is to prove that Mexican University students don't generally give the answers provided by Piaget's adolescent subjects to Probability questions, and to study the ways in which this kind of subjects solve the most classical and "simple" probability problems. This required a new theoretical and methodological perspective, both of which were constructed, in order to understand the psychological foundations of intuitive answers given by adult students to probabilistic problems of classical form. A collateral purpose is the subsequent construction of a theory that could explain the ways adults have to use their formal or intuitive knowledge and the behaviors stemmed from them.

THEORETICAL FRAMEWORK

The definition of Probability underlying this work is the classical one, although it was not imposed upon the subjects nor was there a qualification of "incorrect" when they adopted a subjective or intuitive posture. The framework provided by several researchers enriched this research. Jean

Piaget (1951) reports an evolution in children's understanding and solving of several basic problems. According to him, the child in the preoperatory stage generally concentrates in the favorable cases; in the stage of concrete operations he begins to consider both quantities but is able only to solve situations in which only one varies; and in the stage or formal operations he distinguishes between favorable and unfavorable cases and is able to solve the problems by fraction calculation. In other works (1972), Piaget claims that although the ages at which subjects of different cultures attain the three principal stages of evolution that he describes can vary, the succession of these is immutable. Nevertheless, as it is proved in this research, Mexican adults can show a mosaic of two or even three stages, which is similar, but seems more complex, to Piaget's concept of *décalage*. Efraim Fischbein (1975) has provided a framework for the concept of intuition, as a form of immediate knowledge that responds to a biological need of action; it has a characteristic globality, extrapolative capacity, structurally and self-evidentness. Kahneman and Tversky (1982) have proved that adults' intuitions in probabilistic problem solving not necessarily coincide with formal or theoretical results. Other essays were also consulted for the experimental part of the research; worthy of mention are Piaget himself (1926), especially in the clinical method of interviewing, Sylvette Maury's research with high-school students, which used a similar experimental framework (1986), and Gérard Noelting's work on strategies used in proportional reasoning (1980).

METHODOLOGY

64 Mexican University students participated in five experiments. The experimental form consists of a classical binary decision in *a priori* mode, corresponding to the model of open urns with simple extraction; it was adapted from Piaget's classical work (1951). Two collections of cards were presented to the subject, white or black at the front, gray at the back; after the subject had seen the front of both, each collection was separately turned upside down and shuffled. The subject was asked which of the collections he or she would choose (or if it was the same) if the desired result was a black card; he or she was also asked to explain his o her reasons. The composition of both collections was then changed for a new problem. Each subject was given between 20 and 40 problems. For this experimental form a methodology was constructed, which is a fundamental part of the work. It consists of three intertwined lines for the research conduction, and methods for the analysis of the results.

The first methodological line is the construction of the situations: the specific amount of black and white cards in each collection for each problem posed. They were built in order to be able to detect the different strategies considered in the second methodological line. Each problem was defined as an arrangement of two ordered pairs of favorable (f) and unfavorable (d) cases: $(f_1,d_1)(f_2,d_2)$, and the different possible problems were grouped in categories described by several variables. The *combination* is a succession of the possible results of the order relation between these five elements of the two pairs of the arrangement: the total cases ($n_i=f_i+d_i$), the favorable (f_i) and unfavorable (d_i) ones, the differences ($r_i=f_i-d_i$) and the probabilities ($p_i=f_i/n_i$). The *location* unites the possible results of both probabilities, each in five forms: "surely lose" ($p=0$), "lose" ($0<p<0.5$), "draw" ($p=0.5$), "win" ($0.5<p<1$) and "surely win" ($p=1$). For instance, the situation of the arrangement (1,2)(2,3) is characterized by a combination that shows that $n_1<n_2$, $f_1<f_2$, $d_1<d_2$, $r_1=r_2$ and $p_1<p_2$, and by a location of the "lose–lose" type.

The second methodological line is the construction of categories for the interpretation of the answers given by the subject. Strategies are defined as solution mechanisms for problems of a given class, consciously or unconsciously used by a subject, that have a certain structure or logic, and that could be reproduced for other problems of the same class. A strategy was identified in each answer, which could be of a simple or composed form. Simple strategies can be *centrations* or *relations*: a centration is the observation of only the favorable cases (e.g. "I choose this side because there are more blacks"), the unfavorable ones ("there are less whites") or the total ones ("there are less cards"), whereas in a relation two of these three elements are observed and compared by means of an order relationship ("here there are more blacks than whites"), a subtraction ("if each time I take out a black and a white card, there will be more black cards left here"), or a proportion ("here there are two blacks for each white, and there, to equal this, one black is missing"). Composed strategies include two or more simple strategies joined by a logical operation ("here there are more blacks *and* less whites", or "here there are more blacks, *although* there are also more whites").

The third methodological line includes the methods for the interrogation, which are a modification of Piaget's clinical method called the standardized examination method (reported by S. Maury, 1986) in which the clinical interview is based upon a fixed questionnaire; the method was here used with two subjects at a time. This line also includes the methods for deciphering the answers and revealing the underlying strategies.

The methods constructed for the analysis of the results created categories for strategies, depending on their correctness –for which Vergnaud's (1981) concept of *théorème en acte* was used–, on the kind of situations in which they can be used, and considering that in certain situations they could be interpreted as incomplete expressions of a correct strategy. This classification allowed also a classification of the different situations, according to the results obtained by the subjects in the problems belonging to each situation, and including a measuring of the difficulty of verbally expressing the situation. It permitted as well a classification of the results attained by each subject.

RESULTS

As expected, some of the situations were easier (that is, they led to more correct answers) than others. When analyzed by location, the easiest were those involving two extreme probabilities (surely lose and/or win), followed by situations involving one or two "draws"; the most difficult locations were "win-win" and "lose-lose". When analyzed by combination, the easiest were the ones in which all centration and relation strategies led to the same answer (non-discriminative combinations), followed by combinations in which either the favorable, the unfavorable or the total cases showed an equality, themselves followed by combinations in which centrations led to different answers but subtraction and proportionality led to the same answer. All of these combinations had at least 73% of successes. The combinations involving a proportionality were more difficult, followed by the combinations with an equality in r_j, and then the most discriminative combinations, in which centrations and relations led to different answers: of the latter, only 37% of successes was attained. This order does not correspond to the difficulty of expression; for instance the non-discriminative combinations were of difficult expression, whereas the most discriminative were of easy expression. A joint analysis of both situation variables led to the definition of six levels of difficulty labeled I to VI.

Strategies were analyzed, showing that the one mostly used was the centration in favorable cases, followed by the proportionality and equilibrium relations, and by the centration in unfavorable cases. Correct simple or composed strategies were not the most frequently used. Other criteria allowed measuring the facility in which each strategy reacts with others in composed strategies, and the percentage of times in which it is the dominant strategy in a composition. Generally speaking, proportionality strategies are the most satisfactory (react less), and strategies that lead to the choice of one collection are stronger (are more dominant) than those that lead to the answer "it's all the same".

Subjects were assigned to one of the six levels of difficulty when they had 50% of successes in it. Thus, 23%, 28%, 14%, 16%, 11% and 8% of the subjects were assigned to levels I to VI. This means that less tan 20%

could solve the problems described by Piaget for the stage of formal operations, and at most 66% could solve the problems for the stage of concrete operations. Subjects were found to be inconsistent with themselves: they frequently used different strategies when they were posed the same problem or problems belonging to the same situation.

A "shelves" epistemological model is proposed that accounts for the results obtained by these subjects and their general inconsistency. According to it, the ability to recognize a situation is represented by a shelf supporting the strategies that can be used in the situation. A subject who can correctly solve all of the problems of the kind considered in this work would have a mental structure of shelves and a hierarchy of correct strategies on each one, and would be able to recognize the shelf and choose the simplest strategy upon it, whereas most of the subjects who participated in this experiment would have had only one shelf with all strategies available on it.

This model, which could explain the ways in which adults use their formal or intuitive knowledge, has to be studied in new research projects.

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STUDENTS' REPRESENTATIONS AND TRAJECTORIES OF PROBABILISTIC THINKING

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The purpose of this classroom research was to study students' representations of probability generally, and sample space specifically as they engaged in a process of making, testing, and defending conjectures over time. First the data were scanned to collect students' representations of why several dice games were fair or unfair, and how to make the games fair. From these instances a coding system was developed and then the data were scanned again to establish a trajectory of individual student's representations over the game-playing process. Of interest when considering the students' representations are the ebb and flow of these ideas. When traced over time, the collection of representations created trajectories or paths which appeared to be different, unique, and sometimes contradictory for each student within the socio-cultural context. One question of interest is how can such different trajectories come together so that students can develop a shared meaning of chance events?

Purpose

Recent reports from the National Science Foundation note that many U.S. citizens vastly overestimate the occurrence of catastrophic events such as plane crashes, child abduction, and terrorist bombing. These reports speak to the importance of developing students' intuitions of order in the uncertainty of events that may affect their life choices. The historical development of probability theory, unlike the development of other mathematical ideas, results in a number of paradoxes related to the conflict of different intuitive ideas or a divergence between intuitive ideas and mathematics (Borovcnik & Bentz, 1991). Others agree that students' ideas or representations of probability, particularly ideas of sample space and symmetry, begin from intuitions (Vidakovic, Berenson, & Brandsma, 1998; Zavslavsky, 1998). For this research I considered how the manifestation of these ideas change and develop using a social constructivist framework (Ernest, 1994; Crawford, 1996). The path or route that these ideas take over time in individual students is referred to as a *trajectory*. The term, *representation*, relates to the cognitive realm of mathematical ideas or understanding that an individual student's possesses and is more broadly used than *concept* or *object*. The purpose of this classroom research was to study contexts that involved students in a process of making, testing, and defending conjectures to better understand students' representations and trajectories of probability generally, and sample space specifically. The process of classroom research reflects an anthropological research tradition that documents how students in a classroom think and come to build their mathematical understanding.

Perspective of Classroom Research as a Process

Maher and Davis (1990), Cobb and Bauersfeld (1994), and others have studied students' thinking within the social context of the classroom. Maher, et. al. (1998) describes the classroom research process as an opportunity for students to access their representations of mathematical ideas. In this naturalistic research tradition, students test their representations with other students to either alter or maintain their original representations. The role of the researcher within the classroom context is to find the right moments to introduce a question or another student's ideas into the classroom discourse to change or alter the trajectories of student's thinking. However, the researcher does not actively intervene in the development of students' ideas by giving mathematical information or "correcting" students' observations, conjectures, or conclusions.

Mode of Inquiry

Researchers, replacing the classroom teacher, worked in tandem with 16 eighth grade pre-algebra students in a small urban, private school. Several game-playing situations were used during four, hour-long mathematics classes to engage students in accessing and perhaps altering their representations of mathematical ideas. In the first investigation, students made conjectures about the fairness of a one die game and then tested their conjectures by playing the game with another student. Students played the game multiple times, collecting and recording their data for each game. In the one die game, player A scored a point each time a 1, 2, 3, or 4 were rolled and player B received a point when 5 or 6 were rolled. Students used the same investigation process in the two dice game where player A received a point for sums 2, 3, 4, 10, 11, or 12 and player B received a point for sums 5, 6, 7, 8, or 9. In each case, students were asked to redesign the games if they judged them to be unfair. Class began and ended with whole class discussions of the previous or current investigation. Student ideas were considered in an open-ended manner and students were encouraged to make and test conjectures to promote the development of their thinking about sample space. Students worked in assigned diads to test their conjectures.

Sources of data included video tapes of the whole class discussions and selected student diads, audio tapes of student diads, and individual student's paper and pencil responses. Of research interest were the representations of these early adolescents concerning sample space and symmetry and how their representations were altered based on classroom activities and interactions. Data were analyzed using categorical aggregation of students' representations and then reexamined to look for patterns in ideas over time to define a trajectory of ideas (Creswell, 1998). First the data were scanned to collect representations of why the dice games were deemed to be fair or unfair from all students. From these instances a coding system was developed (Shown in Table 1) and then the data were scanned again to establish a trajectory of individual student's representations over the game-playing process. Due to the page limitation, this paper reports only one student's trajectory.

Results

One die game. All students agreed that the one die game was not fair for several reasons. The issue of whether there would be an effect on the fairness of the game if the numbers were assigned consecutively was raised during class discussion. Students tested their conjectures about this aspect at home as an assignment, bringing back their results the next day. Zeke's and Kris's comments about the one die game are fairly representative of most students' thinking in terms of having equal numbers assigned to each player:

> Zeke: No, (the game is not fair) because player A had more chances to win. Player A had 4 numbers and player B had only 2. You could make the game equal by having each player have an equal amount of chances.
>
> Kris: Player A has the greater advantage having 4 numbers to roll on and player B only has 2. Giving each person 3 faces of the diewould make it fair. 1,2,3 and 4,5,6. (After testing the fairness of this game) Player A won AGAIN! Maybe A always wins because it has the numbers 1,2,3. It always or usually lands on 1,2, or 3, probably because of the placement of the numbers on the di.

At the conclusion of this class session, a lengthy class discussion ensued among the students where the prevailing idea was to mix up the assignment of the numbers in the one die game. After testing this conjecture, the students were evenly divided in their opinions. Marcie argued for mixing up the numbers, while Andre argued against this idea:

> Marcie: When player A gets the numbers 1,3,5 and player B gets numbers 2,4,6 the games were a little bit closer. I think this game is very fair because it gives each person the same amount

of numbers and a variety not like 1,2,3 etc.

Andre: It is as fair when you use 1,2,3 and 4,5,6 as 1,3,5 and 2,4,6. They are the same because the odds are 50 to 50 and anybody can win. There was no difference when I rolled the di, with different number selections because it is even.

Another idea that surfaced at different times during the classroom research episodes (See Kris) was that some numbers are rolled more frequently than others. "Two is the most common number"; "Most of the time the numbers landed on 1,2,3, or 4"; "I always roll 10".

Two dice game. It was less clear to the students whether the two dice game was fair and their representations reflect this uncertainty. No student perceived the sample space in the two dice game beyond 21 combinations. JoEllen's representations for the two dice game included equal number of sums, mixing up or spreading out the assignment of sums, more combinations for certain sums, and alternating players or taking turns for assigned sums. A summary of the student representations and corresponding codes are noted in Table 1.

Once the codes in Table 1 were identified, these ideas were traced over time to define the path or trajectory of student representations. JoEllen's conjectures of previous days' activities were used to establish a trajectory for her representations of the second activity and are shown in Figure 1. This trajectory over the four-day problem solving investigation is of interest here in an attempt to: 1) identify the representations associated with each of the two investigations, and 2) trace patterns of these representations over time.

In the one die game JoEllen included the idea that you cannot control the roll of the die: "It's unpredictable so therefore you can't help yourself to get 10 points." She also noted that since player A had more numbers that she would make the game fair by giving each player an equal number of digits. However, she and her partner tested the one die game by taking turns being player A. For the two dice game JoEllen gave the following explanation:

JE : No (the two dice game is not fair). Player A has more numbers that they can get and it's more likely for them to get one of those but on the other hand player B's numbers are easier to get. So in a way the game is fair. Both players have an advantage but it just depends on who has the stronger advantage.

Here we note the tension between JoEllen's two theories from the one die game. On one hand she thinks the two dice game is not fair because A has more sums. Because B's sums occur more frequently, she observes that

Table 1.
Students'
Representations
of Chance Events

EN (Equal Number).	Each player should have an **equal number** of die faces or an **equal number** of sums assigned to him or her to make the game fair.
MU (Mixed Up)	The numbers should be **mixed up** or spread out among the two players because some numbers are more likely to occur than others.
MC (More Combinations)	Certain sums in the two dice game have **more combinations** than other sums. Therefore it is easier to win for the player with these sums.
UP (Unpredictable)	The roll of the dice is **unpredictable** in any instance and therefore any game of chance is not fair because you cannot control the outcome.
HN (Higher Numbers)	**Higher numbers** are rolled with the dice more frequently than lower numbers.
LN (Lower Numbers)	**Lower numbers** are rolled with the dice more frequently than higher numbers.
AN (Alternate Numbers)	If you **alternate the numbers** assigned to the players every other time, then you will be able to make the game fair.
L (Luck)	The games depend solely on **luck** because you cannot control who is going to win the game.
DU (Dice Unfair)	The dice were **unfair** for a variety of reasons including size, color, number placement on the di, and ways of throwing the dice.

perhaps the game is fair after all. Note how JoEllen resolves these two views in an attempt to make the two dice game fair.

JE: I would make it fair by mixing up the numbers so player A has 2,5,4,7 and 12 and player B has 3,6,11,8,9. Both players would have numbers most likely to have rolled. Then switch the players position so each player gets to play each set of numbers.

Unfortunately, we did not interview JoEllen to determine if her omission of 10 was intentional or not (If 10 were assigned to player A then the probability for player A's winning would be 1/2). In either case, JoEllen's approach is intuitive. She seems to have taken the ideas from other students in the class discussion of "mixing up" the numbers as a method of making the game fair. By switching the players' positions she used additional intuitions in her attempt to create a fair game. JoEllen abandoned some of her earlier representations and added new ideas as the context for learning changed from the one die game to the two dice game. In some instances, she delayed using the ideas of others for several days.

Discussion

Of interest when considering the trajectory of students' ideas are the ebb and flow that appear to develop individually for each student. At this point in the analysis of the data, the trajectories of other students in this case study appear to be different, even between partners in the diads. This case study generates several questions for further study. How do students think about the unpredictability of the outcome of rolling dice? What additional activities and questions can bridge students' initial representations about sample space to formal representations? And

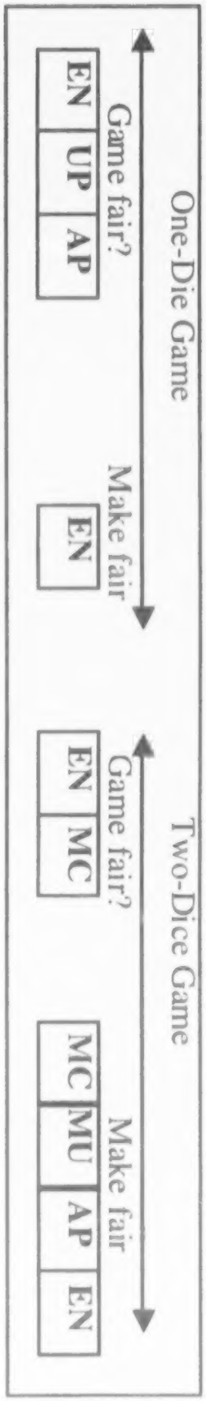

Figure 1. JoEllen's trajectory of representations over time and task.

finally, how can such different trajectories come together so that students can develop a shared meaning of chance events? One possible conjecture within the classroom context is that within any given moment students' representations are variable and unstable.

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MULTIPLICATIVE CONCEPTIONS OF ARITHMETIC MEAN

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We elaborate a multiplicative conception of the arithmetic mean that is grounded in quantitative reasoning. This elaboration serves as a framework for the design and analysis of teaching experiments intended to support students' building this conception of the mean. We discuss insights into students' conceptions and their instructional implications.

Background

Mokros and Russell (1995) distinguish two basic models that have been used in the literature when defining the mean: *fair share* and *balance*. These authors note that works that have used these models seem to omit the notion of *representativeness* as an important characteristic of the mean. Mokros and Russell (1995) view the mean as an indicator of the center of a distribution, one that is useful for summarizing, describing, and comparing different data sets. For them, a powerful understanding of mean is that of a "mathematical point of balance." From this perspective, the different contributions of a data set are balanced with each other, and their point of equilibrium becomes a distinctive characteristic of the collection. Other researchers (Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988) have characterized the mean of a set of scores as "typical" or "representative" of individual scores.

Perspective: Mean as a multiplicative measure

Thompson (1998a) claims that students are misled when the mean of a set of scores is depicted as somehow being "typical" or "representative" of either the individual scores or the set of scores. From an instructional point of view characterizing the mean as typical of individual scores is vague and may direct students to think of the mean as indicating something about individual scores. Characterizing the mean as a summary of a set of scores seems sensible only when it includes a discussion of variance as a summary of the variability of scores relative to a reference value—the mean.

Moreover, we cannot infer anything about how a set of scores is distributed even when we know its mean *and* variance. It could be distributed in any number of ways, many of which have no resemblance to each other.

Thompson (1998a) conceptualizes the mean as a multiplicative notion. His approach grounds instruction in quantitative reasoning and is intended to support conceiving of statistical measures quantitatively. In his theory of quantitative reasoning (Thompson, 1994) the notion *quantity* is a conceptual entity that a person constructs when conceptualizing situations: one thinks of a quantity when conceiving an attribute of an object as measurable. A quantity is schematic, it involves an object-image, a conceptualized attribute of the object, a tacit understanding of appropriate units of measure, and a quantification process—a process by which one directly or indirectly assigns numerical values to the attribute. Thompson's conception of the mean is rooted in images of quantifying an attribute. He stresses the importance of having students first conceive of group performance as an attribute of a group that they imagine themselves measuring. Then students need to create the mean as an adjustment on the measure of group performance, an adjustment that will allow them to compare groups of unequal sizes on an attribute that is sensitive to group size. In order for this to seem sensible to students, they need to have an underlying sense of "individual contributions to the group attribute". That is, one must imagine that in a group each individual performs and that each performance contributes to the group performance. As one "runs through" the members, one accumulates a total, aggregate measure. These images and conceptions support an understanding of the mean as a ratio that measures *group performance relative to the number of contributors in the group.* If students are mindful that they are measuring *a group performance* when they add up scores, then "divide by the number of contributors" can be introduced as a natural normalizing adjustment to their measurement procedure to take different group sizes into account. In this sense the mean is like an average rate (Thompson, 1994); the measure of the group contributions *per* contributor is conceived to be the same as the amount contributed by each of *n* contributors if each were to contribute equal amounts. The quantification process points to an overlap between mean as multiplicative measure of group performance and the "fair share" notion mentioned by Mokros and Russell (1995). But, they are different in that the core conception of the former is measure of group performance and relative contribution per member, whereas measurement is not an issue in "fair share."

The aspect of these conceptions that makes the mean a multiplicative idea is that in comparing measures of group attributes one must understand that aggregate measure and group size are *simultaneously* accounted for in any comparison (Inhelder & Piaget, 1969; Thompson, 1994; Thompson, 1998b).

Purpose and method

The aim of our study is to investigate middle-school students' conceptions as they engage in problem tasks intended to support their understanding the arithmetic mean multiplicatively. In this paper we document and discuss the reasoning of four students—Marissa, Nick, Joy, and Kurt—as they confronted situations requiring them to compare the performance of unequal-sized groups.

Subjects were part of a group of 7th and 8th-grade students at a middle school in Nashville, Tennessee that participated in clinical interviews and teaching experiments. These experiments typically involved one to three 50-minute sessions in which students worked in pairs on one or two problems. Two researchers were present during the sessions; their role was to encourage the students' joint participation in reasoning about the problems, and to encourage them to publicly explain their reasoning. Each session was videotaped and field notes were taken. Data were analyzed from the perspective of mean as a multiplicative measure, as elaborated above.

Results

In the individual interviews, Nick and Marissa (8th graders) were presented with two data sets showing the distances ridden by individual members of two cycling teams in a contest. The teams had an unequal number of members (6 and 9). The problem was to give criteria for choosing the winning team. The intent of this problem was to raise the issue in students' minds of the need to account for the difference in group sizes when comparing groups on a common characteristic. We wondered whether students would recognize the usefulness of the mean in this situation. Neither Marissa nor Nick mentioned the mean of scores as a criterion. Both Marissa and Nick seemed to be thinking of the performance of each bicycle team; they added up the scores in each team but excluded three of the scores in the bigger team—truncating the data was their way of accounting for the size difference.

In retrospect, we conjectured that this first task may have facilitated this additive-like normalization. We therefore designed a subsequent task for the teaching experiment in which students would have little opportunity to reason in this way. The task was also intended to focus students' attention on group performance, consequently the text of the problem contained no information about individual contributors. Additionally, we wanted to set a problem that would not evoke an automatic response to use or calculate the mean. In this task (see figure 1) students were asked to use sample data to help them devise a criteria by which to select a winning troop of Girl Scouts nation-wide.

In this problem both Marissa and Nick sensed that considering only the number of boxes sold or only the amount of money earned would not be a fair way of determining a winner. Nick's strategy to solve this problem was to divide the number of boxes by the number of troop members. He viewed this as a way to express the number of boxes sold as a multiple of

"The National Office of the Girl Scouts of America is presenting an award to one troop of Girl Scouts for participating in the annual cookie sale. It is your job to advise the National Office. How should they select the wining troop?"

Here is an example of the kind of information that is available to the National Office:

Town	Population	Number of troop members	Amount of money earned	Number of boxes sold
Amesbury	15,00	20	480	120
Westport	37,000	200	3,060	1,020
Kendal	25,000	100	2,275	700
Ellesmere	72,000	300	3,250	650
Union City	1,000,000	500	4,500	1,000

Figure 1. The "Girl Scout Cookie" problem statement and data used by students

the size of the troop, and he apparently took this multiplicative comparison as an indicator of the group's performance.

Researcher: So can you explain your method again?

Nick: Yes, it's really just like dividing, I guess, cause am I was just looking at it like, Amesbury (inaudible) sold, I am trying to say this right, six times the number of troop members.

While Nick's strategy can be considered identical to calculating the mean, he seemed to interpret the product of the division as a multiplicative comparison of magnitudes. The idea of "contribution per contributor" did not seem to be part of his reasoning. When trying to help Nick explain his method, Marissa did seem to have the latter interpretation:

Researcher: What does that tell you? What are those numbers telling you?

Marissa: How many boxes each person would've had to have sold

Researcher: What do you mean each person would've had to have sold?

Marissa: About, how, from the number of members in the troop, how many boxes each person in the troop would've had to have sold to come up with that number of boxes.

But Marissa did not view this "equal distribution" as a scaled measurement of group performance that allowed her to make comparisons. During the session she made it clear that she did not consider "Nick's way" to be an adequate way of comparing the troops. Instead Marissa proposed to solve the problem by giving two awards, one for troops in towns with populations less than 500,000 and another for those in towns with populations greater than 500,000.

When the 7th-grade students, Joy and Kurt, first engaged in this task neither one saw a need to normalize the groups in order to compare their performance. These students thought that the winning criterion should simply be "number of boxes sold". Kurt argued that troop size should not be a consideration because the purpose of selling cookies is to collect funds, so the troop with the highest sales should win. After we pushed the students to consider whether this was a fair criterion the discussion led to their considering the comparison of number of boxes sold and troop size. Kurt recognized this as an average, and both students had an image of average that is consistent with Marissa's: they interpreted it as a hypothetical equal distribution of the total boxes sold, but did not see it as a normalized measure of troop performance. Kurt seemed to think that average was just a different criterion from number of boxes sold, one that was unrelated to group performance.

These observations inspired us to design a task intended to have Joy and Kurt confront the problematicity of comparing the performance of groups of unequal sizes in a situation that did not explicitly involve a competition. The problem asked to compare crime statistics in the cities of San Diego (SD) and Tijuana (TJ) in order to decide whether the smaller town (TJ) had a higher incidence of crime (see figure 2 for an abridged version of the statistics examined by students).

San Diego		Tijuana	
Total crimes	54,421	Total crimes	45,868
Population	1,214,000	Population	813,380

Figure 2. Crime statistics for the cities of San Diego and Tijuana

At first, students focused only on the total incidents of crime. Only after being asked whether TJ could be considered a "safer" place than SD did their conceptions entail a consideration of population sizes. They noted that the difference in total crime between the cities was not proportional to their population sizes, and they appeared to take this as indicative of a difference in *safeness* between the two cities. Joy's comment illustrates their thinking: "I'd probably say that SD, it, to me it, SD looks safer to me 'cause it has so many people. And then TJ just has 800,000 people with 45,000 crimes". Students' idea of how to deal with this problem seemed to be based on hypothetically equalizing their populations and proportionally increasing or decreasing the number of crimes:

Kurt: Uhh, yeah if you could just like find a way to show like population evenly between the two. Like just say give them each a million people but took the crimes that are still there and, like some way figure what it would be with a million people by either adding or subtracting.

Researcher: In each city, each having a million people?

Kurt: Yeah , in each city having a million people, and what the crimes would be using the statistics you have now. And so each city would be even as far as population.

Conclusion

Our students demonstrated relatively sophisticated multiplicative reasoning and their understanding of "average" was more than purely procedural, it seemed consistent with the "fair share" conception mentioned by Mokros and Russell. In spite of this, conceiving of a normalized measure of group performance was non-trivial for them. With the exception of Nick, even when students formed the ratio of group total to group size it was unclear to them what this ratio was an indicator of; it did not appear to constitute a *measure* of group performance relative to group size. In addition, students' strategies for dealing with the difference in group sizes suggests that they did not see that this ratio inherently accounts for this difference.

Our challenge in these experiments was to design instruction that would support students' thinking about *group* performance and their envisioning the need to consider group size when comparing group performances. The results of these experiments point us in a clear direction for the future: to focus on first having students come to conceive of group performance as a *measurable* attribute and then to support their conceiving of the mean as an appropriate *measure* of that attribute.

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ACHIEVEMENT IN MATHEMATICS (AIM) AND MATHEMATICS SELF-CONCEPT (MSC): MULTIVARIATE, MULTILEVEL MODELS

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This study demonstrates the use of multivariate, multilevel statistical models to assess how classrooms affect students' mathematics outcomes such as their achievement in mathematics and their mathematics self-concept. The study found a significant variation among classrooms on their mathematics achievement levels and on their students' mathematics self-concept indicating the relative effectiveness of classrooms in terms of improving students' achievement in mathematics and their self-concept in mathematics. The significance of students' mathematics self-concept on their mathematic achievement, especially on the mathematics achievement of females and students from low SES families was also manifested in the study. Mathematics instructional practices where teachers frequently use real life and practical experiences, and where students are often put into small groups demonstrated to be important instructional techniques for raising students' mathematics self-concept.

Over the past years, reform initiatives in mathematics education have emphasized the significance of affects in mathematics learning. The contention is that affects, such as, beliefs about self and mathematics achievement are important outcomes of mathematics learning. An aspect of self belief that has been widely studied, is mathematic self-concept or students' confidence in mathematics. These studies indicate that students with positive self-concept tend to have high scores in mathematics and that, the low mathematics achievement of females and students from low socioeconomic (SES) background is attributable to their low self-concept or confidence in mathematics (McLeod and Otega, 1993; Reyes, 1984). Self concept in mathematics is defined as individuals' belief or confidence in their ability to learn and perform well in mathematics (Ma and Kishor, 1997).

Mathematics self-concept and confidence in mathematics are particularly important in mathematics learning in the classroom, in the sense that success in mathematical problem solving engenders a self belief in students about their abilities in mathematics (Fennema, 1989) and vice versa. This is the basis for the call to mathematics teachers for innovative instructional practices that promote students interest and self-concept in mathematics.

My main interest in this paper is to determine whether mathematics classrooms have effects on students' mathematics achievement and mathematics self-concept, and the extent to which a possible differences in mathematics achievement between males and females, and among students from varying socioeconomic (SES) backgrounds are attributable to differences in their mathematics self-concept. The objective is to provide empirical evidence to support the idea that, promoting a positive self-concept among mathematics students is a desirable instructional goal for enhancing learning in mathematics classrooms.

Most of the research on how classrooms or classroom practices affect students confidence in mathematics had been very descriptive with virtually no empirical studies on how classrooms affects students self concepts in mathematics and their mathematics achievement. This is largely due to the complex nature of modeling classroom effects, especially, when it involves more than one learning outcome. However, recent developments in statistical models and multilevel statistical software, have made research on classroom effect possible.

Another major objective of this paper is to demonstrate the use of multivariate multilevel models in assessing how classroom affects students self-concept in mathematics. In classroom effect research, simple multilevel models involve separate regression equations for each classroom which yields a set of intercepts and regression coefficients (gradients) that can be regressed on classroom level variables. The procedure is the same in multivariate multilevel models, except that, a lower level is introduced into the model which allows researchers to examine separate multilevel models for each outcome along with the correlation between the outcomes. This lower level simply uses dummy variables to distinguish the outcome variables (see Goldstein, 1995). The multilevel statistical program, MlwiN, version 1.02.0003, was used for all the analyses.

Data and Variables

I employed the Canadian population 2 data from the Third International and Mathematics Study (TIMSS) for all the analyses. Population 2 is the 13-year-old students in Canadian grade 7 and 8 mathematics and science classrooms. The mathematics achievement score was standardized (mean = 0, standard deviation = 1) and re-scaled such that the average score for grade 7 was 7 and the average score for grade 8 was 8 into a "grade equivalent" (GE) scale. Since there are ten months of schooling in a year, one can express these scores in terms of months of schooling. The socioeconomic status variable (SES) is a standardized composite score of variables describing the level of education of the parents of a child, and the number of educational-related materials in a child's home. Gender was coded as .5=female and -.5=male. The self-concept variable is four point Likert-scale describing the extent to which students feel they are good in mathematics. The item was re-scaled and standardized such that high scores indicate positive feelings and lower scores mean negative feelings.

Instructional practices

Students responded to a wide range of questionnaires characterizing their classroom instructional activities (see below for the description of the items for these questionnaires). Two constructs (small grouping, and problem solving) describing the instructional activities in classrooms were derived from these questionnaires by taking the aggregated classroom mean of the items representing each construct.

Description of the items for the instructional variables

Small grouping:

BSBMSGRP	Work in pairs or small group in class.
BSBMSMGP	Work together in small groups on problems or project.

Problem solving:

BSBMPROB	Show how to do mathematics problems.
BSBMEVLF	Mathematics problems with everyday life things.
BSBMRULE	Explain rules and definitions.
BSBMPRAC	Discuss practical mathematical problem related to real life.
BSBMEG	Try to solve examples related to new topic.

These variables were used as proxy for students' perception of the extent of the described instructional practices in their classrooms. The mean of these constructs at the classroom levels is a measure of the magnitude of these instructional activities in a particular classroom. The items were coded such that higher numbers represent frequent occurrence of a particular instructional practice in a classroom.

Models

The analyses involved four models. Model 1 was the null model which simply partitioned the total variation on each of the outcome variables, AIM and MSC, into within classroom and between-classroom variance as is usually done in analysis of variance (ANOVA). TIMSS Canada sampled only one classroom per grade so that the within school and within classroom effects are confounded. Grade, a dummy variable coded as, 1 = grade 8, 0 = grade 7, was added into the equations in model 2 to serve two main purposes:

1) adjusting for grade effects, and 2) ensuring that the total variation on an outcome variable was partitioned into between and within-classrooms rather within schools. The students' background variables were added into the equations in the third model to determine the effect of these variables on AIM and MSC. The last model was designed to model the extent of the effect of MSC on gender and SES differences in mathematics achievement, and also determine the influences of classroom mathematics instructional practices on MSC. The multivariate structure was set off in the last model because the covariance between MSC and AIM is meaningless with MSC in the equation as an independent variable. This model is equivalent to separate multilevel models for AIM and MSC.

Analyses

Table 1 displays the parameter estimates, the variance components, and the parameter correlations of the models. Model 1 indicates a statistically significant (p<.01) variation among classrooms on students mathematics achievement scores and on their mathematics self-concepts. These variations (mathematics achievement = 1.441, mathematics self-concept = .049) explained about 21.9% and 4.9% of the total variation on mathematics achievement and on students' mathematics self-concept respectively. This means that classrooms do make a difference on students' mathematics achievement as well as on their mathematics self-concept. The model also shows a positive correlation between students' AIM and their MSC as well as between classroom levels of AIM and their levels of MSC.(see the last two roles in Table 1) indicating that students with high mathematics achievement tend to be confident in mathematics, and classrooms with high mathematics achievement levels tend to have high mathematics self-concept. Consistent with the scaling of the mathematics achievement score, the difference between grade 7 and grade 8 on their mathematics achievement is about a year of schooling (see model 2). However, grade 7 students seem to have more confidence in their mathematics than grade 8 students. The difference is about 6.4 percent on the MSC scale. Models 3 revealed that the backgrounds of students affect their AIM and their MSC. The predicted AIM and MSC scores of females and students from low SES families are significantly lower than their counterparts. The SES difference is about 6 months of schooling on AIM and 18.6 percent on the MSC scale, while, the gender difference is about 2 months of schooling on AIM and 16.8 percent on MSC scale. Model 4 demonstrates the magnitude of the effect of mathematics self-concept on mathematics achievement. The effect, .94 (p<.01) indicates that, a unit increase in a student's mathematics self-concept is likely to increase the mathematics achievement of a student by over 9

Table 1

Parameter Estimates, Variance Components and Parameter Correlations

	model 1		model 2		model 3		model 4	
	AIM	MSC	AIM	MSC	AIM	MSC	AIM	MSC
Intercept	7.246**	-.001	7.753**	-.034*	7.790**	-.024	7.813**	-.026
Gender					-.026	-.168**	-.044	-.169**
SES					.574**	.186**	.400**	.187**
Grade			-.997**	.064**	-.062*	.063**	1.003**	1.061**
Self-Concept							.941**	
Small Grouping								.028**
Problem Solving								.049**
Variance Components and Parameter Correlations								
Variance Components:								
Among Classrooms	1.444**	.048**	1.182**	.047**	.999**	.044**	.995**	.039**
Within Classrooms	5.152	.934	5.154	.934	4.885	.898	4.085	.898
Intra-class correlation	.219	.049						
Classroom variance Explained (%)			18.1	2.1	30.8	8.3	31.1	18.8
Parameter Correlation:								
AIM*MSC								
Among classrooms	.144		.229		.109			
Within Classrooms	.430		.430		.405			

*significant at p<.05

**significant at p<.01

months of schooling. Mathematics self-concept also reduced the SES mathematics achievement gap from .57 to .40 (about 30 percent) and the gender gap in AIM to a statistically non significant value. The last column of the table 1 shows a positive effect of classroom instructional practices on mathematics self-concept. The mathematics achievement level is high in classrooms where students are often put into small groups and where mathematics problem solving often involved real and practical experiences of students. The effect reduced the variation among classrooms on their MSC from .044 to ,039 and therefore explaining about 11.4% of the variation.

Conclusion

This study has demonstrated that the low mathematics achievement of female students and students from low SES families is partly attributable to their low confidence in mathematics. The study supports the idea that raising the confidence levels of students should be an important mathematics instructional objective as this would likely lead to high mathematics achievement levels for all students. This objective could be achieved through mathematics instructional practices that incorporates problem solving activities involving real and practical experiences of students and where students are often put into small groups.

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INCONSISTENCIES IN PRESERVICE TEACHERS' THINKING ABOUT PROBABILITY

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This paper reports some preservice teachers' inconsistent probabilistic reasoning while solving a variety of probability problems in a test-like written setting and an interview setting. The inconsistent reasoning was mainly because of the tension between their formal and informal thinking. The preservice teachers' thinking appeared to be inconsistent especially in the interview setting when the investigator probed them further about their initial responses. Although all the participants were aware of probabilistic concepts of independence and randomness, the tasks and settings of the investigation played an important role on how they viewed these concepts.

Many researchers have shown considerable interest in students' thinking about probability (Fischbein & Gazit, 1984; Konold, Pollatsek, Well, & Lohmeier, 1993; Shaughnessy, 1992; Tversky & Kahneman, 1982). One of their major findings is that many school and college students use inconsistent reasoning in solving probability problems. Konold and his colleagues (Konold, 1989; Konold, Pollatsek, Well, & Lohmeier, 1993) have identified informal conceptions of probability demonstrated by some first year college students. They also found that college students' responses on probability problems in the test like situation and the interview situation substantially varied. In spite of a large amount of research carried out on students' thinking about probability with school and first year college students, there is a lack of research on preservice and inservice teachers' thinking about probability, who are preparing to teach these content areas in schools.

It is important to investigate teachers' thinking about probability because their thinking often have a large impact on students' reasoning (Thompson, 1992). In the 1998 International Group for the Psychology of Mathematics Education held in Stellenbosch, South Africa I reported that secondary school preservice teachers with reasonably strong background in mathematics chose classical, frequentist, or subjective conceptions of probability based on tasks and settings of the investigation (Koirala, 1998). This paper provides more insights in this line by focussing on inconsistent probabilistic thinking held by preservice teachers of secondary school mathematics.

Methodology

The participant and the setting

This study focused on probabilistic reasoning held by four preservice teachers of secondary school mathematics at the University of British Columbia, Canada. Each preservice teacher had a relatively strong background in mathematics and probability because they had taken at least 10 undergraduate mathematics courses including a course in probability and statistics.

Each preservice teacher was asked to solve probability problems in two different settings. First, they were asked to solve problems in a written setting, similar to a test-like situation. Second, they were asked to solve problems in an interview setting. The researcher had analyzed participants' written responses before asking problems in the interview setting. This allowed the researcher to check their consistency of probabilistic thinking.

Although five problems were included in each setting, not all problems were designed to check for consistency. Only four problems (two from the written setting and two from the interview setting) that allowed for consistency checking are reported in this study. The problems were selected and modified from the work of previous researchers (Fischbein & Gazit, 1984; Konold, Pollatsek, Well, & Lohmeier, 1993; Shaughnessy, 1992; Tversky & Kahneman, 1982). The problems in the written setting were called the birth sequence problem, and the five head problem. The birth sequence problem asked the participant to select the most likely sequence (BGBBBB, BBBGGG, GBGBBG, BGBGBG, or all sequences equally likely) of births in a family. The five head problem required preservice teachers to decide the sixth outcome if the previous five outcomes were all heads when a fair coin was tossed. The birth sequence problem and the five head problem used in the written setting were similar to the head-tail sequence problem and the lottery problem in the interview setting. The head-tail sequence problem and the lottery problem asked in the interview setting are provided in the next page.

Head-tail sequence problem

Which of the following *orderly* sequences is *most likely* to result from flipping a fair coin six times? (Heads = H, Tails = T).

a) HTHHHH
b) HHHTTT
c) THTHHT
d) HTHTHT
e) All four sequences are equally likely
Give reasons for your choice.

Lottery Problem

Two students in a probability class have opposing views about selecting numbers for a lottery 6/49. One student prefers to choose consecutive numbers like 1, 2, 3, 4, 5, 6. Other student thinks that chance of getting a sequence of six consecutive numbers in a lottery is smaller than getting a random sequence of any six numbers. Both students come to you for an advice. How would you advise them?

Method of data analysis

Each participant's choices and written episodes to the birth sequence problem and the five head problem were collected by the researcher. Their responses to these problems were analyzed to look for consistency between their answers and written justifications. Each episode of the problem solving in the interview setting was audiotaped, transcribed, and analyzed to identify themes emerging from the responses. Their responses to the problems in the interview setting were analyzed to check for consistency within each setting and between settings. The main focus of the analysis was to explore what types of reasoning the preservice teachers used in solving the problems posed and whether or not their reasoning were consistent within and between the settings.

Results

Preservice teachers' reasonings were inconsistent within problems and across settings in this study. The most inconsistent participant in the study was John (pseudonym) who had taken 16 undergraduate mathematics courses including two courses in probability and statistics. His response to the birth sequence problem in the written setting was influenced by the representativeness heuristic (Tversky & Kahneman, 1982), even though he said that "having a boy is as equally likely as having a girl" and also "each birth is independent of any previous births." Despite his awareness of independence and randomness in the birth sequence task, John thought that the sequences BBBGGG, GBGBBG, and BGBGBG would be more likely than the sequence BGBBBB. Mathematically all these four sequences are equally likely.

John repeated this conception in the head-tail sequence problem asked in the interview setting. In the beginning of the problem, John stated that all the sequences were equally likely. Upon further probing, however, he was not confident about the equal likelihood of all these outcomes. When the researcher asked whether the outcomes would be equally likely if the total number of trials was 20 and there were 19 heads and 1 tail in one sequence and 12 heads and 8 tails in the other sequence [shows the examples of sequences HTHHHH..., HTHHTHTHHT...], John replied that he would change his answer. John thought that when the number of trials was increased up to 20, then the sequences with roughly half heads and half tails would be more likely than the sequences with an unbalanced number of heads and tails, for example, nineteen heads and one tail. Like in the head-tail sequence problem, John's responses to the lottery problem were also inconsistent.

The other participants in the study also demonstrated similar inconsistencies. In the lottery problem Jane, like John, thought that the consecutive sequence 1, 2, 3, 4, 5, 6 would be less likely to occur than a random sequence of any six numbers. In the five head problem of the written setting, Jane and Alan predicted that the sixth outcome would more likely to be a tail because they thought that the head and tail should balance out when the sample size was large enough. Their response to the head-tail sequence problem in the interview setting was also similar. They reasoned that if the sample size was large enough "about fifty percent of each flip was going to be heads and fifty percent was going to be tails." Both of them stated that the sample size in the birth sequence problem and the head-tail sequence problem was not large enough and so chose the statement "all sequences were equally likely" as their correct response.

The participants in the study basically expressed their beliefs that the same number of heads and tails can not be expected in a sequence when the number of trials was small. If the number of trials was small, they thought that the sequences, whether the occurrence of heads and tails appeared to be balanced or not would have the same probability of happening. However they thought that if the number of trials was large enough, the sequences with approximately the same number of heads and tails were more likely to occur than the outcomes with an unequal number of heads and tails. Generally the preservice teachers demonstrated their awareness of independence and randomness but did not look at each event independently, especially if the number of trials was larger. Their faith in the law of large numbers overshadowed their knowledge of independence and randomness.

The investigator also asked, "what would be their response if the question was to find out the least likely sequence instead of the most likely one?" Jane chose HTHHHH because the length of the string in this sequence was too long compared to the other three. The fourth participant Anne also said that the sequence HTHHHH was the least likely one. It was interesting that the participants chose HTHHHH as the least likely sequence even though they stated earlier that there was no most likely sequence. When the phrase was changed from "most likely" to "least likely" the preservice teachers were using their informal reasoning. If they had used the formal reasoning of independence and randomness their response to the problem would have remained the same even when the phrasing was changed.

It is interesting to note that the preservice teachers' probabilistic reasoning were similar to those of school and college students as reported by other researchers (Fischbein & Gazit, 1984; Konold, Pollatsek, Well, & Lohmeier, 1993; Shaughnessy, 1992; Tversky & Kahneman, 1982). Although the preservice teachers' initial responses were usually correct in the interview setting their reasoning often contradicted. The inconsistencies were even more common when the interviewer changed the phrasing of the question such as from "most likely" to "least likely.

Conclusions

The results of this study indicate that preservice teachers struggled to find a resolution between their formal and informal thinking. They indicated that they used formal and informal reasoning depending on tasks at hand. They found it easy to make a decision if their formal and informal thinking did not conflict with each other, but when they conflicted they lacked confidence in their decisions. The major cause of inconsistency in their probabilistic reasoning was the conflict between their formal and informal thinking.

It is interesting that these preservice teachers were aware of these inconsistencies most of the time. They did not feel bad about their inconsistencies. Rather they admitted that human beings are inconsistent in their thinking. They commented that their thinking differed dramatically when they solved problems in their everyday lives and when they solved problems in their university examinations.

The findings of this study have important implications about the nature of probability and its teaching in schools, college and university levels. If people with strong background in mathematics are not consistent in their probabilistic thinking why would ordinary people and students with little mathematics background? Although, the study focussed on only four preservice teachers' thinking, and therefore has some limitations, it certainly raises questions regarding the ability of current system of teaching probability to help individuals developing consistent thinking. Psychologists, mathematicians, and mathematics educators should be engaged in a dialog to decide whether or not school and college mathematics courses need to focus on helping students develop a consistent mathematical thinking whether that may be in everyday world or academic world.

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THE TEACHER'S ROLE IN SUPPORTING STUDENTS' DEVELOPMENT OF STATISTICAL REASONING

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The purpose of this paper is to highlight the teacher's proactive role in supporting students' ability to reason about data while developing statistical understandings related to exploratory data analysis. In doing so, I will present an episode taken from a seventh-grade classroom in which I participated in a twelve-week teaching experiment. One of the goals of the teaching experiment was to investigate ways to proactively support middle school students' development of statistical reasoning. The goal of the instructional sequence then became that of students engaging in instructional activities in which they both developed and critiqued data-based arguments. In this setting, the teacher's role was viewed as critical in supporting shifts in both the students' ability to engage in data analysis and to reason statistically about the data they were analyzing.

The purpose of this paper is to document the teacher's proactive role in supporting students' ability to reason about data while developing statistical understandings related to exploratory data analysis. The aspects of the teacher's role highlighted in this paper include supporting students' mathematical development by (1) planfully sequencing and orchestrating whole-class discussions in order to support mathematical shifts in students' ways of reasoning, (2) redescribing and notating students' offered solutions and explanations to ensure understanding and to highlight significant aspects and (3) working to ensure that students' activity remains grounded in the situation-specific imagery of the context of the investigation.

In this paper I will clarify how the afore mentioned aspects of the teacher's practice can provide opportunities to support shifts in students' ability to reason mathematically by presenting an analysis of an episode taken from a seventh-grade classroom in which I participated in a twelve-week teaching experiment. One of the goals of the teaching experiment was to investigate ways to proactively support middle-school students' ability to reason about data while developing statistical understandings related to exploratory data analysis. The image that emerged in the course of developing an instructional sequence was that of students engaging in instructional tasks in which they both developed and critiqued data-based arguments. In this setting, the teacher's role was viewed as critical in supporting shifts in both the students' ability to (1) engage in data analysis and (2) to reason statistically about the data they were analyzing.

Instructional Sequence

As the research team began to design the instructional sequence, we attempted to identify the "big ideas" in statistics. Our plan was to develop a single, coherent sequence and thus tie together the separate, loosely related topics that typically characterize middle school statistics curricula. In doing so, we came to focus on the notion of distribution. This enabled us to treat notions such as mean, mode, median, and frequency as well as others such as "skewness" and "spread-outness" as characteristics of distributions. It also allowed us to view various conventional graphs such as histograms and box-and-whiskers plots as different ways of structuring distributions. Our instructional goal was therefore to support students' gradual development of a single, multi-faceted notion, that of distribution, rather than a collection of topics to be taught as separate components of a curriculum unit. In formulating hypotheses about how the students might reason about distributions, one of our primary goals was that students would think about data sets as entities that have properties in their own right rather than as collections of points (e.g., Hancock, Kaput, & Goldsmith, 1992; Konold et al., in press; Mokros & Russell, 1995). We conjectured that if students did begin to think about data in this way, they could then investigate ways of structuring data sets that would help them identify trends and patterns.

As we worked to outline the sequence, we reasoned that students would need to encounter situations in which they had to develop arguments based on the reasons for which the data were generated. In this way, they would need to develop ways to analyze and describe the data in order to substantiate their recommendations. We anticipated that this would best be achieved by developing a sequence of instructional tasks that involved either describing a data set or analyzing two or more data sets in order to make a decision or a judgment.

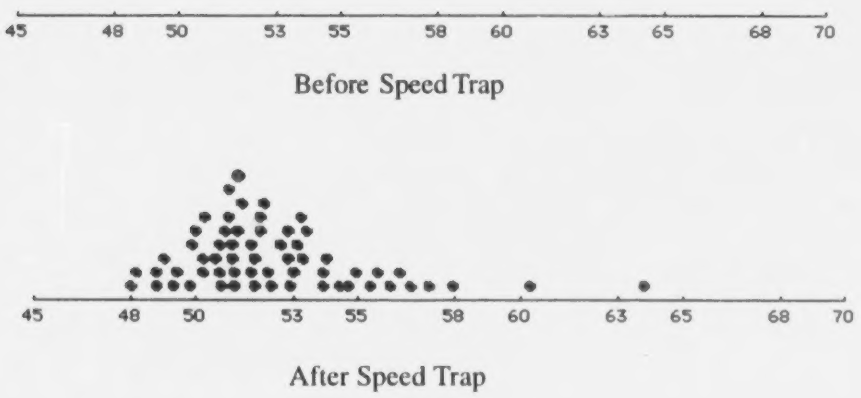
Classroom Episode

The format for the class sessions typically involved an initial discussion about the task situation. During this discussion, students raised questions about the problem at hand and data that would need to be collected in order to answer the question posed. Against that background, they would discuss methods of data collection. Afterwards, students would work in pairs at the computer conducting analysis of the data. Students analyzed the data using computer minitools that were developed as integral aspects of the instructional sequence. These tools were intended to provide ways for students to both organize and structure the data sets so they could then use the results of these efforts in their analysis. Their work at the computer was followed by whole-class discussions in which they would explain and justify the results of their analysis. In order to facilitate the discussion, a computer projection system was used to allow the students to project the data sets on the white board as they discussed their ways of organizing and structuring the data to support their argument.

As the students engaged in tasks from the instructional sequence, the teacher focused explicitly on ways to support her students' mathematical development. This can be seen in an episode that occurred near the midpoint of the teaching experiment. At this point in the instructional sequence, students were working to analyze data using the second of two minitools. In this tool, a data set was shown as collection of dots located on an axis (i.e. an axis line plot). In addition, two data sets could be viewed simultaneously for comparison. The tool also offered a range of ways to structure data. Two of the options can be viewed as precursors to standard ways of structuring and inscribing data. These are organizing the data into four equal groups so that each group contains one-fourth of the data (precursor to the box-and-whiskers plot) and organizing data into groups of a fixed interval width so that each interval spans the same range on the axis (precursor to the histogram). However, the three other options do not correspond to graphs typically taught in school. These involve structuring the data by (1) making your own groups, (2) partitioning the data into groups of a fixed size, and (3) partitioning the data into two equal groups. The least sophisticated option simply involved dragging one or more bars to chosen locations on the axis in order to partition the data set into groups of points. The number of points in each group was shown on the screen and adjusted automatically as the bars were dragged along the axis. The key point to note is that this tool was designed to fit with students' ways of reasoning while simultaneously taking important statistical ideas seriously.

The particular investigation that is the focus of this analysis involved analyzing data on driving speeds on a very busy road in the city. As drivers are known to speed on this particular stretch of highway, the police department had decided to set up a speed trap to try to slow the traffic. Students discussed what information would be necessary to determine if the speed trap was effective. After much discussion on both issues of safety related to speed and on the specifics of this particular problem, students were asked to compare data on the speeds of sixty cars before the speed trap was set up with the speeds of sixty cars a month later to decide if the speed trap was effective (see Figure 1).

After students had analyzed the data using the computer minitool, they were asked to develop a written argument that could be submitted to the Chief of Police. In doing so, students attempted to find a way to organize



Before Speed Trap

After Speed Trap

Figure 1. Data on cars before and after speed trap.

and describe the data so that they could make a recommendation. In the subsequent whole-class discussion, students read their reports as part of their explanations. The first report was presented by Janine.

Janine: If you look at the graphs and look at them like "hills" then for the before group it is more spread out and more are over 55. If you look at the after graph then more people are bunched up closer to the speed limit which means that the majority of the people slowed down.

Teacher: Okay, Janine said if you look at this like hills. . . now think about this as a hill (draws a hill over the data in the first data set) and think about this as being a hill (draws a hill over the second data set) see what Janine was talking about? Before the speed trap the hill was spread out but after the speed trap the hill got more bunched up and less people were speeding.

Kent: They were slowing down. I want to compliment Janine on the way that she did that. I couldn't find out some way to compare and I think that was a good way.

In reviewing the students' reports in preparation for the whole-class discussion, the teacher had judged Janine's solution to be significant with respect to her mathematical agenda. It was, in fact, the first time that a student had worked to find a way to talk about a data set as an entity and was clearly a step toward reasoning about distributions. For these reasons, the teacher purposely began the whole-class discussion by focusing on Janine's solution in an attempt to initiate shifts in the other students' ways of reasoning toward more global views of the data sets. During the discussion, the notion of a data set as a distribution of data points began to emerge as the students discussed Janine's "hills" interpretation. Previously, the focus in whole-class discussions had been on the number of data points in parts of data sets. This change is significant in that it offered opportunities to build on students' current understandings to support shifts toward more sophisticated ways of reasoning.

The teacher also took a very proactive role in ensuring that the other students in the class understood Janine's way of organizing and structuring the data. She did so by highlighting the important aspects of Janine's report by actually drawing the "hills" over the data sets as she verbally recast the explanation so that everyone would understand. In doing so, she also indicated that she particularly valued Janine's way of reasoning about the data.

The next report selected was presented by Kim and consisted of structuring the data into equal intervals as shown in Figure 2. In her argument, Kim maintained that by focusing on the number of cars whose speeds fell in the range of 50 to 55 miles per hour and 55 to 60 miles per hour, one could see a shift towards a slower speed. She noted that of the cars in the range of 50 to 60 miles per hour, before the speed trap 19 cars were traveling between 55 and 60 miles per hour. After the speed trap, only 7 cars were traveling between 55 and 60 miles per hour. She argued that this was a significant shift.

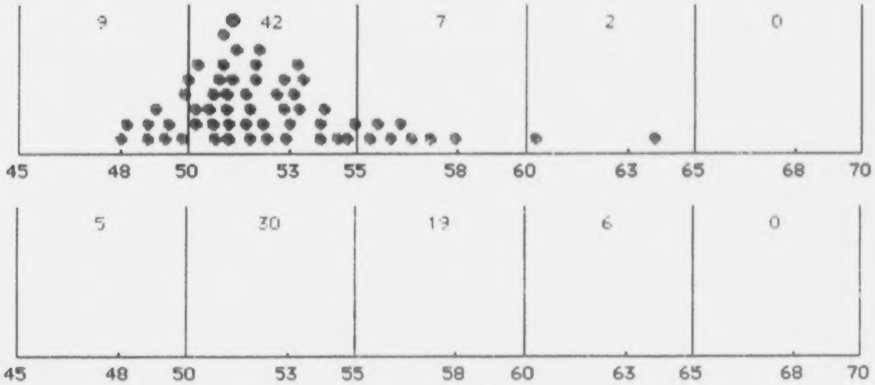


Figure 2. Speed data in intervals of five

After Kim finished, the teacher redescribed and notated Kim's solution. Kim's solution was also judged by the teacher to be significant because, although adequate in this particular task situation, the use of direct additive comparisons would be problematic in situations where the data sets were composed of unequal N's. As a result, the teacher wanted to highlight the process of using additive comparisons so that it could be problematized in later task situations. It was hoped that this would then create a need for a shift toward multiplicative ways to reason about how to structure data sets in order to make a comparison.

It is important to note that the two analyses that were shared in this class session were carefully selected to fit with the teacher's mathematical agenda. Both reports offered opportunities for the teacher to support shifts in the students' ways of reasoning about data while building from their contributions. This is very different from simply allowing all students to share their way. It requires that the teacher make judgments about the significance of each solution and how it fits with the mathematical goals. This aspect of teaching is extremely important in that it presents opportunities for mathematical shifts to emerge out of the students' problem-solving efforts.

Throughout the whole-class discussion, the teacher continually focused the students' attention on issues concerning the data on the two groups of cars. Earlier discussions had often consisted of students talking about how many "pink dots" were greater than a certain value compared to the number of "green dots" (the minitool distinguished two data sets by using pink and green for the data points). By working to ensure that the students' conversations were grounded in the context of the investigation, the teacher was able to support the students' ability to engage in actual data analysis instead of simply acting on dots or numbers. In addition, the students were developing ways to create mathematical arguments that were grounded in the context of the investigation at hand.

Conclusion

The way of conceptualizing the teacher's role that I have described in this paper is not intended to make light of the extreme complexity of teaching. It is, instead, intended to *acknowledge* this complexity while attempting to provide a frame for thinking about the numerous and diverse aspects of effective teaching. In reflecting on Simon's (in press) description of the Mathematics Teaching Cycle, he notes the "inherent tension between responding to the students' mathematics and creating purposeful pedagogy based on the teacher's goals for student learning." In a similar manner, I am attempting to point to issues of pedagogy that can frame the decision-making process while acknowledging that the mathematics must be the central focus in all decisions. However, I do not want to be perceived as offering a prescriptive list of effective techniques that, when employed, will improve student learning. Instead, my goal is to attempt to identify aspects of practice that offer ways to effectively support students' mathematical development. In this way, these critical aspects can offer a starting point for analyzing what teachers might *do* as they work toward advancing their mathematical agenda.

Acknowledgments

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WHY SAMPLING WORKS OR WHY IT CAN'T: IDEAS OF YOUNG CHILDREN ENGAGED IN RESEARCH OF THEIR OWN DESIGN

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Abstract. This research is part of a multi-year project investigating the potential of children's data-based scientific research, through a combination of classroom-based educational design experiments and associated laboratory studies. The subjects in this analysis consisted of second, fourth and fifth graders in three elementary school classrooms (n=75). Shortly after the children had designed and implemented a research project with a small group of fellow students, the author conducted a structured interview that probed their ideas concerning interpretation of the data in their respective study, warranted generalizations and extrapolations, how they could strengthen their study, *etc.*. Surprisingly few children thought they could make inferences about the population on the basis of a small sample, thus not falling prey to the buggy *law of small numbers* identified by the judgement and decision-making literature. This propensity is attributed to their acute awareness of variability, a recurrent theme across most of their rationales of why sampling couldn't work. Adequate sample size and robust sample trends constituted the most common necessary conditions children articulated for justifying inference from sample to population.

The NCTM (1989) recommends that children at the kindergarten through grade four level should collect and interpret data and explore concepts of chance. This recommendation accords with the contention of the American Association for the Advancement of Science (1993) that children should not only study about science but actually *do* science. The concept of sampling, in conjunction with the correlative idea of the Law of Large Numbers, is fundamental to the understanding of the design, implementation, and interpretation of empirical studies.

The adult judgment and decision-making research literature indicates that ideas of sampling and the Law of Large Numbers are non-intuitive even to adults (Fischbein & Schnarch, 1997; Konold, Well, Lohmeier, & Pollatsek, 1993; Shaughnessy, 1992). While a growing literature examines children's understandings of chance, randomness, and probability (*e.g.*; Fischbein, 1975; Kuzmak & Gelman, 1986; Metz, 1998; Piaget & Inhelder, 1975), negligible research in the developmental literature examines children's understandings of sampling and the Law of Large Numbers. Of even more importance to educators, negligible work examines what young children can grasp of sampling given instructional support. Our multi-year, classroom- and laboratory-based research project aims to scaffold children's grasp of fundamental constructs of statistics and probability in the context of research design, data interpretation, and study revisions and also examine the understandings children manifest in interaction with the curriculum.

Methodology

Following Ann Brown's (1992) model of *educational design experiments*, children's emergent statistical understandings are being investigated in the context of classroom-based teaching experiments and parallel laboratory studies. Whereas the classroom-based teaching experiments support study of the longitudinal development of children's statistical understandings within an enriched learning environment, the laboratory studies enable our close analysis of children's thinking under more controlled conditions.

The top-level goal of our project is to narrow the gap between scientific inquiry as practiced by scientists and scientific inquiry as practiced by children in elementary school classrooms. The other curriculum design principles formulated to support this top-level goal include:

* Scaffold the full scope of data-based research, including preliminary explorations of the domain, formulation of researchable questions, planning of experiments, subsequent collection, representation and analysis of data, preparation of research posters, and revision of research plans.

* Concentrate children's science study in a small number of strategically selected domains.

* Scaffold understandings of uncertainty and chance, especially in the interpretation of scientific phenomena and the design and revision of research studies.

* Gradually transfer control of the inquiry process to the individual child.

We have developed curriculum in animal behavior and botany on the basis of these design principles, within which children have worked in "research teams" of pairs (or occasionally triads) to conceptualize their own questions, and then to design, implement, critique and redesign their investigations. We have also draw upon units of the *Investigations in Number, Data and Space* curriculum to formulate research projects in less time- and less semantically-demanding contexts where the teacher aims to focus on issues such as data representation. Following the completion of their research projects, the author conducts a laboratory-based *post-research study interview* with each research team. This protocol was developed to elucidate the children's understanding of uncertainty, chance and causality vis à vis their research design and data interpretation. The protocol probed: *(a)* their conceptualization of the question they researched and their research methodology; *(b)* their conceptualization of their results ("What did you find out? " How sure are you that .." *(c)* the possibility of increasing their confidence level ("Is there a way you could be more sure that ...?" "Any other way?"...); *(d)* any improvements they thought they could make in their study ("When scientists finish a study, they frequently try to figure out how they could have improved their study. Can you think of a way that would make your study a better study? Why would that make your study better?"); *(e)* their ideas of a question and corresponding methodology for a subsequent study; *(f)* the generalizability of their current findings ("I wanted to ask you another question about the study you already did. Do you think your results tell you about just the [plants / crickets] you studied or others as well? Why do you think that .....?"; *(g)* if they do not think their study already informs them about the population, possible changes in their research design to more adequately support inference from the sample to the population (" Is there a way you could set up your study so that it would tell you not just about the [ones] you studied, but others as well, or don't you think there is? [If a child suggests a way] Why would ....?"; and [where they have not suggested increasing the sample size, the weak scaffold, "Would it help if you use a lot more [plants/ crickets], would that help you learn about how [plants/ crickets] that you haven't studied or don't you think it would? Why/why not?"). These interviews, as well as all laboratory studies and all project-relevant instruction, were videotaped to enable close analysis by multiple viewers.

The educational design experiments are being implemented in a rural elementary school in Southern California, with a middle SES population (with 34% receiving free or reduced-cost lunch). During the first three years of its existence, the project was centered in two teachers' classrooms; one lower elementary teacher who has shifted between first and second grade and a fourth/fifth grade teacher. Analyses reported in this paper are based on seventy-five subjects, comprised of all members of the straight second grade classroom (*n*=21) and the two fourth-fifth grade classrooms (including 15 fourth graders and 16 fifth graders in the first year and 13 fourth graders and 16 fifth graders in the second year).

The author and her research assistant analyzed all the videotapes. We developed coding categories of children's rationales for why sampling doesn't work; why sampling does work; where qualified, the particular conditions under which sampling works; and their ideas about the nature of the inferences that sampling supports. These categories were developed through a modeling process, with new ideas about how to better represent the children's interpretations requiring a re-examination of all previously analyzed videotapes. In those cases where both the author and her research assistant did not find the evidence for a particular interpretation of a particular child's thinking compelling, the particular instance was excluded from the analyses as reported below.

Results and Discussion

At least by the end of the post research study interview, 41% of these children (31 out of 75) argued for the power of sampling (in some veridical, albeit always incomplete form). This group consisted of just 14% of the second graders, 33% of the fourth graders and 56% of the fifth graders in the 1996-1997 fourth-fifth grade cohort, and 38% of the fourth graders and again 56% of the fifth graders in the 1997-1998 fourth-fifth grade cohort. Arguments of these 31 children concerning how one would need to implement the sampling to make it work or statements about limitations on warranted inferences provide further indicators of their respective understandings of the construct. Eighteen of the 31 argued that you had to have an adequate sample size for the sample to give you information about the population. Ten contended that a sample told you about the population only if the trend reflected in the data from the sample was robust. Eight contended that you have to replicate your study before you could say anything about the population. Seven probabilistically qualified the information about the population that one could draw from a sample. Seven came up with the idea of stratification, arguing that you needed to draw from different subpopulations that could well behave differently (*e.g.*; crickets raised in the lab versus crickets from the wild).

At the other end of the continuum, a small minority of second graders (two individuals from two different research teams) appeared to regard gaining knowledge about the population as a whole as unimportant or even appropriate (*e.g.*, "We're only *supposed* to study our crickets, nobody else's."). As Reif and Larkin (1991) emphasize, the goal-structure of the scientific domain diverges from the goal structure of everyday cognition; with maximizing of generality and consistency constituting a main goal within scientific cognition and "adequate prediction and explanation" constituting a subgoal of everyday cognition —— with "leading a good life" as the central goal. In a similar vein, these children appear not to have appropriated the goal as learning about crickets in general, but rather are conducting their study within the classroom cultural constraint of focusing solely on the materials given to them. Furthermore, the strong tendency for second graders to immediately name the crickets they drew to study (and even one research team's framing of their question in terms of these individuals: "Will Charlie and Suzy be more active alone or with other crickets?") suggests a "getting to know you" enterprise, again contrasting with the scientific goal-structure of generality.

The adult judgment and decision-making literature (Tversky & Kahneman, 1971) would lead us to expect that many children would view inference from the sample to population as straightforward and nonproblematic (as parodied in the Tversky & Kahneman's phrase the *law of small numbers*). Surprisingly, we seldom observed this orientation. Indeed, just two of the 21 second graders and three of the 28 fourth graders ever manifested this view. We conjecture that its infrequency stemmed from the children's relatively in-depth exploration of the domain and their acute awareness of the variability therein.

The children's statements about why samples did not tell them about the population suggest conceptual issues that need to be incorporated into instruction. Among the 37 children who at some point in their protocol posed an argument against the power of sampling, 13 contended that you only know about the subjects you observe; 12 posed the closely related argument that you only know about a population when you test every member of the population; and 19 argued sampling cannot work where there is variability in the population.

Conclusions

In conclusion, the concept of a sample is fundamental to research design, data analysis and, more generally, reasoning under uncertainty. Given the size of our sample, the author remains extremely tentative about their relative frequency—particularly under alternative instructional conditions. However, the study has identified the existence of a range of understandings and misunderstandings about the relation between a sample and population that emerge under these instructional conditions. These results suggest issues that could get in the way of children understanding of why and how sampling works; including a goal-structure which excludes generality, the belief that variability within the population negates the functionality of sampling, and a failure to grasp how sampling can be informative of the population. More generally, these findings point to the conceptual challenge of coordinating patterns and uncertainty—an integration crucial to the construction of ideas of randomness, probability and sampling.

Current research in the project classrooms is aimed in several directions. We are developing a statistical strand within the science modules that emphasizes: *(a)* frequentist probabilities; and *(b)* mapping of distributions of repeated trials of different sample sizes, viewed against the mapping of the distribution of the population. We are also making some changes in the science domain research-focused units, including: *(a)* the children's access to a much larger number of organisms, so that their studies can include a more adequate sample; *(b)* a stronger emphasis on data representations in which discrete data points are obscured; and *(c)* inclusion of "thought experiments", in the form of narratives of research accounts from top scientists within the domain they themselves are studying to structure the children's reflection on these conceptual issues of statistics, as well as the big ideas of biology.

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MIDDLE SCHOOL STUDENTS' AWARENESS OF THE RELATIONSHIP BETWEEN EXPERIMENTAL AND THEORETICAL PROBABILITY: MAKING THE CONNECTION BETWEEN DATA AND CHANCE

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For too long, we have seen the consequences of teacher-centered middle school learning environments: formula-driven representations of probability concepts force unsuccessful students to develop strategies for coping without understanding. One means of effecting innovation involves curricular change. Results of this study illustrate that, while typical middle school students are unaware of the relationship between experimental and theoretical probability, appropriate cognitive activity engaging students in probability simulations can enrich such conceptual development.

World-wide curriculum reforms in school mathematics (e. g., Australian Curriculum Corporation, 1994; National Council of Teachers of Mathematics, 1989) have advocated broadening the scope of probability in the middle school curriculum. In particular, such reform documents have called for students to carry out simulations of random phenomena, and to compare experimental results to the mathematically-derived probabilities. The advocation of simulations to foster probabilistic reasoning is largely predicated on students' awareness of the relationship between the experimental probability of an event (i. e., using relative frequencies to determine the likelihood of an event) and the theoretical probability of an event. In general, the relationship between the two concepts results from the fact that, for a given event, experimental probability will more closely approximate theoretical probability as the number of trials increases.

Purpose of the Study

The present study sought to investigate middle school students' understanding of the experimental probability of an event and the theoretical probability of an event. Additionally, the study sought to determine the role simulations of random phenomena play in fostering students' understanding of the relationship between these two key probability concepts.

Theoretical Frameworks

The investigation was based on two theoretical positions. The first is a pedagogical orientation that acknowledges the importance of teachers' knowledge of student cognitions (e.g., Fuys, Geddes, & Tischler, 1988; Shulman, 1986). The second was a cognitive framework (Jones, Thornton, Langrall, & Tarr, 1999) that captures the manifold nature of students' probabilistic reasoning and, in particular, identifies four levels of thinking with respect to the theoretical probability of an event and the experimental probability of an event. The cognitive framework was used to generate assessment protocols, instructional tasks, and to evaluate the impact of instruction on students' probabilistic thinking.

Methodology and Data Sources

Twenty-three students from one sixth-grade class participated in the study. The researchers used a cognitive framework (Jones, Thornton, Langrall, & Tarr, 1999) to develop a five-day instructional program that comprised a series of problem-solving tasks, key questions, and writing prompts. Each probability task required students to carry out simulations of random phenomena, and to compare experimental results to theoretical probabilities. In this study, classroom instruction was provided by both researchers who attempted to elicit students' probabilistic thinking and to engage students in classroom discourse.

Two mathematically-equivalent forms of an interview protocol were used to assess students' cognitions prior to- and one week after instruction. The instructional program was evaluated with respect to growth in students' probabilistic thinking by using a Wilcoxon Signed Ranked Test (Siegel & Castellan, 1988). Using multiple sources of qualitative data – audiotaped interview assessments, videotapes of instruction, students' journals and worksheets of case study students, and two researchers' journals – case study analysis was undertaken to identify patterns and changes in students' thinking with respect to the theoretical- and experimental probability of an event. The process of triangulation by data source (Miles & Huberman, 1984) was used to provide repeated verification of qualitative research findings.

Results of the Study

The instructional program yielded varied success in fostering students' understanding of the two key concepts, experimental and theoretical probability. Following instruction, marked changes in students' thinking levels were observed. With respect to the experimental probability of an event, the Wilcoxon Signed Ranks Test detected a significant difference ($\underline{z} = 2.03$, $\underline{p} < .05$) in students' level of thinking prior to instruction and their level of thinking on the post-instructional assessment. Although qualitative analysis revealed evidence of students' ability to make connections between experimental and theoretical probabilities following instruction, such growth in students' probabilistic thinking was not uniform.

Prior to instruction, five of six case-study students demonstrated limited awareness of the relationship between experimental and theoretical probability. Emily exemplified students' limited ability to realize that unlikely events become more difficult to occur as the trials in an experiment increase. In the baseline assessment, students considered four sets of outcomes of repeated flips of a colored chip: a) 3 out of 4 white, b) 6 out of 8 white, c) 12 out of 16 white, and d) 24 out of 32 white. Although 75% of the outcomes are white in each sample, this proportion is least likely to occur in the largest set of trials, d). The following excerpt from the initial assessment illustrates Emily's misconception as she sought to determine which set of outcomes was most- and least likely to occur in repeated flips of the colored chip:

> I: Which of these four sets of outcomes do you think is the most likely to occur, or do you think all four sets are equally likely?
>
> E: Well, they're all three-fourths so they all have the same chance because in each row there's four and there's only one red and three whites.
>
> I: Which of these four sets do you think is the least likely to occur, or do you think all four sets are equally unlikely?
>
> E: They're all equally unlikely. It seems like there should be two reds and two whites [in each row] if there's the same chance for each color.

Growth in Emily's thinking was evident on the post-instructional assessment as she demonstrated an awareness of the long-term behavior of repeated flips of the colored chip. In particular, she recognized that larger samples were more likely to reflect the parent distribution. The following is an excerpt from the post-instructional assessment:

> I: Which do you think is the most likely to occur, or do you think all four are equally likely?
>
> E: [Points to 3 out of 4 white].
>
> I: Why is this one [3 out of 4 white] the most likely?
>
> E: Well this one [3 out of 4 white] is unlikely, so that means that if it would happen *again* like this one [6 out of 8 white], it would be *even more unlikely*. And if it were to happen *again* like that one [12 out 16 white], it would be *even more unlikely*.

I: Which of these four sets do you think is the least likely to occur or do you think all four sets of outcomes are equally unlikely?

E: That one [Points to 24 out of 32 white].

I: And why would that be the most unlikely?

E: Because, well, Kevin brought that up in class. Since there's an even chance [for each side], where there's three whites and one red, it would probably happen once instead of like a bunch of times. It would be harder to get one red out of four flips all those times [24 out of 32 white] instead of just this one time [3 out of 4 white] because it would have to happen eight times.

The notion that smaller samples are more likely to yield results not representative of the parent population eluded most case-study students prior to instruction. This was particularly evident in an assessment task in which students determined whether their chance of winning an unfair game was influenced by the number of trials. More specifically, the game involved a spinner partitioned into two sectors, blue comprised 60% and yellow 40% of the circle. After playing a "Winner Takes All" format of the game, students considered whether playing the game to 3 points, or to 10 points influenced their probability of winning. Blake was typical of case-study students at baseline; he demonstrated little awareness of the relationship between the number of trials and the probability of winning the spinner game:

I: Let's say that we play the game again so that the winner is the first to score three points. Compared to last time when we spun it only one time, are your chances of winning better, the same, or are they worse than when we played before?

B: It's the same because there's still more blue than yellow so I'll probably still win.

I: What about my chances of winning? Does it make a difference to my chances of winning if we play to three?

B: No, because my space is still bigger.

I: Let's say that instead of playing the game to three points, we played it so that the winner was the first to score ten points.

Does playing the game to ten points make any difference to your chance of winning?

B: Well, yeah. Like in going to one [point], if it lands on mine then I would win; but if we played to ten, then I could be ahead and you could come back and win. I could still win but your chance [of winning] is better now.

Following instruction, Blake demonstrated his awareness that large samples are more likely to produce results that reflect the parent distribution. In the following excerpt from the post-instructional assessment, Blake realizes that the number of trials does indeed influence the probabilities associated with the unfair game:

I: We're going to start over only now we're going to play the game so that the winner is the first one to score three points. Compared to the first time we played, are your chances of winning better, worse, or are they the same as they were before?

B: With three times I have a better chance [of winning] because, like, when we did that game where we had 12 and 2 [The Race Game, Day 1 of instruction]... well, with one try anything could happen but with three [spins] I probably have a better chance of winning.

I: What if we played the game so that the winner was the first to score ten points? Do you think you have a better chance of winning, a worse chance of winning, or the same chance of winning if we played it to three points?

B: Better. A better chance because... with three, it's still not that hard for you to win. You could easily just get three right here [points to the yellow sector]. But with ten, you'd get more chances to land on blue, and it'll probably land on blue more because you know that [blue] has more of a percentage.

Analysis of case-study students' learning revealed that growth in Emily's and Blake's thinking may have occurred as the result of instructional tasks. In "The Race Game," students were assigned a race car numbered 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. Working in pairs, players took turns rolling two dice and if the sum equaled the number of his or her race car, then that car advanced one mile (space) toward a finish line ten miles (spaces) away. Each pair of students kept track which race car was first to cross each of the ten mileposts. Results from individual simulations were pooled and whole-class discussions ensued. Through such discussions, students came to realize that Race Cars 2 and 12 had a greater probability of winning when the race track is shorter or, stately alternatively, smaller samples are more likely to yield unusual results than in a larger number of trials.

Discussion

In a single trial with two outcomes, one more likely than the other, an unlikely event *can* occur. However, as the number of trials increases, the ratio of unlikely to likely events should decrease. Heretofore, research has not documented the extent of middle school students' awareness of this relationship between experimental and theoretical probabilities. Results of this study illustrate that while most middle school students were unaware of this relationship prior to instruction, appropriate cognitive activity can enrich their conceptual development. Post-instructional interviews with Emily and Blake documented growth as the result of instruction and concomitant mental activity. Such findings clearly have pedagogical and epistemological ramifications for the middle school mathematics curriculum. Curriculum designers, textbook authors, and classroom teachers may wish to note that none of the instruction was formula-driven or teacher-centered. The rich learning environments in this study enabled students to do what was necessary in order to make connections between data and chance.

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PROBABILISTIC TEACHING ELEMENTS FOR ELEMENTARY SCHOOL CHILDREN FROM 5 TO 8 YEARS OLD.[1]

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This report is part of a main study project, which is intended for analyzing the understanding capability of the probabilistic idea in students from 5 to 8 years old. This qualitative research led us to identify which were the answers and justifications used by the pupils in order to make a decision, also what kind of difficulties they had, regarding problems which implied extraction with or without replacement, and if they could identify the sample space. This research involving 16 children from 5 to 8 years old. In the six activities we used marbles in boxes, roulettes and dice. Some relevant results are as follows: in the boxes and roulette activities only Luis (7, 0) is consistent referring to complement of C; 4 children made reference to all the sample space elements when tossing a dice. Thirteen children observed the sample space reduction when they made extraction without substitution. The students' justifications allowed us to make some suggestions in order to design the activities for probabilistic teaching to young pupils, such as: a) to propose activities which lead to the identification of both a certain event and an impossible one, just as Sciolis suggests (1991); b) to offer group activities in which the participants collectively identify the sample space, because, according to Jones, et. Al. (1997), and as we can confirm through this researching, it is a problematic concept; c) to promote the reflection about the sample space continuance and variation when doing extractions with or without substitutions; d) to encourage consideration of such results in order to draw a distinction about the sequence.

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PROBABILITY AND ARITHMETIC: AN EPISTEMOLOGICAL STUDY IN MIDDLE SCHOOL

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Nowadays, the teaching of Stochastics has acquired relevance in the elementary education. Heitele (1975) emphasized the need to associate the teaching of Stochastics to that of Arithmetics and Geometry so to deter difficulties that university students have, such as a defective manipulation of fractional numbers when calculating, reporting and interpreting probabilities (Garfield and Ahlgren, 1988). The present work refers to 10 to 15 year old pupils' understanding of probability, while being also in the process of learning fractional numbers.

Information arose from two sources: 1) **General**: A questionnaire was designed in two versions, both consisting of ten questions, and it was applied to 145 students. The first version (Arithmetics) considered Kieren's model (1988). The second version posed questions about probability whose answers supposed the use of the fractions in the first version. For instance: During the weekend, among 45 pupils, 27 went to the movies; 18 studied, and 9 went to the movies and studied. Questions (Arithmetics): *What fraction of the total of students went to the movies*? (Probability): *If we choose at random a pupil, what would have been his activity*? 2) **Specific**: semi-structured interviews to eight students selected from the answers given to both versions of questionnaire were carried out by focusing on their answers given to the probability version. Some results are: Students who did not have difficulties to identify events and to express their probabilities with fractions showed an unstable use of fractional numbers within arithmetical situations. Students who operate successfully fractional numbers for arithmetic situations does not necessarily indicate their consistent grasping, since their use for probabilistic situations may be incorrect.

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THE READING AND TREATMENT OF BIVARIATE DATA IN CONTINGENCY TABLES

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The general objective of this research has been to explore the possibilities of representing information, and its characterizing relationships, in alternative ways other than the symbolic-algebraic representation; these alternative ways are meant to constitute a supporting tool for solving probability and statistics problems, and also to improve the understanding of the concepts that such problems imply.

Duval's (1993) work on representation registers provided us with a framework that we used to analyze the relationships among the different ways in which a mathematical object can be represented. This paper focuses on the particular objective of comparing the legibility of *contingency tables* with the legibility of other formats used to present the information.

We applied questionnaires to 492 high-school students. Each situation in the questionnaire possessed the same mathematical structure but presented, within a familiar context, information of a bivariate nature in different formats, including the contingency table. After reading the information, the task consists in answering several questions which ask to clarify the relationships among the data (for instance, how many of these possess a certain characteristic, but not some other?, etc.). The purpose of varying the different formats in questions of a similar structure was to contrast their different degrees of legibility, which would manifest themselves in the percentages of correct answers for each format. There were three ways of presenting the information for the problem which we will analyze here: compacted data, a list of bivariate data, and the contingency table.

Our finding was that the presentation in the form of a contingency table was not the most legible for the students who took the exam. There were 77% correct answers to the questions in which the information was provided through a "list of bivariate data"; 76% with "compacted data"; and only 30% with contingency tables.

Our conclusion is that, in spite of its apparent simplicity, the use of contingency tables presents complexity degrees which we must understand before we can formulate strategies which will then be deliberately introduced in the teaching of probability and statistics.

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HOW 5 TO 6-YEAR-OLD PUPILS INTERPRET RECTANGULAR DIAGRAMS[1]

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In this study six activities about combinatory are included in order to, among other purposes, identify how young children interprete rectangular diagrams. The activities were developed with 8 children between 5 and 6 years old, in which the use of every-time-more-complex rectangular diagrams was promoted; grouping them into 3 types: a) given the ordinated pairs, determine which are the elements that correspond to each group; b) given the elements of each group, draw the ordinated pairs; c) given some elements of each group and some ordinated pairs, determine which are the missing elements and ordinated pairs. When the children were asked what was being combined, according to the rectangular diagram type a), that showed six ordinated pairs, as a cartesian product of 3 x 2 elements, all of them interpreted that it was 6 skirts and 6 blouses; with rectangular diagrams type b) by developing activities that included diagrams type a) and b), all pupils could complete the two rectangular diagrams type c). The previous discoveries allow us to reaffirm some contributions such as English's (1991), who stated that children can acquire a systematic method for forming n x m combinations; and we agree with Fischbein (1975) at assuring that by using adequate pre-figuration methods such as the generative figurative methods (like rectangular diagram) it is possible to prepare the arrival to the next stage of development.

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SUPPORTING STUDENTS' STATISTICAL DEVELOPMENT IN A TECHNOLOGY-INTENSIVE CLASSROOM

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Our purpose in this poster session is to present classroom episodes which emphasize the crucial role of two computer-based minitools developed as part of a seventh-grade classroom teaching experiment focusing on the area of statistics.[1] The role of the computer minitools was essentially twofold in nature: 1) supporting students' emerging mathematical notions and 2) providing students with tools for data analysis.

Current discussions about the role of technologies in supporting students' understandings of data and data analysis are often cast in terms of what has been defined as expressive and exploratory computer models (cf. Doerr, 1995). In one of these approaches, the expressive, students are expected to recreate conventional graphs with only an occasional nudging from the teacher. In the exploratory approach, students work with computer software that presents a range of conventional graphs with the expectation that the students will develop mature mathematical understandings of their meanings as they use these conventional inscriptions. The approach we employed when designing the computer-based minitools for the classroom teaching experiment offers a middle ground between the two approaches. In this sense, the approach introduced particular tools and ways of structuring designed to fit with students' current ways of understanding, while simultaneously building toward conventional graphs (Gravemeijer, et al., in press).

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Problem Solving

THE EMERGENCE OF STUDENTS' PROBLEM SOLVING BEHAVIOR: A COMPARISON OF TWO POPULATIONS OF UNIVERSITY STUDENTS

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Abstract: This study investigated the problem solving behaviors of two populations of university students. Schoenfeld's theoretical model, including four major problem solving components (i.e., resources, heuristics, control and beliefs) provided the framework for this investigation. Comparison of the behaviors of graduate and second semester calculus students while attempting complex problems revealed that the graduate students in this study were much more likely than the second semester calculus students to rely on the mathematical knowledge learned in school, when solving an unfamiliar problem. The graduate students also report and were observed being more reflective and persistent when attempting a complex problem. In addition, the graduate students report that their choice to voluntarily and frequently bring a problem into their consciousness greatly enhances their effectiveness and success in completing complex problems; whereas the second semester calculus students report only working on problems when directed to do so by their instructors.

Introduction

NCTM (1989) states that developing effective problem solvers should be a primary goal of instruction. In Shaping the Future (1996), the authors call for undergraduate education to teach not only mathematical facts, but also the methods and processes of solving complex problems. Despite these calls, recent studies suggest that both public schools (Cogan, 1996) and universities (Schoenfeld, 1989a, Carlson, 1997) are failing in this regard.

An explanation of people's behavior in mathematical situations, including an explanation of why they are successful or unsuccessful in their attempts to solve mathematics problems needs to consider four categories (i.e., resources, heuristics, control and beliefs) (Schoenfeld, 1989b). This study investigated these aspects of second semester calculus and beginning graduate students' problem solving behaviors.

Resources, says Schoenfeld (1989b), are the mathematical facts and procedures potentially accessible to the problem solver. Heuristics are the board range of general problem solving techniques (e.g., working backwards, drawing pictures, looking for symmetries). Control refers to the cognitive actions taken when selecting and implementing a solution approach. One's control decisions determine the efficiency with which facts, techniques, and strategies are exploited. Control refers to the cognitive actions taken during the initial engagement period, the execution of the solution approach and the verification phases of solving a problem. These cognitive actions are often characterized by observing: acts of reflection and non-reflection; the process of selecting and implementing resources and strategies; and the decisions made regarding resource (e.g., time, content) allocation. Schoenfeld (1989b) contends that purely cognitive behavior is rare and the performance of most intellectual tasks takes place within the context established by one's perspective regarding the nature of those tasks. Belief systems, says Schoenfeld (1989a), shape cognition and determine the perspective with which one approaches mathematics and mathematical tasks, even when one is not consciously aware of holding these beliefs. Belief systems prompt more deep-seated convictions than emotions and attitudes (Schoenfeld, 1989b), and are characterized by such statements as "learning mathematics is mostly memorization" (p. 344) and "some people are good at math and some just aren't (p. 353).

Methods

The subjects for this study were 74 second semester calculus and 45 graduate students from two large public universities in the United States. The data was collected during the last two weeks of the second semester calculus students' course, and upon completion of the graduate students' second semester of graduate level mathematics. Three data sources were utilized during data collection. A quantitative instrument, the *Views About Mathematics Survey* (VAMS) (Carlson, 1997), was administered to assess and broadly classify these students' beliefs about knowing and learning mathematics. The results from this assessment provided data to identify trends and differences in students' beliefs for these two groups.

The problem solving behaviors of 12 high performing students from each group were further studied by observing their behaviors while completing five complex mathematical tasks. Initially each subject was asked to provide a written response to each of the five problems, while talking through their solution approach. Immediately after completing their solutions these students were prompted to describe their solutions and were further probed to explain their motivation for various decisions made during the process of completing the problem. Analysis of the interview transcripts involved multiple readings of each transcript, with efforts made to identify and classify specific response types and general problem-solving behaviors. In addition to observing students' use of control, heuristics, and access of content knowledge, students self-reported beliefs were compared with the beliefs exhibited during problem solving. After compiling the results for both groups, the behaviors of the two groups were compared.

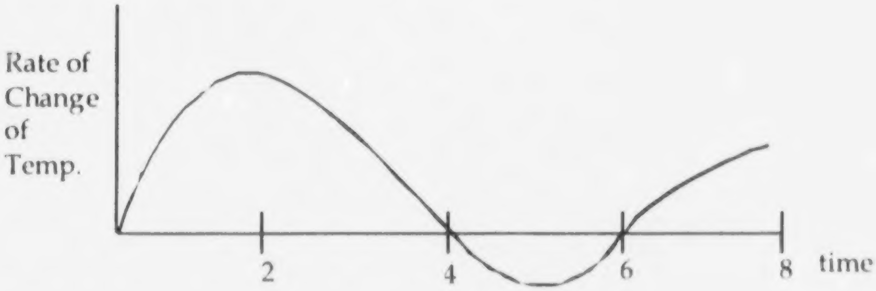
Each of the interview subjects were also prompted to describe the experiences in their background that had influenced the development of their problem solving behavior. These interviews were analyzed (Carlson, 1999) using the Strauss and Corbin (1998) open and axial coding techniques. The results of this analysis were also compared with behaviors exhibited during problem solving in an attempt to identify important factors that contribute to the emergence of effective problem solving behavior.

Results

A comparison of the VAMS responses for the 74 second semester calculus students and 45 graduate students revealed that the graduate students hold more expert beliefs as defined by VAMS (Carlson, 1997). When using VAMS profiling strategy, 83% of the graduate students were classified as having an expert profile, compared to 23% of the second semester calculus students. Among the beliefs assessed by VAMS, the most noted differences between graduate students and second semester calculus students were conveyed by the graduate students reporting: much greater persistence and confidence; greater expectation of "sorting out" information on their own; the belief that solving problems involving mathematical reason is enjoyable; and the belief that problem solving requires both general problem solving techniques and knowledge of content, rather than merely memorizing what they believe is important.

A discussion of the responses for the two groups while completing two of the five problem solving tasks follows.

> ***Problem 1.*** *Given the graph of the rate of change of the temperature over an eight hour time period, construct a rough sketch of the graph of the temperature over the same eight hour time period. Assume the temperature at time, t=0, is 0 degrees Celsius.*



None of the 12 second semester calculus students interviewed were observed planning their solution attempt; rather they immediately began constructing a response, with little evidence of reflecting or utilizing heuristics to simplify their approach. In the process of constructing the temperature graph, over half (7 of 12) showed no evidence of analyzing the changing rate information displayed by the graph. These same 7 students either reversed or omitted the concavity changes, and even with prompting, these students showed little ability to attend to the covariant nature of the graph. They did not appear to access their knowledge of first semester calculus, rather they appeared to resort to algebraic reasoning, by justifying their graph with statements like, "positive rate graph, positive slope", followed by the construction of an increasing straight line or concave up graph on the interval from time = 0 to 4. The 3 subjects who provided a correct response also utilized algebraic reasoning when describing their approaches. The remaining second semester calculus students repeatedly attempted to construct a temperature graph that had the same shape, roots, maximum and minimum as the "rate" graph. They appeared to be mechanically generating their response, with little reflection apparent regarding the reasonableness of their responses. However, 9 of the 12 graduate students provided a completely correct graph and appeared to be analyzing the situation using concepts of calculus. Each of these students provided a correct temperature graph, while fluently discussed increasing, decreasing, and concavity changes in the context of the information provided. As a group, they employed a variety of strategies to verify the reasonableness of their solutions. The remaining three graduate students omitted one of the concavity changes, and when prompted to explain their answers, corrected the error, and provided a justification that utilized calculus.

When completing this problem, the graduate students were more likely to utilize calculus content, exercise control (e.g., reflect, verify), and employ heuristic strategies (e.g., set up tables, drew pictures). They also exhibited greater persistence during their attempt and confidence in their ability to generate a correct response.

Problem 2. *Assume $F(x)$ is any quadratic function.*

True or False: $F(\frac{x+y}{2} < \frac{F(x)+F(y)}{2}$ *Justify your answer.*

Both groups of students appeared to have difficulty recognizing how to interpret this question. Only two of the second semester calculus students and five of the graduate students elected to algebraically represent the two expressions, with only two of these five graduate students arriving at a correct solution. When prompted to verify their solutions graphically, these second semester calculus students made little progress, with only two of the five students proceeding to compare the two expressions in the context of the graph. Further probing of the second semester calculus students revealed that 6 of the 12 interview subjects did not recognize in F( ) as the input to the function, with five of the second semester calculus students discontinuing their pursuit of a solution after announcing they were unable to make progress. However, once the graduate students were prompted to examine the situation graphically, they constructed various parabolas and compared the two expressions in the context of various graphs. When difficulty arose, the graduate students persisted (from 18-37 minutes) until they formulated a response, and employed various strategies to "sort out" information on their own. They were also frequently observed making and verifying conjectures during the problem solving process and reflecting on the effectiveness of their approach. In addition, the graduate students offered only solutions that appeared to have a logical foundation, in comparison to the second semester calculus students who frequently offered "nonsense" type responses. For both groups of students, once a method was initiated, they rarely changed course.

When prompted to describe the experiences and influences that have contributed to their problem solving development, several notable differences were observed between the two groups. 10 of the 12 graduate students interviewed praised a mentor, most frequently a high school teacher, for creating a non-intimidating environment where questions were encouraged and guidance was provided in learning to approach complex problem; while 7 of the 12 second semester calculus students conveyed frustration about the frantic pace and algorithmic delivery of their high school mathematics curriculum. Most (8 of 12) of the graduate students also mentioned their passion for working through hard problems. They conveyed that they had learned to appreciate the power of carrying a problem with them, describing their tendency to think about problems in places like the shower, while driving and eating, and when awakening in the middle of the night. Such expressions of passion were not mentioned by their undergraduate peers. Finally, the graduate students were much more likely to cite persistence and confidence as attributes that have facilitated their problem solving success.

Conclusions

- Both groups had some difficulty accessing their known resources; although the difficulty was more pronounced in second semester calculus students. The second semester calculus students were observed reverting to more elementary tools (algebraic reasoning) and concepts when confronted with a problem they perceived as complex.

- *The graduate students in this study applied more varied heuristics during problem solving than did the second semester calculus students. They more frequently utilized strategies of working backwards, setting up tables, drawing pictures, etc.*

- The graduate students exhibited greater use of control during their solution attempts than did the second semester calculus students (problem 2). They were more likely to devise a plan for their solution approaches, they more frequently accessed their resources, they were more reflective during their solution attempts.
- *Differences were observed in both the self-reported and observed beliefs of these two groups. Among the beliefs assesses by VAMS, persistence and confidence were much more prevalent among the graduate students in this study. The graduate students also reported and displayed an expectation that they could complete complex mathematical tasks, and were careful during their problem solving attempts to offer responses that appeared to have a logical foundation. These traits were not apparent in the second semester calculus students and have not been widely observed in high performing undergraduate students (Schoenfeld, 1989a, Carlson, 1998).*

- The graduate students in this study expressed that their problem solving success had been positively influences by their willingness to bring a problem into their consciousness frequently over extended periods of time.

This study provided numerous insights into the complexities of applying one's mathematical knowledge during problem solving, while increasing our understanding of the psychological aspects of learning mathematics. The results suggest the need to further investigate the role of control and its manifestation during problem solving. As well, it is suggested that mathematics instructors and curriculum developers attempt to involve students in learning situations that provide experience in reflecting, conjecturing, verifying and persisting. Providing students an opportunity to work on a complex problem over several weeks, while orchestrating opportunities for regular cognitive engagement, may also help to develop students' power in solving more complex problems than they previously believed were possible. Lastly, it is suggested that teachers devise a shared language for discussing the fine points involved in problem solving with their students. This shared vocabulary should enable teachers to direct students while in the process of attempting to complete complex problems.

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STUDENTS' UNDERSTANDING OF GRAPHICAL AND SYMBOLIC REPRESENTATION OF FUNCTIONS AND ITS RELATIONSHIPS

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This study reports what high students showed when asked to work on tasks that involve the use of various representations. The analysis focused on small groups' responses to a set of questions in which they explored mathematical connections between graphical and symbolic representations of a problem that addressed issues of variation, approximation, and optimization. Results showed that few students made proper connections between representations and the initial situation or problem. In general, they exhibited competence in procedures such as expressing an area or solving quadratic equations, but experienced difficulties in interpreting or connecting those procedures.

If you have a string of length 50-cm, what are the dimensions of the rectangle of maximum area that you can enclose with your string? If an apple orchard occupies one acre of land, how many trees should it contain so as to produce the largest apple crop? These are typical examples studied in pre-calculus or calculus courses in which students have opportunity to display various mathematical resources. Tactical decisions that students eventually grasp and apply to approach these problems include the selection of important variables and conditions attached to the situation, the representation of the situation with a formula, and the use of diverse rules or techniques to find the solution. Each phase of the solution process involves also strategic decisions in which students need to choice and access proper mathematical resources to carry out a solution plan and explain relationships between different representations of the situation and its solution. When the students' competence focuses on only reporting the final solution of a proposed problem, it is not clear whether they actually identify appropriate connections between representations used to achieve the solution and the original problem or situation. For instance, What does its graphical representation tell you about the phenomenon? In particular, the points of intersection of the graph and the Cartesian axes. What are the advantages or disadvantages in the use of graphical, symbolic, or table representation of the phenomenon? These are the types of questions that students should discuss and examine while working with problems. In this context, mathematics learning goes further than studying only a particular set of content, it involves learning a way of thinking in which it is important to identify, explore, and communicate mathematical relationships. "If students have a solid foundation in mathematical thinking, they will be prepared for a wide array of high-powered courses designed to meet the interests and needs of the entire spectrum of students" (Cuoco, 1998, p.104). In particular the idea of representation plays an important role to examine, understand, and communicate mathematical results.

What type of representation helps students understand mathematical relationships? What mathematical resources do students need to recognize and examine the potential of diverse representations? Do different representations entail different ways to explore mathematical relationships? What type of questions should students pursue in order to inspect relationships between symbolic and a graphic representation? This study reports results shown by high school students who worked on activities in which they were asked to examine relationships between graphical and symbolic representation of a problem. In particular, the extent to which they were able to make connections between the problem or situation and its mathematical representation. Special attention was paid to the interpretation of the graphs in terms of their relationship with symbolic representation and the mathematical criteria used to support the solutions.

Rationale and Importance of Instructional Tasks

The design of tasks which help students explore mathematical relationships by using diverse representations has been an important goal in mathematics instruction. The context in which the task is embedded and the mathematical resources and strategies needed to work in the task are important aspects to consider for its implementation in the classroom. However, the way the task is actually presented, and the questions that students may focus and pursue during the implementation are crucial elements to document the students' engagement and interest in the task (Santos, 1996). Thus, a task that involves the use of various ways to visualize useful information and mathematical relationships may offer an interesting challenge for the students. In some cases the process of working on an initial task shows other features or mathematical connections that allow the use of other mathematical resources and as a consequence the initial task will be transformed into a more robust one. This activity appears frequently during the development of mathematical ideas, that is, focusing on a task or mathematical problem initially functions as reference to launch and explore other mathematical relationship. In this perspective, the scope of this paper addresses two aspects:

(i) The identification of mathematical ideas that emerge during the process of dealing with an initial task. This includes the use of basic mathematical resources and ways to support particular results or relationships, and

(ii) The importance of relying on both the uses of analytical and graphical representations to examine a set of questions that appears attached to the solution process of the task. Here, the use of determined software often helps to give meaning and sense to mathematical properties.

What mathematical qualities should a task include to promote students' mathematical thinking? This has been an important question in the work done in mathematical problem solving. Indeed, Schoenfeld (1985) suggests that students should be encouraged to work on nonroutine tasks in their learning of mathematics. Furthermore, it is recognized that the type of questions and ways shown to pursue those questions during the solution process of a routine task may also transform the initial natúre of that task. In this perspective, two routine tasks are used as reference to illustrate attributes or mathematical resources that appear as important to respond a set of new related questions. In addition, aspects of the solution process that include the importance of identifying key information of the task, ways to represent it, and the need to constantly revise initial conditions and new connections of the tasks are discussed throughout their solution processes.

Subjects, Methods, and Procedures.

A group of 40 grade ten students worked on mathematical tasks designed to explore their competence in the use of graphical and symbolic representations that involved mathematical resources studied previously. In particular, there was interest to document the type of explanations or arguments utilized by students to support their responses to a set of questions embedded in the task. The class was divided into 9 small groups. The instructor monitored the students' interactions and asked each group to report their written responses. Some of the students' reports were discussed with the whole class. The small groups' responses were organized in episodes showing diverse type of mathematical qualities attached to their arguments. The analysis of the information (mainly students' written reports) focused on documenting whether students eventually grasped advantages and disadvantages in the use of graphical and symbolic representation of a problem or situation.

The Task and the Initial Instructional Goal. Variation, maximum value, and representations were basic ideas to discuss with students through a task in which they had to examine the behavior of rectangle drawn on the first quadrant (Cartesian System) with one fixed vertex on the origin of the coordinate system and the opposite vertex located on the graph of $y = -2x + 13$ (Figure 1).

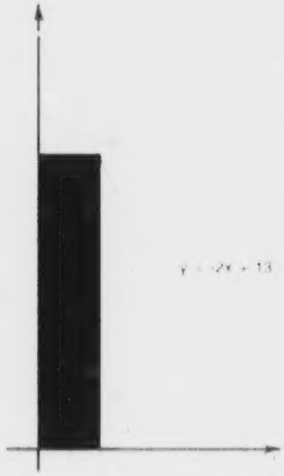

Figure 1.

The students were asked to work on three sets of questions, which addressed the main themes included in the task. Examples of the type of activity that each small group required to work included:

(i) On the Cartesian System graph the equation $y = -2x + 13$ (Show all your work)

(ii) Draw rectangles on the first quadrant so that one of its vertexes lies on the origin of the coordinate system and the opposite vertex located on the graph of the equation. How many more rectangles can you draw having the same condition?

(iii) Show how you can calculate the area of three rectangles you drew above. Explain how you determined the information needed to calculate those areas, that is, how you got the length of each side of the rectangle. Make a table in which you show the length of the sides of each rectangle and the corresponding areas.

(iv) If x represents the length of the side of the rectangle, which rests on the x-axis, then how can you express the area of this rectangle as a function of the side x?

(v) Draw a graph, which corresponds to the above expression. Describe the behavior of the graph in terms of the side and area of the rectangle.

Results and Discussion

Although all the students recognized that the graph of $y = -2x + 13$ was a line, only two small groups relied on two points to draw the graph, the other groups utilized between 7 and 15 points to graph the equation. Students' criterion to judge the pertinence of the graph seems to be based on the number of points used to graph the equation rather than examining the minimal information needed to picture the expression.

It was evident that students showed competence in identifying the symbolic representation of the function area $A = x(-2x + 13)$ but some experienced difficulties when required to examine connections of that representation with the original situation. For example, even though all the small groups' reports included the algebraic representation for the area of the rectangle, only three of them employed it to respond questions related to the existence of rectangles with a determined area. Are there rectangles with area equal to 15? Are there rectangles with area equal to 30? (How many). Explain in detail how you got your response. Only three small groups based their responses on solving the equations $x(-2x + 13) = 15$ and $x(-2x + 13) = 30$. They reported that there were two rectangles with area equal to 15 whose bases were 5 and 1.5; and concluded that the second equation did not have real roots and as a consequence there were no rectangles having 30 as area.

Two small groups reported that there are two rectangles with 15 as area. They supported their response by finding two numbers (5 & 3 and 3 & 5) whose product equals 15. For these students there was no need to check their answer. Four small groups responded that there was only one rectangle with area 15, its dimensions were 5 and 3.

For the second question (area 30) two small groups answered that there were two rectangles with that area whose dimensions were 5 & 6 and 6 & 5.

All the small groups were able to graph the function area by finding points on the plane, which satisfy the expression; indeed, they showed a table including values of x and its corresponding area. However, six small groups did not recognize relationships between these two questions: On (0, 18) draw a parallel line to the x-axis, does this line intersect the graph of $A = x(-2x + 13)$? If your response is YES, then determine the coordinates of those points (justify your answer). The other equivalent question posed to the students was: How many rectangles are there with area equal to 18? What are their dimensions? Two groups responded the first questions by looking at the graphic and reported two points (2, 18) & (5, 18) the other groups relied on the equation to find the corresponding points. For the second question, the 6 small groups listed dimensions that included 1 & 18; 2 & 9; 3 & 6; 4 & 4.5; 4.5 & 4; 6 & 3; 1 & 18, 18 & 1; 6 & 3; etc. It seemed that their goal was to find two numbers with product 18. It might be that students identify and access proper mathematical resources when a question include terms (intersect the graph, determine the coordinate) which involve the use of certain procedure; other ways, it is difficult for the students to establish connections. Only two groups of students recognized that there was only one case in which the area of the rectangle reached a maximum value. Here, the argument provided by the students was geometric and based on the properties of the parabola. They identified the points of intersection of the parabola and the x-axis and determined the middle point between those two points. They stated that the maximum value was reached at the value of A in the middle point. That is, when $x = 3.25$ the value of A is 21.125 and the dimensions of the rectangle are 3.25 & 6.25. Again, it seemed that the graphic representation helped students access a procedure to find the maximum value. However, when they were asked to examine the roots of the equation $x(-2x + 13) =$, where represented the maximum value of the area.

The students, in general, assigned a particular number to the right side to solve the equation. In particular, no body was able to identify the relationship between the existence of only one root of the quadratic equation and the maximum value. That is, students did not realize that it was not necessary to have the value of to find the value of x in which the parabola reaches its maximum value. Here, it was important to relate the discriminat of the quadratic equation and the roots. For instance it the discriminat is equal to zero, then there is only one root.

Conclusions

It is clear that the extent in which students make sense of information attached to the different representation is an indicator of the type of understanding achieved by the students. Paying attention to the students' mathematical resources and interpretations in various contexts helps instructors observe difficulties in their approaches to the problems. It is clear that the students processes to grasp a robust understanding in the efficient use of various representation takes time and should be an overarching goal of instruction. In addition, it seems that students should also be encouraged to interpret the original situation or problem via their representations and reflect on questions that addressed important connection of the phenomenon and its representation (What does it tell you the axis intersections? What does it mean the maximum/minimum value in terms of the phenomenon? What does it mean the increasing interval in terms of the behavior of the phenomenon? etc.)

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CROSS-NATIONAL COMPARISONS OF REPRESENTATIONAL SKILL FOR PROBLEM SOLVING

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Abstract: Flexible use of multiple representations has been described as a key component of competent mathematical thinking and problem solving. In this study, 6th-grade American students are compared to 3 samples of Asian 6th graders (Chinese, Japanese, Taiwanese) to determine if the well-documented mathematical achievement of students from these Asian nations may be due in part to a greater understanding of mathematical representations. The results showed that, among all groups, Chinese students generally scored highest on the representation tasks and, except on items about the visual representations of fractions, all Asian samples scored significantly higher than the American sample. The results are discussed in terms of possible instructional antecedents and textbook differences.

Many studies have shown that children from Asian countries have more advanced mathematics skills than their age peers in most other countries of the world, including the United States (McKnight et al. 1987; Pursuing Excellence 1996; Robitaille, & Gardner 1989; Stevenson & Lee 1990; Stevenson, Stigler, Lee, Lucker, Kitamura & Hsu 1985). However, the complexity of mathematics learning (Schoenfeld 1992) is such that there is still much to be learned from cross-national studies about how to enhance particular areas of mathematical competency such as mathematical problem solving. There is a growing body of research literature suggesting that despite their relative lack of basic mathematics skills, children from the United States are relatively good problem solvers (Cai 1995; Mayer, Tajika and Stanley 1991; Silver, Leung and Cai 1995; Tajika, Mayer, Stanley and Sims 1997). Thus more comparative research is needed to understand mathematical problem solving as a multi-faceted process influenced by aspects of school instruction and informal experience that have not yet been fully identified. In this study we focused upon students' competency with different mathematical representations, a skill that has been suggested to undergird competent problem solving (Mayer 1987).

As outlined by Mayer and Hegarty (1996), the competent problem solver engages in a comprehension process with three phases-translation, integration, and planning. During the translation phase, the problem solver interprets the problem statements into internal propositional statements. In the integration phase, the problem solver creates a problem model from these propositional statements. A rich problem model is a representation that integrates both the numerical information presented in the problem and the relations between the objects that are relevant to the problem. A rich problem model enables the problem solver to choose an appropriate computational representation. The final phase of problem solving is the execution phase in which the answer to the problem is found.

Most large comparisons of mathematical competency have required students to solve problems, drawing upon all the different phases of problem solving. If students were lacking in some of the procedural knowledge needed in the final phase, it was not possible to detect their competency in the earlier phases. This problem has been addressed by using a multiple choice test to look at students' competency with the different phases (Cai 1995; Mayer, Tajika and Stanley 1991; Tajika, Mayer, Stanley and Sims 1997) or by having students solve novel open-ended problems not directly related to the school curriculum (Cai 1995; Silver, Leung and Cai 1995; Tajika, Mayer, Stanley and Sims 1997). The results have been somewhat contradictory since American students were found to be better at the phases of problem solving in which representations are used, but an examination of the representations actually used by students when solving problems found that the Asian samples often used more sophisticated representations.

For this study, three aspects of representational competence are examined. The first is students' flexibility in moving between representations of the same sort. An example of this on our test is being able to express fractions as the combination of other fractions. This type of skill has been linked to mathematical competence in areas such as mental computation (Sowder 1992). The second aspect that we test is being able to move between different kinds of representations. Dreyfus and Eisenberg (1996) argue that flexibility in moving across representations is a hallmark of competent mathematical thinking. An example from our test is being able to express a fraction as a decimal, a division problem and a proportional relationship. The third aspect we examine is skill with visual representations. The use of visual representations has been found to differ significantly between American samples and those from Asian nations in earlier studies (Cai 1995; Silver, Leung, & Cai 1995) with American samples favoring visual representations such as diagrams.

Methods

Sample. The participants in this study included 895 sixth-grade students from four nations: 223 from the People's Republic of China, 224 from Taiwan, 177 from Japan, and 271 from California, USA sampled to include both urban and rural areas and high and low achieving schools.

Instrument. A test of mathematical achievement was developed for this study, with two different kinds of items. The test included solution items and sets of matching representation items. The solution items consisted of two types—either straight forward computations or complex problems. For each problem solving item, there were between 5 and 7 corresponding representational items presented to the students. The students were asked to judge whether each representational item was right or wrong for the matching solution item.

Procedure. Students were administered two forms of the math achievement test in sessions two months apart. At the first testing session, they were given two of the solution items and three of the representational items on Form A. At the second session they were given the other three solution items and the remaining representational items on Form B of the test.

Analysis. The solution items were scored as either right or wrong. In addition, a Total score was created for each of the representational sets of items by adding together the scores on each individual representational item given for that question. Mean scores for each solution and representational item were calculated for each national group. An ANOVA using nation as the grouping factor was run on all individual items (both solution and representational items), and post-hoc contrasts were conducted for each significant ANOVA result.

Results

Scores and Rank Orders on Solution Items. For the total sample, scores ranged from a low of 36% to 80% on the solution items. Among the four nations, China ranked first on three of the five items, and Taiwan ranked second on four of the five. Japan ranked third on three of the five. The U. S. ranked last on all items.

Scores and Rank Orders on the Representational Problems. Overall, students had more difficulty with the representational items than the problem solving items. However the pattern between nations was similar to that of the problem solving items: Taiwan and China ranked first or second on most items, with Japan showing the third strongest performance overall. Students in the U. S. had low performance on the representational items, scoring below fifty percent on over 1/3 of the questions and ranking last among the nations on about 70% of the items.

ANOVA on Individual Items. Each of the five solution items had significant national differences. Twenty-eight of the thirty representational items also had significant national differences.

Flexibility Within a Representation. There were nine items that requested the students to judge the equivalence of different fraction items. The Chinese sample did consistently best on these items with mostly high scores above .5 with two exceptions. Interestingly, the Japanese sample had a very mixed performance on these items. For five of these items the Japanese students had a performance that was statistically equal to that of the Chinese and Taiwanese students while on four of the items they performed significantly worse than the Chinese and Taiwanese students, and approximately equal to the American students. The American sample scored equal to or below chance of most of these items.

Flexibility across Formal Symbolisms. The items related to this type of flexibility were quite mixed and included fraction to decimal, fraction to proportion, multiplication to addition, multiplication to a number line, and simple number sentences to algebraic number sentences. On the average these were the most difficult representational tasks and performance within seemingly similar items was uneven for all nations. China and Taiwan showed the highest results on these more difficult items, although even their performance was often little better than chance. They tied for highest on seven items. Japan scored the highest on one item, and there were no significant nation differences on two other items.

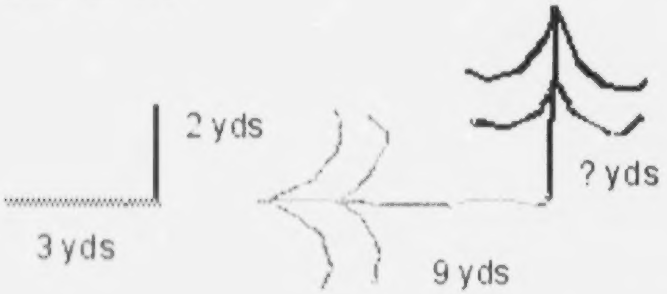
Visual Representations. Five of the representational items were visual, i.e. diagrammatic, representations of the problem solving items On two of these items the United States sample scored significantly better than the samples of the other nations. Both of these items were diagrams of rectangles divided into different shaded regions to represent a fraction. However, on the other three visual items, the United States had the lowest accuracy score. Unlike its performance on the other items on the test, China did relatively poorly on the visual items. On three of the items China had the lowest score or tied for the lowest score with either Japan or the US. China also scored significantly lower than the other nations on two of the representations for the problem solving item about finding the area of a garden cut by two paths and was not the highest scorer for any of the representational items for this problem.

Discussion and Summary

As in many earlier achievement studies, the Asian students in this study scored above the American students on most of the test items. The gap in performance between the American students and the three Asian groups was even greater for the representational items than for the solution items. On the problem solving items, students in Japan, Taiwan, and China were two times as likely as students in the U. S. to get an item correct. When adjusting for chance guessing on the representational items, Japanese

Solution Items
(Students solve the problems)

***A1** A pole 2 yards high casts a shadow 3 yards long. The shadow of a tree is 9 yards long. How tall is the tree?



***A2** Give the answer to the following problem in decimals.

3 ÷ 6 =_______

Figure 1: Examples of Test Items
(Continues on next page)

Corresponding Representational Items
(Student jmarks as right or wrong)

B1 The same problem is stated. "Now decide if each of the following statements are right or wrong For each statement check the appropriate box." (There are two boxes labeled Right and Wrong for each item.)

*a. $\frac{2}{3} = \frac{h}{9}$

*b. $2 : 3 = h : 9$

*c. $3 + 2 = 9 + h$

*d. $\frac{2}{9} = \frac{h}{3}$

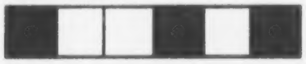
*e. $2 \times 9 = h \times 3$

B2 The fraction $\frac{3}{6}$ may be represented in several different ways. Decide if each of the following examples are right or wrong representations of $\frac{3}{6}$.

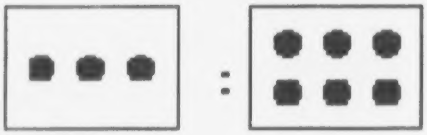
*a. $3 \div 6$

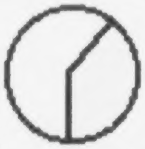
*b. .50

*c.



*d

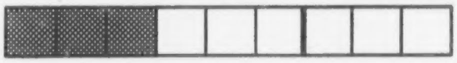




*e.

*f. Paul has half a dozen doughnuts. He wants to share them equally with his friend Joani.

*g.



students were almost three times as likely to get an item correct as American students, and Chinese and Taiwanese students nearly five times as likely to answer correctly.

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INSTRUCTIONAL IMPACT ON U.S. AND CHINESE STUDENTS' SELECTION OF STRATEGIES AND REPRESENTATIONS IN SOLVING A PROBLEM INVOLVING ARITHMETIC AVERAGE[1]

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This study examined the impact of teaching algebra on U.S. and Chinese students' selection of solution strategies and representations in solving a problem involving arithmetic average. The results of the study support the hypothesis that students who have formally learned algebraic concepts are more likely to use algebraic approaches to solve problems than those who have not formally learned algebraic concepts. In fact, U.S. 8th-grade students who have formally learned algebraic concepts used algebraic representations as frequently as Chinese 6th-grade students did. On the other hand, results of the study suggest that the Chinese students' less likely to use pictorial representations than the U.S. students cannot be explained by their opportunity of learning algebra. Further studies are needed to explore how U.S. and Chinese teachers view and use pictorial representations in their classroom instruction, and then to understand why the Chinese students are less likely to use pictorial representations to solve problems than the U.S students.

It is widely accepted that the "golden ring" of educational research is to improve the learning opportunities for all students. Cross-national studies in the teaching and learning of mathematics provide unique opportunities to explore how we can improve students' learning. The purpose, then, of cross-national studies is to provide information about how we can improve students' learning in mathematics. Recently, several cross-national studies have used open-ended assessment tasks to explore students' thinking and reasoning involved in U.S. and Asian students' mathematical problem solving in addition to examining the correctness of answers (e.g., Becker, 1992; Cai, 1995, 1998a; Silver, Leung, & Cai, 1995). These studies revealed many important similarities and differences about students' thinking and reasoning.

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For example, Cai (1995, 1998a) has examined U.S. and Chinese sixth-grade students' thinking and reasoning in mathematical problem solving and problem posing. He found striking differences between U.S. and Chinese students' use of solution strategies and representations in their solutions. The U.S. students frequently used verbal or pictorial representations while the Chinese students more frequently used symbolic representations. In fact, the Chinese sixth-grade students rarely used pictorial representations in their solutions. When both U.S. and Chinese sixth-grade students used the symbolic representations to show the solution process of a problem, the U.S. students' solutions were based on arithmetic while the Chinese students' were more often based on algebra, which is a more advanced representation (Dreyfus, & Eisenberg, 1996). Cai also explored how students' use of solution strategies and representations is related to their level of overall problem-solving performance. He found that those students who used algebraic or arithmetic representations are better problem solvers than those who used pictorial or verbal representations (Cai, 1998b).

These studies suggest that U.S. students prefer to use verbal and visual representations; while Chinese students prefer to use symbolic representations. However, no study has been conducted to examine the possible causes of the differences in the use of solution strategies and representations between U.S. and Chinese students. Because of the central role the classroom instruction plays in the development of students' thinking (Rogoff & Chavajay, 1995), one may hypothesize that the difference is due to the variations of classroom instruction between U.S. and China. In fact, it is well-documented that for the same concepts, Chinese students may be formally introduced earlier than U.S. students (e.g., Cai, 1995). For example, in China, students in their 5th or 6th grade start to formally learn concepts of variables, equations, and equation solving. In contrast, most U.S. students will not learn these concepts until their 8th or 9th grade. Therefore, it might be possible that Chinese 4th- or 5th-grade students who have not formally learned algebraic concepts may use similar visual or verbal representations as U.S. 6th-grade students do. On the other hand, U.S. 8th- or 9th-grade students who have formally learned algebraic concepts may use symbolic representations as Chinese 6th-grade students do. The purpose of this study was to test the hypothesis. To test the hypothesis, it is appropriate and necessary to examine U.S. and Chinese students' mathematical thinking and reasoning from a developmental perspective.

Methods

Subjects

Subjects of this study consist of 196 4th-grade, 213 5th-grade, and 200 6th-grade Chinese students from four typical schools in Jishou City, Wunan Province; and 115 6th-grade, 109 7th-grade, and 110 8th-grade U.S. students from four typical schools in Pittsburgh Metropolitan Area, Pennsylvania. The Chinese 4th- and 5th-grade students have not formally learned algebraic concepts, but the 6th-grade Chinese students have had. The U.S. 6th- and 7th-grade students have not formally learned algebraic concepts, but the 8th-grade U.S. students have formally learned algebraic concepts. The unmatched grade selection for the U.S. and Chinese samples allows us to examine the solution strategies and representations of U.S. and Chinese students who have or have not formally learned algebraic concepts.

Tasks

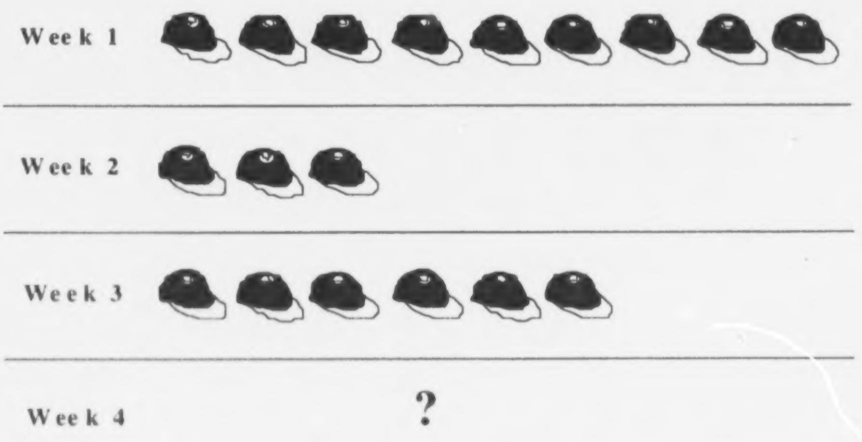
This study used a set of four open-ended assessment tasks. These tasks are open-ended in the sense that students need to produce their own answers and explain how they got their answers. This paper only reports the results from one of the tasks, involving arithmetic average, shown in Figure 1. To solve this problem, students cannot directly apply the averaging algorithm in the traditional "add them up then divide" way. Since knowing the "average" is an uncommon situation, a correct solution requires the flexible and reversible application of the algorithm. The use of algebraic or arithmetic representations may allow students to mentally represent this "reversibility" more easily than through a use of pictorial or verbal representations. Therefore, students may have advantages if they use arithmetic or algebraic representations to solve this problem involving arithmetic average.

Data Coding and Analysis

Data coding and analysis were completed using a classification scheme adapted from Cai (1998b). In particular, each student's response to this problem was analyzed from four perspectives: (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. To ensure a high reliability, 60 student responses to the task were randomly selected (10 responses from each grade level in each nation) and were independently coded by two raters. The inter-rater agreements are 100% for coding numerical answers, 92% for coding mathematical errors, 94% for coding solution strategies, and 89% for coding representations.

- *The Task Involving Arithmetic Average*

Angela is selling hats for the Mathematics Club. This picture shows the number of hats Angela sold during the first three weeks.



How many hats must Angela sell in Week 4 so that the ***average*** number of hats sold is 7?
Show how you found your answer.

Figure 1

Results

Table 1 shows the percentages of U.S. and Chinese students getting correct or incorrect answers, having appropriate solution strategy, and using various representations by grade level. Students' grade level is associated with their correctness of numerical answers for both the U.S. sample (χ^2 $(2, N = 334) = 37.44$, $p < .001$) and the Chinese sample ($\chi^2(2, N = 609) = 60.48$, $p < .001$). For both samples, the higher grade level students are in, the higher percentages of them got the correct answer for the problem. For those U.S. 8th-graders and Chinese 6th-graders who have formally learned algebraic concepts, over 90% of them obtained the correct answer for the problem. Cross-nationally, the percentage of Chinese 6th-grade students getting correct answer for the problem is higher than that of the U.S. 6th-grade students, 7th-grade students, and that of the U.S. 8th-grade students ($p < .01$). However, the difference between U.S. 8th-grade students and Chinese 6th-grade students was reduced to eight percentage points from

Table 1

Percentages of U.S. and Chinese Students Getting Correct Answer, Using Each Solution Strategy, and Each Representation

	CH4th	CH5th	CH6th	US6th	US7th	US8th
ANSWER	(n=196)	(n=213)	(n=200)	(n=115)	(n=109)	(n=110)
Correct	68	83	98	54	77	90
Incorrect	32	17	2	45	23	10
STRATEGY	(n=141)	(n=151)	(n=198)	(n=70)	(n=86)	(n=104)
Leveling	6	3	1	9	6	7
Using Average Formula	91	87	98	41	64	78
Guess-and-Check	4	11	1	50	32	15
REPRESENTATION	(n=180)	(n=190)	(n=197)	(n=102)	(n=97)	(n=109)
Verbal	23	9	0	27	32	14
Pictorial	1	0	0	11	8	8
Arithmetic	62	88	68	60	54	51
Algebraic	14	3	32	2	6	27

the 44 percent points difference between U.S. 6th-grade and Chinese 6th-grade students. The majority of the U.S. and Chinese students who did not get the correct answer were due to their incorrect use of the computational algorithm. These students tried to directly apply the computational algorithm of the average to solve the problem, but the application was incorrect. For example, a student added the number of hats sold in week 1 (9), week 2 (3), and week 3 (6), then divided the sum by 3, and got 6. Since the average was 7, the student added 3 to the sum of the numbers of hats sold in the first three weeks, then divided it by 3 ($18 + 3 = 21$, $21 \div 3 = 7$), and got 7, and then gave the answer 3.

Table 1 also shows the number of U.S. and Chinese students in each grade level who used appropriate solution strategies. As a grade level advances, the higher percentage of the students in that grade level have shown evidence of using appropriate solution strategies. In particular, only 72% (144 out of 180) of the Chinese 4th-grade students used appropriate solution strategies, but 99% (198 out of 200) of the Chinese 6th-grade students used appropriate solution strategies. Similarly, only 61% (70 out of 115) of the U.S. 6th-grade students used appropriate strategies, but over 95% (104 out of 110) of the U.S. 8th-grade students used appropriate strategy. Cross-nationally, the percentages of U.S. 8th-grade students and Chinese 6th-grade students who used appropriate strategies were almost identical.

Three appropriate solution strategies were identified and are described as follows:

> *Leveling*: The student used "leveling-off processes" to solve the problem. The student viewed the average (7) as a leveling basis to "line up" the numbers of hats sold in the week 1, 2, and 3. Since 9 hats were sold in week 1, it has two extra hats. Since 3 hats were sold in week 2, 4 additional hats are needed in order to line up the average. Since 6 hats were sold in week 3, it needs 1 additional hat to line up the average. In order to line up the average number of hats sold over four weeks, 10 hats should be sold in week 4.
>
> *Using Average Formula*: The student correctly used the average formula to solve the problem arithmetically (e.g., $7 \infty 4 - (9 + 3 + 6) = 10$ or algebraically (e.g., $) = 7 \infty 4$, then solve for x).
>
> *Guess-and-Check*: The student first chose a number for week 4, then checked to see if the average of the numbers of hats sold for the four weeks was 7. If the average was not 7, then the student

chose another number for the week 4 and checked again, until the average was 7.

Table 1 shows the percentages of U.S. and Chinese students in each solution strategy. For those who used appropriate solution strategies, the vast majority of the Chinese students in each grade used the Averaging Formula to solve the problem. Using Average formula is also the most preferred solution strategy for the U.S 7th-grade and 8th-grade students. A considerable number of U.S. students used the Guess-and-Check strategy, but the percentages of students using Guess-and-Check strategy decreased from 50% for the 6th-graders to 15% for the 8th-graders. The Chinese 6th-grade students are less likely to use Leveling or Guess-and-Check strategies than the Chinese 4th- and 5th-graders. A slightly larger percentage of U.S. than Chinese students used Leveling strategy.

Four categories were used to evaluate and classify representations in a student's explanation: verbal (primarily written words), pictorial (a picture or drawing), arithmetic (arithmetic expressions), and algebraic (algebraic expressions). Developmentally, the percentages of Chinese students using symbolic representations (arithmetic or algebraic) increased from 76% for the 4th-graders, to 91% for the 5th-graders, and to 100% for the 6th-graders; while the percentages of students using verbal representations decreased from 23% for the 4th-graders, to 9% for the 5th-graders, and to 0% for the 6th-graders. Similarly, the percentages of U.S. students using symbolic representations (arithmetic or algebraic) increased from 62% for the 6th-graders and 60% for the 7th-graders, to 78% for the 8th-graders; while the percentages of students using verbal representations decreased from 27% for the 6th-graders and 32% for the 7th-graders to 14% for the 8th-graders. In particular, over 30% of the 6th-grade Chinese students used algebraic representations, but only 14% of the 4th-grade Chinese and 3% of the 5th-grade Chinese students used algebraic representations. Only 2% of the 6th-grade U.S. students and 6% of the 7th-grade U.S. students used algebraic representations, but nearly 30% of the U.S. 8th-grade students used algebraic representations.

It is clear that for both samples, the students who have formally learned algebraic concepts were more likely to use algebraic representations than those who have not formally learned algebraic concepts. It is interesting to note that percentages of U.S. 8th-graders and Chinese 6th-graders using algebraic representations are very close. For those U.S. and Chinese students have been formally taught algebraic concepts were less likely to use verbal representations than those who have not been formally taught algebraic concepts. About 10% of the U.S. students in each grade used pictorial representations, but there is no Chinese except for two 4th-graders using pictorial representations in their solutions.

Discussion

Although research in cognition has shown that students' experiences out of school have a substantial effect on their learning and problem solving (e.g., Lave, 1988; Resnick, 1987), classroom instruction is still considered a central component for understanding the dynamic processes and the organization of students' thinking and learning. This study examined the impact of teaching algebraic concepts on U.S. and Chinese students' selection of solution strategies and representations in solving a problem involving arithmetic average. The results of the study support the hypothesis that students who have formally learned algebraic concepts are more likely to use algebraic approaches to solve problems than those who have not formally learned algebraic concepts. U.S. 8th-grade students who have formally learned algebraic concepts used algebraic representations as frequently as Chinese 6th-grade students did. For both samples, students who have formally learned algebraic concepts are less likely to use verbal representations to solve the problem than those who have not formally learned algebraic concepts. Chinese 4th-graders who have not formally learned algebraic concepts used verbal representations as frequently as U.S. 6th-grade students did.

On the other hand, the instruction of the algebraic concepts seems to have a little impact on students' uses of pictorial representations. Chinese students, have or have not formally learned algebraic concepts, did not use pictorial representations. U.S. students who have formally learned algebraic concepts are equally likely to use pictorial representations to solve the problem as those who have not formally learned algebraic concepts. Therefore, the finding that the Chinese students are less likely to use pictorial representations than the U.S. students cannot be explained by their opportunity of learning algebra. Since teacher's beliefs and knowledge have impact on students' thinking, further studies are needed to explore how U.S. and Chinese teachers view and use pictorial representations in their classroom instruction, and then to understand why the Chinese students are less likely to use pictorial representations to solve problems than the U.S students.

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REPRESENTATIONS PRODUCED BY SECONDARY EDUCATION PUPILS IN MATHEMATICAL PROBLEM SOLVING

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Summary: This study describes and classifies the representations produced by secondary education pupils (13 to 14 year olds) in mathematical problem-solving. We analyze the relationship or association between the representations used by the pupils, the proposed problems and the correct and incorrect processes used.

In problem-solving two general phases are identified: problem representation and problem solution (Kintsch & Greeno, 1985; Mayer, 1985; Newell & Simón, 1972; Riley, Greeno & Heller, 1983). In the case of mathematical verbal problem-solving, these two processes have been analyzed in more detail. Problem representation has been characterized by means of two substages: (a) problem translation and (b) problem integration. The problem solution phase has been characterized by means of two substages: planning and execution that includes selecting the process to continue and execute the necessary calculations to obtain a numeric answer (Mayer, 1985). A similar distinction between the subprocesses of problem representation has been made by Kintsch and Greeno (1985). When the solver reads the problem it builds a "textbase", or mental representation of the problem text. This mental representation expresses the semantic content of the problem. Taking it as starting point the solver builds a problem model that integrates the information from the textbase to express the mathematical situation of the problem (Lewis, 1989).

About the representations Silver (1987) points out that "the problem task itself has several representations associated with it, including at least the mathematics that it represents and the forms of representation (e.g., natural language, graph, equation) used in its description" (p. 42). This paper deals with precisely these representations. The representations play a crucial part in the resolution of problems as diverse authors have shown. Carpenter and others (1988) indicate: learning how to represent situations with mathematical symbols is a fundamental objective in mathematical syllabus instruction.

During their school years children learn how to represent problems in different ways: in a graphic, numeric and symbolic way. Some investigators have carried out investigations on the effects of training in these representations (Lewis, 1989; Willis & Fuson, 1988; Wolters, 1983). Others have investigated the relationship between the semantic structure of the problem and their symbolic representation (Bebout, 1990; Carpenter, Moser, & Bebout, 1988).

Also, Duval (1993) pointed out that resorting to several representation registrations seems characteristic of human thought if compared to animal intelligence, on the one hand, and artificial intelligence on the other. What characterizes human thought in contrast to animal intelligence is not so much resorting to a semiotic system to communicate (language) but resorting to diverse representation systems: language and graphic images (drawing, painting, diagrams,...)

Our intention is to examine which form of representation secondary education students prefer, when solving mathematical problems after a normal school syllabus. We try to check the hypothesis that *types of problem representation depend on the contents of the problems.*

When approaching a verbal problem, the solver must bring into play a mental representation that must be transferred to the paper where he is writing, using a physical representation system. We have classified the representation systems according to the five categories that the students approach the verbal problems with: *Numeric, Numeric-graphic, Graphic, Graphic-algebraic* and *Algebraic*. We add the *Non answer* category to include no answers. We have chosen these categories because we think that they allow a general description of the situation peculiar to students in their evolution from the arithmetic stage to the algebraic one. The criteria used to define the categories are:

- Numeric representation, only arithmetic operations are used.
- Numeric-graphic representation: the numeric operations are accompanied by graphs.
- Graphic representation, is when only graphs are used.
- Graphic-algebraic representation, when equations are accompanied by some graphs.
- Algebraic representation is when there are only algebraic equations in the solution.

The use of one representation system or another for the resolution of problems will give rise to possible groups. To ascertain these groups or classes, either of problems or of subjects, according to the representation systems used, will provide us with vaiid information that implies a better knowledge of the obstacles that certain types of problems present, besides showing that different types of solvers may exist and what their characteristics are.

The objective of the study is to identify the types of representations children choose when they solve problems and analyze if there is a relationship between the problems and correct and incorrect processes used. To achieve this objective we have carried out an investigation design in which three variables intervene.

- PROBLEM VARIABLE. With eight different levels that express the problems proposed to the students.
- REPRESENTATION VARIABLE. With six different levels that express the type of representation used by students in the solution of the problem: without representation, numeric, graphic-numeric representation, graphic, algebraic, graphic-algebraic.
- PROCESS VARIABLE. With two different levels depending on whether the process is correct or incorrect

The variables have been studied repeatedly with the same subjects; a factorial design of independent measures has been used for the variable problem; together with the repeated measures in the rest of the variables.

Method

Subjects

The subjects were 192 pupils from secondary education (13-14 year olds) from three secondary schools in Grenada (Spain). Six groups in all participated, two from each school. The sample pupils had not received specific instruction for the problems in the tests. The schools are located in an area that is considered lower middle-class. The number of pupils who solve each problem is equal in each group.

Instruments

Two paper-and-pencil tests consisting of 4 problems were elaborated (see Table1). The 8 problems have been selected according to the following criteria:

- they are restricted to numerical thinking
- the problems admit a plurality of representations and processes of solution
- the problems are adapted to the first secondary education cycle and belong to the block of numbers
- with one or two quantitative relationships
- the relationships are linear or quadratic
- the numbers used are natural or fractions
- the natural numbers used are less than 100 and if they are greater than 100 finish in exact tens
- the fractions used are proper fractions with denominator £ 10.

Table 1. Problems included in each of the tests

Test 1	Test 2
Problem 1. The side of a square measures 8 cm. If the perimeter of the square diminishes in 8 cm to form a new square, in how many square centimetres has the area of the square diminished?	*Problem 5.* The Garcías go on a trip. Mr. García has to drive 434 kilometres from Madrid to Granada. On the way they decide to stop for a soda, although they still have 4 times more kilometres ahead of them than the distance already travelled. How many kilometres do they still have to travel? And how many kilometres have they already travelled?
Problem 2. To tile a floor we have square tiles that are 30 centimetres in length and weigh 150 grams. If each tile costs 72 pesetas, how much will it cost to tile a rectangular room 7,2 m long by 5,4 m wide?	*Problem 6.* In a carpentry there are two types of wooden planks: some long and some short. If we put a long plank in a row together with two short ones, they measure 210 cm. The long plank measures 30 cm more than the short one. How much does each wooden plank measure?
Problem 3. 5/8s of the members of the school choir are girls. So that the number of girls is equal to the number of boys 12 more boys should be incorporated. How many members does the choir consist of?	*Problem 7.* Juan has a square table. On one side he places a 30 cm ruler and 15 clips in a row. On another side he also places a 50 cm ruler and 10 clips in a row. If the perimeter of the table is 360 cm. How much does each clip measure?
Problem 4. Luis and Pedro are in a bookstore. Luis then visits the bookstore regularly every 3 days at twelve in the morning and Pedro every 4 days also at twelve o'clock. How many days will have to pass by since the first visit for them to coincide again?	*Problem 8.* A snail tries to go up a wall 10 metres high. During the day it goes up 2 metres and during the night it descends1 metre. How many days will the snail require to reach the end of the wall?

Procedure

We respected the composition of natural groups in the classroom in the application of the tests. Each group was divided in two in such a way that half of the class solved Test 1 and the other half Test 2. The purpose of this division was to avoid fatigue in the students. Therefore, each student solved four problems. The date of application of the tests was from 20 April to 30 April 1998.

The tests were distributed alternately in such a way that two students sharing a table didn't have the same test. The tests were carried out at the end of term and there was no time limit. Students took between 15 and 40 minutes to do the tests.

Results

We classified the written productions of the pupils in response to the eight problems according to the type of representation that they used and according to whether the process employed was correct or not. The independent variables used in the statistic analysis were:

- the *problem* variable, refers to each one of the eight problems used in this study, with values from 1, 2, 3, ..., to 8.
- the *representation* variable, refers to the approaches used and takes the following values: 0-Non answer; 1–Numerical; 2–Graphic-numerical; 3–Graphic; 4–Graphic–algebraic; 5–Algebraic.
- the *process* variable, depending on whether the process is correct or incorrect, receives a value of 1 for the correct process, and 0 otherwise.

The crossing of these three independent variables gave rise to a contingency table: PROBLEM x REPRESENTATION x PROCESS. This contingency table was analysed by means of a log-lineal model and we obtained the following results.

PROCESS and REPRESENTATION effect

From a total of 768 answers, 409 were correct processes (53.3%) and 359 were incorrect (46.7%). We did not obtain significant differences between the frequency of correct and incorrect processes.

We obtained significant differences in representation variable (chi-parcial=369.529, p = .0000). The most frequent representations were numeric representation (30.7%) and graphic-numeric representation (32.8), followed by algebraic representation (16.1%) and finally graphic representation that only represented 3% of the total answers.

PROBLEM * PROCESS effect

In the study we used eight problems that were numbered from 1 to 8. The statistical analysis showed a significant association between PROBLEM and PROCESS variables (Chi-square partially = 178.873, p = .0000). The percentage of correct and incorrect processes varied significantly from one problem to another. According to the estimate of parametres, problems 3, 8 and 6, were those that contributed in greater measure to erroneous processes. In problem 3 the value was z=1,73 and in 8 the value was z=0.99. While in problems 2, 4 and 5 exactly the opposite happened; they being the problems that most contributed to the frequency of correct processes; in problem 4 it was particularly significant (z = 2.15). In problems 1 and 7 the percentage of correct and incorrect processes was balanced.

PROBLEM *REPRESENTATION effect

Association between PROBLEM variable and REPRESENTATION variable vas significant (Chi-square partially = 601.072, p = .0000), whereby the representation used by the students was conditioned by certain problems.

In all the problems a percentage were left unanswered that oscillated between 4 and 13 percent. None of the eight problems were associated significantly with this level of representation variable. Problem 4 had the greatest value (z=1.3) within the log-lineal model, but still without great importance.

It was numeric representation that prevailed in problems 4 (54%) and 8 (53%) followed by graphic-numeric representation. However, within the log-lineal model we only found a significant z-value in problem 8 (z=2.36).

Graphic-numeric representation was significantly the most frequent in problems 1 (81%) and 2 (67%). The next most frequent representation was numeric.

The use of an exclusively graphic representation was not very frequent. In problem 8 the highest percentage reached was only 14%. Nevertheless, in the estimate of parametres, the association between problem 8 and graphic representation (z=4.59), proved to be significant and exclusive.

Graphic-algebraic representation was the one that appeared with most frequency in problem 6 (32%). There was significant association between them (z=5.54), and also problems 5 (z=3.10) and 7 (z=2.56).

The use of exclusively algebraic procedures was most frequent in problems 3 (51%) and 5 (41%) and occupied second place in problem 6 (26%). Significant association was observed in the three cases being stronger in problem 3 (z=6.54), followed by problem 5 (z=4.80) and then problem 6 (z=2.72).

Although in problem 6 the percentage of algebraic procedures (26%) and graphic-algebraic (32%) was balanced, the one that carried most weight within the log-lineal model used was the graphic-algebraic one (z=5.54) followed by the algebraic one (z=2,72).

REPRESENTATION * PROCESS effect

Association between REPRESENTATION and PROCESS variables was significant (Chi-square=167.295, p = .0000). We previously show that there was a balance between the number of correct and incorrect processes. If the representation variable intervened, the representations with a larger percentage of successes were the graphic-numeric ones (39.4%), followed by the numeric ones (29.8%), the z-values were significant in both. If we stick to the incorrect answers, non answer was the level that carried most weight within the log-lineal model (z=5.37).

PROBLEM * REPRESENTATION * PROCESS effect

Since there was significant association among PROBLEM, REPRESENTATION and PROCESS variables, we describe what the association is between PROBLEM and REPRESENTATION variables, for correct and incorrect processes (see table 2).

In problems 1 and 2 there was a majority choice of numeric and graphic-numeric representations. In problem 1 there was a high percentage (81%) that tended towards graphic-numeric representations, however, only half of them provided the correct answer. That is to say, in this problem graphic-numeric representation was what led to erroneous solutions to a greater degree. The representation that was shown to be most effective in obtaining a correct solution was numeric representation. However, in problem 2 in a similar situation (67%), the percentage of correct answers based on graphic-numeric representation was very high, with a ratio of 57 correct to 10 incorrect. Although in smaller measure numeric representation also contributed to the success.

Purely algebraic representations prevailed in problems 3 and 5. In problem 3 the percentage of correct processes within those that choose algebraic representations was approximately a third that is also in consonance with those that chose numeric representations. It was one of the problems where the greatest number of non answers was obtained. In problem 5 graphic-algebraic and algebraic representations were more effective in obtaining the correct answer than in problem 3, the percentage was approximately 76%. While numeric representations produced a large number of errors. This problem also produced a high index of non answers.

In problem 4 the pupils predominantly used numeric and graphic-numeric representations thereby obtaining the correct solution in a

Table 2. PROBLEM * REPRESENTATION * PROCESS in percentages

	REPRESENTACIÓN											
	0		1		2		3		4		5	
	PROCESS		PROCESS		PROCESS		PROCESS		PROCESS		PROCESS	
PROBLEM	0	1	0	1	0	1	0	1	0	1	0	1
1	4	0	3	11	40	41	1	0	0	0	0	0
2	4	0	8	19	10	57	2	0	0	0	0	0
3	14	0	21	8	3	1	0	0	2	0	33	18
4	13	0	0	54	0	31	0	1	0	0	1	0
5	11	0	11	9	2	9	0	0	4	13	10	31
6	7	0	21	2	6	4	1	0	11	21	4	22
7	11	0	9	17	13	19	5	0	5	10	2	8
8	8	0	45	8	21	4	4	10	0	0	0	0

significant way in the case of the numeric ones (z=2.61) and almost significant in the graphic-numeric ones (z=1.86017) in such a way that there was a very small percentage of errors. This problem gave rise to a great number of non answers. Most of the incorrect answers were non answers.

In problem 6 the representations that proved to be the most effective to reach the correct solution were significantly the algebraic ones (z=2.4), followed by the graphic-algebraic ones. In contrast, the students that tended towards numeric representations significantly failed almost in their entirety (z=3.15). In terms of percentages 87% of those that chose this option obtained incorrect answers. The errors, although in smaller measure, were present in graphic-numeric representations. The polarisation of correct and incorrect answers around the algebraic and numeric procedures respectively is to be noted.

In problem 7 algebraic representation is the one that is shown to be most effective in obtaining the correct answer. 80% of the pupils that chose this representation obtained the correct answer. It was followed by numeric-algebraic representation in which 2/3 of those that chose this option guessed correctly. Numeric representation proved to be the next most effective. Within the limits of purely graphic representations was where the greatest degree of errors was observed: 100% of those that chose graphic-numeric representation were slightly below average in the number of correct answers obtained.

In problem 8 it was observed that students in the sample had chosen numeric representation for the most part (53%), continued by graphic-numeric representation (25%). However, neither of these representations had led to the correct solution in most cases: The ratio of successes to failures is 8 to 45 in numeric representations and 4 to 21 in graphic-numeric ones. Meanwhile 109 of the 14 students that chose the graphic representation solved the problem correctly. Therefore problem 8 had two clear connotations in its solution: a) it significantly induced students to make errors when they used numeric representations (z=2.29), and graphic-numeric ones (z=2.42), and b) graphic representations provided a high percentage of correct choices that was significant (z=2.25).

Conclusions

The analysis of the written productions of the students included in the sample when they solved the proposed problems allowed us to observe that they used more than one representation type. We classified the representations types in "Non answer", "Numeric", "Numeric-graphic", "Graphic", "Graphic-algebraic" and "Algebraic", that express general categories of representations. It is obvious that subcategories appear in each one of them, but to confirm our hypothesis it is not necessary to specify them.

Statistical analysis, using the log-lineal model has confirmed our hypothesis. Therefore, one can affirm that for those subjects of the sample and the problems in question

> *There is a significant association between the proposed task, in this case the problems, the representation type that the children used, and the correctness or incorrectness of the written processes that they used when they solved the problems.*

This has important implications in as much from the theoretical point of view as from the practical point of view for mathematics instruction. From the theoretical point of view it shows us the multiplicity of representations: when people solve problems, they use more than one representation. Also, the task imposes on us the most appropriate representation type. It is obvious that if a pupil is immersed in a period of school instruction in which numeric or algebraic representations are used, a clear preference is shown for one or the other representation type. But when the resolution is not directly linked to a period of school instruction the subjects choose the representation that they freely find most appropriate in solving the problem. The children in the sample had had experience and they had practiced carrying out representations (arithmetic, graphic and algebraic), but the test was not related to these periods of instruction in such a way that they could not perceive that it was an exam requiring a specific skill that they had been taught and that it was necessary to put into practice at that moment.

The confirmation of the hypothesis is so much more general as no significant differences were found between the number of correct (46,7%) and incorrect (53,3) processes, which suggests that the hypothesis may be as valid for the successes as well as for the failures when solving problems.

The frequency of the different types of representations that were used in the resolution of the problems was significantly not the same. The most frequent representations were graphic-numeric representation, and numeric representation, followed at a certain distance by algebraic representation. Exclusively graphic representation was not very frequent. We think that this is a consequence of the problems used and the education system in which symbolic representations are preferred in detriment to graphics that usually accompany the symbolic ones in problems solving. There is also significant partial association between the problems and representations used by the students, as well as between the problems and the adaptation or not of the process chosen in order to obtain the solution to the problem. The type of representation used is also associated in a significant way with the correctness or not of the chosen process.

For the above reasons we believe that the findings in this paper are of practical importance when applied to school curriculum. It is not sufficient just to bear in mind numeric or algebraic representation in mathematical problem solving. It is necessary to include graphic representations as a way of obtaining solutions to problems as well as preparing appropriate problems. With certain problems it is important to bear in mind the role that graphics play in facilitating their comprehension and there are even problems whose correct solution is easier to obtain by means of graphic strategies than numeric or algebraic strategies. For example, in problem 8, the students that used the graphic strategy of drawing ascents and descents obtained the correct answer in greater proportion than those that used a numeric strategy. The latter did not perceive that it was unnecessary to take in to account the last day, because they were carried away by the inertia of the calculation and did not notice the last descent, whereas the students that used graphic representation didn't fall into that error.

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THE JOINT CONSTRUCTION OF PROBLEMS AND SOLUTIONS IN COLLABORATIVE BILINGUAL GROUPS

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A teacher-researcher collaboration investigated collaborative problem-solving among 122 middle school students in five bilingual classrooms. Analysis of quantitative data indicated that working in small groups was effective in increasing individual performance on a written test of mathematical problem solving. An analysis of videotaped data of 12 students focused on differences in interaction patterns among students who benefitted most and least from the small group work. In general, students who provided more elaborated remarks and explanations in the groups showed greater gains on the written test, although there were variations in this pattern. Implications of these results, particularly for students whose participation in the groups may have been reduced because of lack of English proficiency, will be discussed.

Objectives

Problem solving is a central activity in mathematical practice and, as such, has received a great deal of attention from the mathematics education research community. Much of the research on this topic has considered problem solving from the point of view of the individual student. Even studies of students working with others to solve problems have often described the path toward a solution as a unitary phenomenon, with little differentiation among the roles and participation of the different participants. Instead, the focus has often been on the mathematics itself, i.e., the content of the problem on which the group is working (e.g., Schoenfeld, 1985). Although some researchers and theorists have investigated both mathematical content and communication within small groups (e.g., Hoyles, 1985; Noddings, 1985), there remain many unexplored questions concerning how students interact while solving challenging mathematical problems. In addition, cognitive and social aspects of problem solving are influenced by an increasingly important variable, namely, the diversity of background knowledge and native language present in many classrooms. This diversity raises additional issues, since students whose first language is not the language of instruction face the task of interpreting and understanding not only a second language, but concepts expressed in a linguistic register (vocabulary and language structures) specific to mathematics, again, expressed in a second language (Moschkovich, 1996).

The research reported in this paper focuses on mathematical problem solving among bilingual middle school students working in small, cooperative groups. The study was carried out in five classrooms in which the majority of the students were native Spanish speakers. The study included both quantitative and qualitative methods of gathering and analyzing the data. This report will summarize the quantitative results as a way to set the context for a more in-depth look at patterns of interaction and problem solving among the students. The focus of the qualitative analysis will be on the different levels and types of participation of a small number of students, in relation to their performance on a written test of problem solving.

Theoretical Framework

The research is based on the sociocultural framework, derived from the work of Vygotsky, and in particular, the theoretical process of internalization in which cognitive activity is first developed socially, in interaction with others, and later internalized in the individual. Data on students' joint problem solving through social interaction, as well as their individual performance on written problems, can shed light on this internalization process. Previous research examining students working in groups on mathematical tasks has examined the relationships between individual student achievement and variables related to group interaction. For example, Webb (1991) summarized seventeen studies of small group interaction in mathematics classes, and found that "(1) giving explanations to teammates is positively related to achievement, and (2) receiving non-responsive feedback from teammates is negatively related to achievement" (ibid., p. 382). Cohen (1996) found similar results, and further stated that, "it is *only* under the conditions of a true group task and an ill-structured problem that interaction is vital to productivity (ibid., p. 8). Such tasks "require resources (information, knowledge, heuristic problem-solving strategies, materials, and skills that no single individual possesses...[and] are open-ended non-routine problems for which there are no standard procedures" (loc. cit.). The tasks used in the current study are this type of ill-structured problem, with the students working on true group tasks.

Methodology

The larger study took place in five middle school classrooms and involved 122 students. The data for the qualitative analysis is based on the work of 12 students, five boys and seven girls, in two classrooms taught by the same bilingual teacher. Three of the girls and four boys were native speakers of Spanish; the remaining students were English-only speakers.

All of the girls whose native language was Spanish, as well as one of the boys, were categorized as having limited English proficiency (LEP). The remaining three boys were categorized as having fluent English proficiency (FEP). At the time that the study took place, during the spring of the school year, instruction took place primarily in English, and, apart from occasional side conversations, the language used in the small groups that were videotaped was English.

The students worked in small groups, generally with four members, but occasionally with three or two. The groups engaged in 20 minutes of collaborative problem solving for four days a week over a period of four weeks. The problems used in the groups were presented as a set of clues, all of which needed to be read and understood to solve the problem. During the collaborative problem solving, the students were instructed to each read their own clues, to listen to each others' ideas, and to work together until they had reached a solution that they all agreed on. When a group agreed on a solution, the teacher would ask them to justify their solution by stating how it satisfied all of the clues. There were no explicit roles assigned to individual members of the groups, and each group thus worked out its own way of sharing the task and deciding on strategies.

Data

A total of 17 twenty minute collaborative problem solving sessions were videotaped and transcribed. In addition, there was a writien pre and post-test made up of problems similar to those solved in the small groups (presented in both English and Spanish), which all 122 students completed individually. The change in the mean score between the pre and post test was analyzed statistically, using the t-test. The videotapes were analyzed both holistically and in 30-second segments, in which each student's level and type of participation was coded.

Results

The quantitative analysis indicated an increase across the entire group of 112 in the students' mathematical problem-solving skills, as measured by the written pre- and post-tests. The pre-test mean for the entire group was 8.94, and the post-test mean was 10.93., out of a possible 21 points. This increase of almost 2 points for the group as a whole was statistically significant ($t=4.39$, $p < .001$). In terms of subgroups, there were statistically-significant increases for the English-only group as well as the Limited English Proficiency group, but not for the Fluent English Proficiency students (although this was a small group, comprising only 16 students).

Examining the performance of the twelve students who were videotaped, seven students showed increases in their scores, an average of 4.3 points (two of the seven were English-only speakers; the rest were Spanish speakers). Five students' scores decreased, an average of 2.8 points (three were English-only speakers; two were Spanish speakers). The students who were videotaped represented a wide range of abilities; their rankings on the post-test ranged from first to 109th, with an average ranking of 56.

Although the quantitative analysis indicated that the class as a whole benefited from working collaboratively in small groups, this was not true for all students. Furthermore, it is important to investigate what aspects of working in small groups might account for their effectiveness, and how to increase their benefits for all students. Thus, the analysis of the videotapes focused on two groups of students: those who seemed to benefit most from the experience of working in small groups, and those who seemed to benefit least, as indicated by their relative performance on the pre- and post-tests. The videotapes were coded, in 30-second segments, for level and type of participation by each student. The patterns of participation by each group of students were examined, as well as more general patterns of interaction revealed on the videotapes.

A total of 11 episodes of problem-solving were analyzed in depth; the episodes ranged from 7 to 23 minutes in length. The students whose participation was examined in depth are described in Table 1.

Table 1: Students with greatest and least increase on test

Student	Sex	Language	Pre-test	Post-test	Difference
Greatest increase:					
NB	F	Eng. only	4	14	+10
NG	M	Eng. only	5	13	+8
MA	F	LEP	0	5	+5
Greatest decrease:					
CT	F	Eng. only	17	13	-4
FP	M	FEP	10	7	-3
LR	M	FEP	14	11	-3

The analysis focused on the level and type of participation by each student; that is, was the student engaged or disengaged from the activity of the group? Did the student ask questions? Did the student make brief remarks or suggestions, or more elaborated explanations or arguments? Did the student work individually at certain points, withdrawing from the activity of the group?

Figure 1 presents the results of the analysis, given as the total percent of time each subgroup demonstrated a particular type of participation, across all coded episodes.

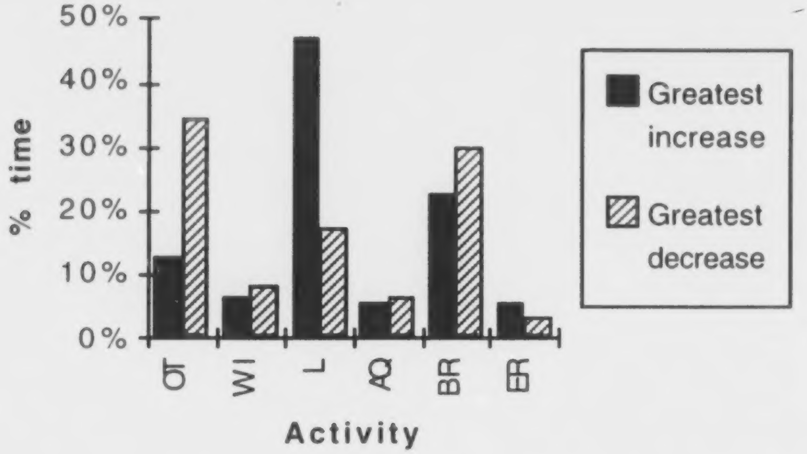

Figure 1: Patterns of participation by subgroup

The figure illustrates certain differences between the patterns of participation of each subgroup. For example, those students whose scores showed the greatest decrease tended to have the highest percentage of time spent in off-task talking or silent disengagement, not a surprising finding. Students with greater increases spent more time listening as the group worked on solving the problems.

The other categories seemed to show roughly equal amounts of participation for both subgroups. However, when the data is examined for individual students, and when the details of the students' interaction are examined on the videotapes, there are some striking variations. For example, the individual with the greatest absolute increase in pre-post test scores, NB, also showed the highest percentage of time spent giving elaborated remarks, suggestions, or explanations (11%), a result consistent with previous findings (Cohen, 1996; Webb, 1991). This student also spent about 12% of her time working independently, often when the rest of her group was either talking off-task or making brief remarks that did not help to advance the solution process. This pattern, in which students either made elaborated remarks that indicated careful reasoning, or worked independently, or both, was found among two other students who performed very well on the post-test.

One LEP student, MA, who began with a score of 0 on the pre-test and raised it to 5 on the post-test, spent 89% of her time listening closely to her group, verbalizing only 4% of the time. Her participation, however, contrasted strongly with another LEP girl, RC. RC's pre-post-test score increased by 1 point; however, she spent 67% of her time in silent disengagement from her group, after an initial period in which her English-speaking teammate assisted her in reading her clues. Clearly, RC was not able to benefit strongly from working in a small group, apparently because of her lack of English fluency, since her group utilized English in its discussion.

Discussion

These results, presented only briefly here, suggest that although many students in these bilingual classrooms were able to benefit from the opportunity to work collaboratively to solve mathematics problems, and the class as a whole showed a statistically-significant increase on the written test, a number of students were not able to gain maximum benefit from the setting. It is clear that simply placing students in groups, even ones which are heterogeneous in terms of mathematical ability and language, does not guarantee full participation by all members and the corresponding opportunity to verbalize, and eventually internalize, powerful methods of problem-solving. Future research should investigate the results of modifying the group setting to increase the participation, and therefore performance, of all students. Possible modifications include the assignment of roles within the groups, which some research has shown to be an effective way of increasing participation of all students (Cohen, 1996). Although clues and materials were available in Spanish, it might also help to create groups in which Spanish is the language used for discussion. It may also be useful to reinforce the norms and "rules" for working in small groups; for example, taking turns, listening to other students, and, particularly, giving reasons for statements and suggestions, since these behaviors seem to be associated with greater increases in problem-solving ability. In general, although working in small groups carries the potential of bringing a valuable diversity of social and cognitive resources to problem-solving, careful planning and pedagogical support is needed so that all students can realize this potential.

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ELICITING STUDENTS' DILEMMAS THROUGH ACTIVITIES THAT INVOLVE POSING QUESTIONS OR REFORMULATION OF PROBLEMS

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This study reports the implementation of instructional activities in which precalculus students were asked to participate in the process of posing questions or reformulating of problems. Estimation, variation, successive approximation, and the use of different representation were the main themes studied during one semester. Results showed that students might initially be reluctant to formulate their questions, but when the students realized the questions can continue with "follow-ups" and discussed just as textbook questions they began to participate. This not only helped the students to better the quality of the questions but also increase the numbers of questions. Use of language, mathematical resources and representations are also analyzed in the paper.

What activities help students to learn mathematics? In particular, what types of activities should the students carry out which are consistent with the mathematics practices? What is the role of the teacher during the development of the class? These are fundamental questions that mathematics teachers face when they discuss ways to improve students learning of mathematics. We agree that the problem solving goal is important however, the analyses of activities to be used by students in order to achieve this mathematical competence is also very important. In particular the activities proposed by teacher in which students are asked to analyze a given information and then pose a question or add more information and then formulate a question need to be analyzed in terms of the mathematical resources that students show in their interaction with the situation. This study documents what students show when they are constantly asked to participate in the process of formulation of questions during their learning experience. Problem posing has been identified as a fundamental activity during the students' mathematical experience.

> ...to the extent students are given the opportunity to create and explore problems of their own choices, they will take more responsibility for their own learning. It may take longer to cover predetermined bodies of knowledge, and in fact less material may be covered, but students will most likely uncover a great deal about

> their style of thinking, attitude towards working with other, and about the purpose of studying and nature of the subject matter as well (Brown & Walter, 1993, p26).

Indeed, Brown and Walter (1983) suggested the use of a strategy called "what if not" in which students are encouraged to explore connections or new situations by changing the original conditions of the initial problem. The work of Brown and Walter is full of examples in which variation of the initial conditions of the problem might lead students to find and explore mathematical relationships that are different from the typical content shown in textbooks. During the process of examining other connections or questions students are required to develop language to express their ideas and ways of presenting mathematical arguments. Santos (1997) in the same vein recommended that students, during their learning experiences, should be engaged in activities that help them identify and examine congruencies or incongruencies of the given information and pose new questions.

Methods and Procedures

Suppose you begin your first pre-calculus class (grade 11) by asking your students: "What is the total surface area of a human body?" Your students then pose different questions such as: "What do you mean by total surface area?" "How can we measure an area which does not correspond to any regular figure?" "Are you going to give us more information?" "What about some data?" "Are you asking us for an approximation?" "Then what is the best estimation?" "How do we know that we have calculated the best?" Since sizes of human bodies are different, "Which one are you asking us to calculate the total surface?" These questions were used as preamble to discuss issues that included the concept of area and its dimensions, estimation, limits, successive approximation and the importance of thinking in different ways to respond to the initial question. For example, a student suggested thinking of a human body as a set of known geometric figures: spheres and cylinders. Thus, the head was assumed to be a sphere, the trunk, arms, legs, fingers, feet, as cylinders. With this information they focused on assigning different dimensions to these figures and to calculate their corresponding surface areas. Another group of students suggested binding the human body in bandages and then measuring the area bandage. Yet other students suggested covering the human body with pieces of paper with known areas (squared sheet) and counting the number of sheets needed to cover the body. These activities were useful in showing how the students approached the problems and also to show the concepts that would be studied during the course, e.g. variation, approximation, area, measurements and dimensions, and proportion.

The design of the course had many activities in which students had to make certain assumptions or create conditions in order to propose different ways of approaching the solution. In addition there were tasks in which students were asked to pose their own questions and follow them up. e.g. an important theme studied during the course was the concept of variation. Here, the students received information showing prices of a product, the behavior of a school of fish and the fluctuation of currency (peso). They were asked to examine the given data and formulate corresponding questions. The information provided to the students was given in formats that included tables, paragraph forms, or actual newspaper texts. Another important activity was to select related problems from textbooks and the teacher deleted the proposed question (s) and with the remaining information the students were asked to pose and to follow them up.

Results

The findings in this paper will have two aspects:

(a) **Global**. The presentation of overall results that describe important events that occurred during the development of the course. The key information considered for this analysis included students written reports, student participation in small group discussions while working on a specific task, and class discussions among the teacher and the whole group.

(b) **Specific.** The analysis in detail of a session in which a small group of students presented and discussed a particular task in front of the class. This activity was videotaped, transcribed and organized in episodes that showed important moments of the activity. We were particularly interested in discussing the language used by the students in posing and follow up the questions, the mathematical resources used when approaching the problem and the different representations used by students during their interaction with the problem.

We observed that when students were asked to pose questions at first they might have felt reticent, " What type of question do you want me to pose?" "Why not just give the question and we will try to answer it?" These were their first reactions. However when the student realized that the teacher gave them credit for their questions, they became confident that they could actually formulate a complete problem. In addition, it was observed that the first type of questions formulated by the students included such terms as "find" the area of a rectangle, "calculate" the numbers a school of fish, or "determine" the slope of a line. Here the students focused on particular cases of the phenomenon being studied. Later it was evident that students formulated questions in which they showed interest in posing more profound questions. For example, a task in which students were asked to examine the motion of a train from a graph representing time versus distance, they posed and followed up questions that include: "Is there constant motion of the train and where does it takes place?" "Where does the velocity of the train increase/ decrease?" etc.

The written reports of students' approaches to the tasks seemed to reflect only partial or incomplete ideas when contrasted with discussions previously held by one of the small group. For example, the issues that students addressed during small group discussions, while working on a task (fig.1) that included a graph of y = x and x =3 on the first quadrant and a rectangle with one vertices on point (3, 0) is fixed and other on the line y = x (figure 1), involved the students' recognition of the existence of various rectangles retaining the original conditions, and the relationship between the points on the line and the sides of the rectangles; however when they posed their questions these did not reflect the above ideas.

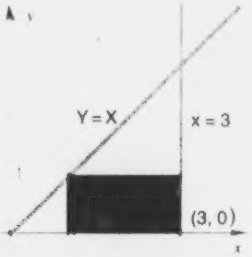

Figure 1

In their written report the group showed the dimensions of three particular rectangles and concluded that their areas and perimeters were not constant. The question that they posed was "At which moment does the rectangle become a square?". The students failed to show important mathematical ideas that they had already identified during the small group discussion. Indeed, the word "moment" might involve other concepts e.g. time and distance which are not present (and are not relevant to the situation) in the original statement of the situation. Since the students have not received opportunities to express themselves in a written manner, this may explain their lack of clarity in the written part. The lack of clarity was also detected in other tasks.

It was noted that different stages were involved in the question formulation process e.g. the students in the first stage, concentrated on superficial or specific information. Based on this the students formulated their questions. They omitted implicit information, or information which they could have inferred from the given data. After this the teacher asked question from the students e.g. what does "maintain a fixed vertice mean". This would help the student realize that there may be other rectangles which could comply with the above condition. The teacher's question opened new horizons to the student and could help them reformulate the original question. This could be the 2nd stage. It seems that if the students had an opportunity to discuss and to explore the information in the given situation, they eventually could begin to see and to explore interesting mathematical relationships and as a result achieve a better understanding of the situation or problem.

Instructional Dilemmas

The role of the teacher was to encourage students to discuss the information, pose questions and to follow up. In general, the teacher, at the end of the small group interaction, identified the mathematical qualities of the questions proposed by the students. For example, in a task that included a table showing monthly changes in the population of a school fish during several months, the students initially focused on the monthly differences of the population and the other data show in the table. When the teacher discussed these ideas. with the students, they eventually examined aspects that included accumulated frequency and its relationships with the initial information. Here it was also important to introduce a mathematical symbol to represent the ideas that emerged from the discussion. These types of problems were used to introduce calculus ideas that included the concepts of limit, derivative and integral. Although students seemed to have grasped the essence of this activity, there is evidence that they want to used this format for all the other tasks. In the other tasks they wanted to find differences, accumulated frequencies and their sums in all situation that included similar data. This event confirmed the students' tendency to examine mathematical data or relationships by following a set of rules or specific formats presented by the teacher.

The language used by the students to express their ideas when posing and following-up questions lacked precision and were often ambiguous. Even when the students spent time preparing the task to be presented in the class and formulated possible questions, some of the answers or explanations did not exactly agree with the situation under discussion or study. Only later when the video segment was analyzed did the teacher realize that the meaning the students were attaching to the situation was different from the meaning given by the professor during the discussion. For example, the students in one of the small groups presented the class with table in which they provided data of coffee price variations during the last two weeks of September of 1998. The table was incomplete and the questions this small group posed and asked from the whole group to discuss were: Can you complete the missing information on the table? What do you have to assume in order to determine the price of coffee for the five days before the 15th and five days after the 31st ? What is the price function? How can you determine the accumulated price? etc. A basic assumption that students made in order to complete the table was that fluctuations in prices were uniform and the price differences were constant. Here, when students tried to determine the prices previous to the 15th, they mentioned that the difference for the two days previous to the 15th was -.31. Here the table showed an increase in price for all reported values. When the students were asked to explain why they thought that the difference was negative, they explained that the sign only indicated that the value was for before the 15th and it was not related to the meaning of the minus sign of the difference. That is they had introduced another reference for the use of the negative sign that was not required to explain the phenomenon.

Conclusions

To what extent problem-posing activities helped students improve their focus to mathematical tasks? The results from the study showed that when students are engaged in this activity they exhibit strengths and limitations in their use of mathematical resources. When a difficulty or misunderstanding of a concept appeared in the students' work, the discussion with other students and with the teacher was important in order to overcome such difficulties. Here, problem posing was seen as a vehicle to detect and attend student difficulties in dealing with the mathematical ideas involved in the situation. It was also observed that problem posing involves the use of language and symbols which students need to understood in order to communicate their ideas. The use of different representations seems to provide tools for students to develop such language. Although students might be reluctant to formulate questions initially, it was noted that if the community, including the teacher, give then credit for the questions posed and continue with the follow-up, they eventually will pose questions that involve significant mathematical relationships.

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THE ROLE OF MATHEMATICAL KNOWLEDGE AND READING PROCESSES IN THE REPRESENTATION AND SOLUTION OF MATHEMATICAL WORD PROBLEMS

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Mayer's (1992) model of problem solving highlights the importance of comprehension in problem solving. Ehri's (1995) model of reading comprehension provides the structure in which the problem solving process may be understood. This study examined reading behavior during the solution of mathematics word problems, and its results suggest the importance of reading behavior within this domain. Forty sixth-grade and 40 seventh-grade students completed a computation test and were videotaped as they solved compare word problems. Analyses included regression with problem solving success as the dependent variable, comparisons between high and low fraction knowledge groups, comparisons for differences due to problem types, and tests of Lewis and Mayer's (1987) Consistency Hypothesis.

Theoretical Framework

This study draws on models of mathematical problem solving (Mayer, 1992) and reading comprehension (e.g., Ehri, 1995; Pressley & Afflerbach, 1995) to examine children's behaviors while solving mathematical word problems. Mayer (1992) proposed a two-stage model of problem solving. The first phase, problem representation, involves translating and integrating text components leading to problem comprehension. The second phase, problem solution, is based upon the representation the problem solver constructs. Evidence indicates that children's errors are frequently based upon their miscomprehension of the word problem (e.g., Cummins, Kintsch, Reusser, & Weimer, 1988).

Ehri (1995), in her model of reading comprehension, postulates that the reader and the text interact depending on bottom-up and top-down processes. Strategic reading behavior facilitates this interaction as the reader monitors comprehension and alters behavior accordingly. Pressley and Afflerbach (1995), in a review of studies examining reading behaviors, have established that expert readers actively interact with the text and employ comprehension strategies when miscomprehension is detected. Similarly, problem solving involves an interactive and iterative process. Based upon an initial understanding of the problem, the problem solver devises a solution plan. This, in turn, may stimulate further understanding, which supports subsequent solution steps. However, in the absence of requisite mathematical knowledge, students are less likely to form a meaningful representation and to solve the problem. Pape (1998) developed a model of mathematical problem solving that incorporates Mayer's (1992) analysis of mathematical problem solving within a model of reading comprehension (Ehri, 1995).

Background Research and Purpose

Based on the premise that problem solving errors depend largely on the miscomprehension of story problems, Lewis and Mayer (1987) examined students' performance on consistent language (CL) and inconsistent language (IL) compare word problems. These problems are three sentences long and include an assignment statement, a relational statement, and a question (Mayer, 1982). The relational sentence presents the quantitative relationship between the known and unknown quantities and may involve addition, subtraction, multiplication, or division. In CL problems, the language of the relational statement matches the necessary mathematical operation. For example, in these problems the phrase "1/n as many" accurately indicates division by n. In IL problems, the relational term is opposite to the mathematical operation needed for solution (e.g., "1/n as many" indicates multiplication by n).

Lewis and Mayer (1987) proposed the consistency hypothesis to explain differences in processing needs on CL versus IL problems. Accordingly, for IL problems, the problem solver reverses the relational sentence requiring the substitution of the opposite relational term. This transformation often results in a reversal error (e.g., dividing by n when the problem requires multiplication by n). Lewis and Mayer found that undergraduate students committed a greater number of reversal errors on IL than CL problems and a greater numbers of errors on IL addition and multiplication than IL subtraction and division problems.

Subsequent research has supported these claims, but several studies (Verschaffel, 1994; Verschaffel, De Corte, & Pauwels, 1992) have questioned the explanation provided by Lewis and Mayer. Verschaffel (1994) suggests that the pattern of errors may be due to the application of a keyword strategy rather than a predisposition for the CL problem structure. He examined fifth-grade students solving one-step addition and subtraction compare problems. Although the findings lend additional support to Lewis and Mayer's hypothesis, Verschaffel offers potential alternative explanations for the results and calls for the use of "new (combinations of) techniques for data gathering and data analysis" (p. 161).

Hegarty, Mayer, and Monk (1995) studied differences in the patterns of behaviors of successful and unsuccessful problem solvers on compare word problems. Undergraduate students' eye fixations were monitored as they read and stated their solution plan. Unsuccessful students committed a larger number of reversal errors on IL than CL problems and focused their eyes on the numbers and relational terms more than successful problem solvers. The authors conclude that although unsuccessful problem solvers struggle to represent problems, they focus on the numbers and relational terms more than other more informative terms in the problems. These individuals were said to use a direct translation versus the more meaningful approach of the successful problem solvers.

The studies reviewed here minimized the effects of conceptual and procedural knowledge by incorporating mathematical content familiar to the participants. For instance, they looked at young children as they solved addition and subtraction problems or undergraduate students. As a result, error rates reported in these studies are quite low. In addition, due to the use of eye fixation procedures, the researchers either limited the difficulty level of the mathematics involved or truncated the problem solving process in order to investigate only one phase of this process.

The present study addresses some of these concerns. First, examining sixth- and seventh-grade students as they solved compare word problems involving all four basic arithmetic operations increased the demands on mathematical knowledge. Second, methodological issues are accounted for by employing videotaped verbal protocols and requiring the participants to talk aloud as they read, solved and recalled the problems. Finally, the present study investigates the effect of differences in patterns of reading/problem solving behaviors.

Methods

Forty sixth-grade and 40 seventh-grade students participated in this study. Demographic data were collected through self-reports, and school records were examined for achievement test scores and report card grades. Participants were videotaped "talking aloud" as they solved and recalled 12 compare word problems and four filler problems. Participants then completed a paper-and-pencil computation test.

The videotaped "talk-aloud" protocols were parsed into discrete behaviors. Patterns of behavior were coded as either predominantly a meaningful approach or a direct translation approach. Success rate, type of error, number of rereadings, initial reading time, total response time, and quality of problem recall were also coded. Interrater reliability for 10% of the participants was calculated. Overall, acceptable reliability levels were found ranging from 0.49 to 1.00 with all but one higher than .65. (See Pape, 1998 for details of protocols, methods, and reliability estimates.)

Results and Conclusions

Four sets of analyses were performed: (1) regression analysis with success rate as the dependent variable; (2) ANOVA procedures comparing low and high fraction knowledge groups; (3) paired t-tests comparing performance across problem types; and (4) paired t-tests were used to test Lewis and Mayer's Consistency Hypothesis. Regression analysis indicated that total computation test score, number of problems solved using a meaningful approach, mean number of rereadings, and mean recall score accounted for 48% of the variance in problem-solving success. Students who scored higher on the computation test, who could recall more elements of the problems, and who used a meaningful approach solved more problems successfully. However, contrary to expectations, a greater number of rereadings was indicative of lower success rates.

Differences were found due to fraction knowledge on problems that contain a fraction of a number in the wording of their relational sentence. The high fraction knowledge group solved a greater number of problems ($\underline{M}_{\text{high}} = 1.83$ vs. $\underline{M}_{\text{low}} = 0.75$, $\underline{p} < 0.001$), used a meaningful approach more frequently ($\underline{M}_{\text{high}} = 1.25$ vs. $\underline{M}_{\text{low}} = 0.61$, $\underline{p} < 0.05$), recalled a greater number of elements and structure of the problems ($\underline{M}_{\text{high}} = 2.27$ vs. $\underline{M}_{\text{low}} = 1.99$, $\underline{p} < 0.05$), and made fewer fraction of a number errors ($\underline{M}_{\text{high}} = 1.14$ vs. $\underline{M}_{\text{low}} = 2.30$, $\underline{p} < 0.01$) than the low fraction knowledge group.

The effect of problem type was examined. Students solved more one-step problems than two-step problems ($\underline{M}_{\text{one}} = 2.08$ vs. $\underline{M}_{\text{two}} = 1.82$, $\underline{p} < 0.05$); CL problems more than IL problems ($\underline{M}_{\text{CL}} = 3.67$ vs. $\underline{M}_{\text{IL}} = 2.57$, $\underline{p} < 0.001$); and addition and subtraction problems more than multiplication and division problems ($\underline{M}_{\text{add/subtr}} = 2.35$ vs. $\underline{M}_{\text{mult/div}} = 1.83$, $\underline{p} < 0.01$). Participants altered their behavior on two-step versus one-step problems (i.e., reread and took longer to solve two-step problems than one-step problems). However, the students did not alter their behaviors (i.e., use a meaningful approach or increase the number of rereadings) as a function of other problem variables hypothesized to be more cognitively demanding.

Finally, Lewis and Mayer's (1987) Consistency Hypothesis was partially supported. Students committed a significantly greater number of reversal errors on IL word problems than on CL word problems. However, there was no main effect of language consistency for initial reading time or total response time. This supports Verschaffel's (1994) contention that students may not necessarily be reversing the relational statement but rather applying key-word strategies.

These findings support the conclusion that reading comprehension factors and mathematical knowledge are both important to successful representation and solution of word problems. Students' ability to transform and actively manipulate information presented in the text of a word problem (i.e., their ability to use a meaningful approach) was found to be a significant factor related to problem representation and solution. Although students who were having difficulty solving particular problems did reread those problems more frequently, rereading alone did not necessarily result in successful problem representation and solution.

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VARIATIONS IN PREFERENCE FOR VISUALIZATION AMONG MATHEMATICS STUDENTS AND TEACHERS

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The research described in this paper is a continuation and part of a wider study, which has been ongoing since 1982. The purposes of this study were to investigate patterns in the preferences for visualization during mathematical problem solving of teachers and high school students, and to investigate some of the pedagogical implications of these individual variations. Mathematical visuality (MV, the extent to which a person prefers to use visual methods in solving non-routine mathematical problems) of students was found to be significantly higher than that of their teachers. However, for teachers a weak positive correlation (0.404) was found between MV and teaching visuality (TV), the extent to which a teacher uses and encourages visual methods in teaching mathematics. The present study used the same preference for visuality instrument to investigate the case of 33 prospective teachers. The mathematical visuality of these prospective teachers tended to resemble that of teachers rather than students, although individual variations were wide.

The theoretical framework of this research was based on Krutetskii's (1976) classification of students into four categories, namely, geometric, harmonic-pictorial, harmonic-abstract, and analytic, according to their preference for and ability to use visual methods in mathematical problem solving. According to Krutetskii, all mathematical thinking involves logical analysis, thus the contrast is not between logic and visual thinking, but these may be regarded as orthogonal axes. The achievement of a student is based on the quality of the logic used, while the preferred type of thinking relates to the visual axis and the need for diagrams, visual imagery, and other visual supports. This framework suggests that students of all four types may be high or low achievers in mathematics according to the quality of the logic they use in their preferred modes of working. This aspect was confirmed in the prior research (Presmeg, 1985). [Note: Krutetskii (1976) based his categories on longitudinal studies of high achievers, his "capable" students, but he extended his model to all achievement levels in other writings (Krutetskii, 1969).] Because of overlap between the categories, it was found more useful in the research, rather than categories, to use a continuum of preferences for visual methods in solving mathematical problems.

The methodology of this research was both quantitative and qualitative. The instrument used was a "Mathematical Processing Instrument" (MPI) that measures preference for visual methods in solving non-routine mathematical problems. The instrument was originally suggested by the research of Suwarsono (1982), but adapted for the different purposes of the original study, and subsequently developed and field-tested for reliability and validity in three countries (South Africa, USA, and Sweden). The results suggest that preference for visualization approaches a normal distribution in most populations (Presmeg & Bergsten, 1995). Unlike Suwarsono's test, the writer's MPI consists of three sections, A, B, and C, where section A consists of 6 "everyday" word problems suitable for high school students, section B contains 12 slightly more difficult problems, and section C has 6 problems suitable for mathematics teachers. Students in grades 11 and 12 do sections A and B (18 problems), and mathematics teachers solve the problems in sections B and C (18 problems). After solving the problems in the way most comfortable for each individual, he or she answers a questionnaire concerning the methods of solution. In the prior research the results of the questionnaires were complemented with numerous interviews. In the present study, the refined test and questionnaire (section B only) were given to 33 prospective teachers in a university mathematics education program, and the results were compared with those obtained in the previous studies. Examples of problems from the three sections of the instrument are as follows (and different examples may be found in Presmeg & Bergsten, 1995).

A-2. Altogether there are eight tables in a house. Some of these have four legs and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs? [5 possible modes of solution, 2 visual and 3 nonvisual]

B-9. A passenger who had traveled half his journey fell asleep. When he awoke, he still had to travel half the distance that he had traveled while sleeping. For what part of the entire journey had he been asleep? [3 solutions, 2 visual and 1 nonvisual]

B-12. Ten plums weigh as much as three apricots and one mango. Six plums and one apricot are equal in weight to a mango. How many plums balance the scales against a mango? [4 solutions, 2 visual and 2 nonvisual]

C-4. An older brother said to a younger, "Give me eight walnuts, then I will have twice as many as you do." But the younger brother said to the older one, "You give me eight walnuts, and then we will have an equal number." How many walnuts did each have? [3 solution, 2 visual and 1 nonvisual]

Evidence and results

To illustrate visual and nonvisual solutions to problem B-9, data from the protocols of two of the 33 prospective teachers in the present study, Ana and Jeremy, will be presented. Ana used a visual method and Jeremy a nonvisual method of solution.

Ana: [Let A be the fraction of the journey for which the passenger was asleep.]

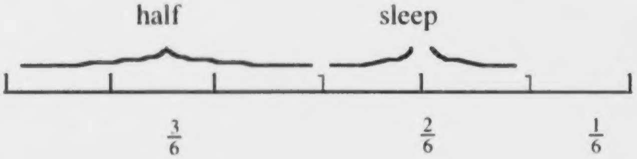
$$\frac{1}{2} + A + \frac{1}{2}A = 1$$

$$\frac{1}{2}A = \frac{1}{2}$$

$$A = \frac{1}{2}$$

Thus the passenger was asleep for one third of the journey.

Jeremy:



One third of journey while sleeping.

Figure 1: Ana's and Jeremy's solutions to problem B-9.

Data from MPI section B were analyzed to identify trends in the preference for visualization of the group of 33 prospective teachers. The data consisted of two scores for each person. A mathematical visuality (MV) score (out of a possible 24) resulted from the allocation of two points for each visual solution of the 12 problems in section B, zero points for a solution which did not use visual methods, including imagery, and one for a solution which was not attempted or was unclear. [MV scores for Ana and Jeremy were 6 and 10 respectively.] A mathematical accuracy (MA) score (maximum 12) consisted of the total number of section B problems solved correctly. [MA scores for Ana and Jeremy were 9 and 11 respectively.]

For the 33 students, frequencies of MV and MA scores are given in tables 1 and 2

TABLE 1

FREQUENCY DISTRIBUTION OF MATHEMATICAL VISUALITY SCORES

MV	1-2	3-4	5-6	7-8	9-10	11-12	13-14	15-16	17-18	19-20	21-22	23-24
Freq	0	1	6	5	10	5	2	1	2	1	0	0

Median MV score = 10 (out of 24).

A scattergraph suggests that the data confirm the previous findings that MV and MA are not significantly correlated. In other words, these prospective teachers manifest a wide range of preferences for visual methods in mathematical problem solving irrespective of their abilities to solve the problems correctly. The data suggest that these prospective teachers have MV scores that are similar to those of high school teachers (which are less visual) rather than those of high school students (more visual). A medians test in the prior research indicated that the medians of 10 and 13 for teachers and students respectively on section B of the MPI were significantly different at the 0.01 level (c^2=6.431, df=1). An implication of the results is that prospective teachers can benefit from the realization that there exist large differences in preferences for visual thinking amongst their peers. Not only this, but they can also benefit from knowing that there is a possibility that the students they will teach will feel the need for more visual supports in learning mathematics than they themselves do. A striking and understandable result of the earlier research was that effective teachers were inclined to teach more visually in the mathematics classroom than their low MV scores suggested. They instinctively knew that some of their students had more need for visual methods than they had themselves.

Conclusions

The results of this research confirm Krutetskii's (1976) suggestion that ability for logical reasoning and preference for visual thinking in mathematics may be considered as independent. The usefulness of a mathematical visuality continuum as a research tool was also confirmed. Prospective teachers often express surprise, not only at the variety of possible methods of solution of the problems in the MPI, but also at the wide range of preferences for visual thinking amongst their peers, as evidenced in the mathematical visuality scores. There follows the realization that they can also expect a wide range of individual preferences amongst the students they will teach.

The results also suggest that "reluctance to visualize" (Eisenberg and Dreyfus, 1991) is a learned phenomenon, a possible artifact of teaching that does not encourage this mode of cognition. The claim that students are reluctant to visualize was challenged (Presmeg & Bergsten, 1995), and the current research confirms that there is no dearth of students who prefer to visualize in their mathematical problem solving. Awareness of their needs amongst prospective teachers is a desirable goal in teacher education.

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HOW A PROBLEM CENTERED CURRICULUM ENHANCED THE LEARNING OF LOW ACHIEVERS

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ABSTRACT: A class of sixth grade low achievers in the rural Southeast United States participated in a problem centered mathematics curriculum for nine weeks. Tasks deemed to be potentially meaningful to students were posed, students worked in groups of two or three, and finally the groups of students presented and defended their solutions to the whole class.

Research Objectives

This experiment was designed to study children who had poor scores on standardized tests in mathematics with hopes of improving their attitudes and achievement. Middle school children were targeted because an abundance of research has emphasized the need to reach low achievers during their *middle school years* – the critical age when children make many permanent decisions about themselves, their abilities, and their future (Garcia & Pintrich, 1995, Manning, 1997, Sigurdson & Olson, 1992, Thorndike-Christ, 1991). The goal was to show improvement using a set of instructional materials designed to promote key number constructions.

Student achievement and attitude were examined by comparing the effectiveness of a problem-centered approach with a conventional approach. The problem-centered curriculum focused on meaning-making in mathematics, whereas the conventional curriculum emphasized learning accepted procedures and then practicing these. Problem Centered Learning was chosen because it has been shown to aid individual constructions and foster meaningful communication in all students (Cobb, Wood, & Yackel, 1991a; Cobb, Wood, Yackel, Nicholls et al., 1991; Thompson, 1985).

Beliefs and Theories Affecting the Study

Low scores on standardized and classroom tests can often be attributed to an inability to make sense of what was taught. Low achievers frequently try to memorize facts and complex procedures without understanding. Some students barely remember methods long enough to pass classroom unit exams because the mathematics did not make sense to them even as the teacher illustrates the procedure on the board. These children are easily identified as low mathematics achievers and receive poor classroom grades.

Others, who retain detailed procedures for a limited time, are able to "pass" mathematics courses in the short run. But they are unable to connect their shallow understandings in any significant way beyond an isolated task at hand. As long as they are practicing procedures just illustrated by the teacher, they might look competent. But these students do not *own* the mathematics; it does not mean anything to them beyond a set of unrelated procedures. Out of context on a standardized test with a variety of problem types (like the Iowa Test of Basic Skills, or ITBS), they are likewise bewildered and often achieve low scores.

The better choice is for children to *use their own insights* to make meaning of mathematics. They have to become empowered. They need to trust their own experiences and realize that there are many acceptable ways to do mathematics. Their way could be a right way. Constructivist theory supports the belief that students develop their own insights by building on past experience (von Glasersfeld, 1995).

People gain self-confidence through assertion, shared interaction with peers, and articulation in a non-threatening environment such as that provided by a problem centered approach. This need to become actively involved in classroom discourse creates a dilemma in the case of most low-achieving students. Because they are accustomed to obtaining the "wrong" answer in mathematics, to maintain some self-respect low attainers learn to avoid humiliation by shying away from offering solutions in public. Unfortunately, their hesitancy to talk in class actually inhibits their learning.

In addition, the kinds of response low achievers get from authority figures is related to their ability to make sense of mathematics. The more often the teacher or perceived "smart kid" says they are wrong, the more often low-achievers regress into just copying the procedures they see without trying to make any sense of it. Thus, a primary challenge in this study was finding ways to get students to open up and recognize the reward of asserting themselves mathematically. An interactive classroom culture (Bauersfeld, 1988) of acceptance is critical as children struggled along a path to understanding.

Experimental Design and Data Sources

A public middle school in the rural Southeast USA agreed to randomly assign twenty-six sixth grade students to an experimental class from a possible selection of one hundred fifty-three low achievers. Twenty-six others were also identified and stayed in the school's regular math classes – these were the control group. All of these students scored below 40% national norm in mathematics on the ITBS in fifth grade.

Students in the experimental problem-centered group attended a special class for the first nine weeks of school and then returned to traditional classes. The mathematical content studied by both groups was identical, ensuring that all children would have equal opportunity to develop the same concepts and skills. Only the instructional strategies used to present mathematical concepts were varied. The author took the role of teacher-as-researcher, or participant-observer, for the experimental group.

All eligible students took a qualifying pre-test during the final week of fifth grade. Teachers agreed that this free-response test represented the mathematics' proficiencies expected at the end of elementary school: pattern matching, number sequencing, fraction concepts, and arithmetic word problems. The pre-test served two purposes: 1) it confirmed a student's identification as a low achiever, in addition to their ITBS score (none of the potential 153 students were able to score above 54% correct on the pre-test), and 2) it gave an independent measure of mathematics achievement for comparison at the end of the experiment. The test was also tied very closely to the mathematical concepts that were taught to both groups in the first nine weeks of sixth grade. A similar post-test was administered after nine weeks, along with parent and student attitude questionnaires, to identify any changes that occurred.

Extensive qualitative data was also collected in order to obtain a thick, rich description of the students. Three students (purposely chosen as high, middle, and low achieving) and their parents were individually interviewed at the end of the study. All students kept a reflective journal throughout the experiment. The researcher kept an observation log, which was triangulated with the observations of another disinterested professional who visited the class.

The Problem Centered Learning model was designed as follows:

> The class begins with a problem posed by the teacher, or perhaps by a student. The class is then organized into small groups (two or three students of similar capabilities) and the students work collectively in groups on the tasks posed. After about 25 minutes, the students are assembled for class discussion in which students present to the class their solutions for consideration by the group which then serves as a community of validators. During the class discussion the teacher is nonjudgmental and the viability of solution methods is determined by the class, not the teacher. In problem centered learning the teacher has three main roles: selecting appropriate tasks based on her knowledge of the students, organizing the groups and listening carefully as they work and finally, facilitating the class discussion (Wheatley, in press.)

In contrast, the control group was randomly distributed throughout the sixth grade with four regular mathematics teachers who used a traditional textbook and lecture and practice methods.

The Effect of the Problem-Centered Curriculum

The results of the study can be divided into three main categories. First, the quantitative analysis of the test scores showed the problem-centered curriculum significantly enhanced the learning of low-achievers. Table 1 contains statistical information used to draw conclusions.

Table 1: Test Results

		Control	**Experimental**
Pre-Test	Mean	27.5	27.3
	Std. Dev.	14.0	11.7
Post-Test	Mean	46.3	60.9
	Std. Dev.	16.9	17.0

Since there was virtually no significant difference between the means of the two groups on the pre-test at the onset of the study, the large increase in post-test scores for the experimental group can be confidently stated as due to the problem centered curriculum. (Using a t-test assuming equal variance, the post-test results yielded a $t=3.11$ and a $p < .01$.)

The second category of results was more qualitative in nature and spoke to the curriculum's influence on student attitude. Over 90 percent of students reported liking mathematics more after participating in the problem centered curriculum. They consistently wrote that "Math was fun," in their student journals. Observations showed that most children participated – articulating enthusiastically, working cooperatively with their peers, and sharing solutions – during every class period. Their demeanor was in stark contrast to the usual non-participatory behavior noted for low achievers.

The parents of children in the problem-centered group also responded that there was an increased interest and enjoyment of mathematics that spilled over into daily life. Students enjoyed doing homework. Interviews supported the findings of journals and observations. Several parents and children felt the nine-week class had made a significant positive difference in their lives, especially in the way they viewed mathematics. On the other hand, surveys and interviews of children in the control group showed that little or no change in attitude had occurred during the nine-week study.

The third category of results should be singled out as a "Description of Tasks" because of their significance not only to this study but also to mathematics teaching and learning in general. Many of the activities developed for the problem centered curriculum were adapted from instructional strategies previously used with younger children, yet meaningful learning occurred in the sixth grade when they were modified to include large numbers, decimals, and fractions. A combination of balances, number squares, and two-way additions required students to analyze each problem, rather than classifying a block of problems into same-type procedures reminiscent of "practicing." When combined with text problems, real-life data collection and graphing, and the requirement to justifying one's solutions to the whole class, students were able to make sense of their mathematics in ways that increased their achievement and improved their attitude.

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RECOGNITION OF STRATEGIES USED BY STUDENTS TO SOLVE WORD PROBLEMS IN ELEMENTARY SCHOOL

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The present study leaded to the recognition of strategies used by fifth grade students (11-12 years) when solving word problems.

Bednarz and Janvier (1996) proposed that to understand the transition from Arithmetic to Algebra it is necessary to reflect on the nature of problems and its relative difficulty. They pointed out the importance of the conditions under which algebraic reasoning appears and the developing of this reasoning within a solving problem context.

In the first part of the research, we selected 15 students (5 of them with a high performance, 5 with middle performance, and 5 with low performance) -through a diagnostic questionnaire- to form groups of three students each, for the experimentation stage. In the second part, 15 videotaped working sessions in-groups were carried out (50 minutes each session) and students solved the total of 22 problems. In the third part of research, a final questionnaire was given to students to explore the progress achieved in the experimentation stage.

During the experimentation stage and in final questionnaire students showed different strategies such as drawings, tables, number line, algorithms, trial and error, systematic trial and error, which allow them to face the problems posed with more favorable conditions. Students used numeric-intuitive strategies, resulting from their previous knowledge, which became in an extension of such knowledge for the construction of new knowledge. Systematic trial and error was the most used strategy.

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THE ROLE AND IMPORTANCE OF STUDENTS' INITIAL PERCEPTIONS IN MATHEMATICAL PROBLEM SOLVING

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What do students attend in their first reading of statement of problems? To what extent their initial understanding of the problem becomes important in the selection and implementation of strategies of solutions? These are research questions that were part of a study carried out with first year university students. An important objective was to document criteria students utilize when they are asked to organize different types of problem and the type of strategies and mathematical resources exhibited by the students during their processes of solution.

A list of 32 problems was given to the 26 students and they were asked to group them in accordance to some criteria defined by them. That is, they had to explain why he or she had decided to group the problems in that way. For this task, the students spent an hour. Later, in another session, the same students were asked to solve five of those problems. This session lasted two hours. Is there any relationship between the students grouping of the problems and their solution approaches?

Results showed that the students, in general, utilized irrelevant features of the problem to group them. For example, a problem that included "figure" in its statement was listed in the geometry group or a problem with the word "root" was referred as algebra group. Other groups proposed by the students involved labels as polynomials, limit, physics, etc. In general, it was clear that students grouped the problems in accordance with subjects or general themes. When students were asked to work on the problems, it was evident that their initial perception was important to approach the problems. For example, in the problem "Does the equation $x^{21} + x^{19} - x^{-1} + 2 = 0$ have real roots in (-1, 0)? (Justify your response)" which, in general, the students identified as an algebra problem; the students' attempt showed that they spent significant time searching the solution by using trial and error, finding a formula, or by factoring the expression. These approaches often led the students to the incorrect use of data or pursue for long time fruitless ways of solutions. Thus, instructor should pay attention and respond accordingly to the students' initial meaning of the problem.

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STUDENTS' STRUGGLE TO RECONCILE THEIR ACTIVITY-BASED WITH THEIR PROCEDURE-BASED SOLUTIONS: IMPLICATIONS FOR INSTRUCTION

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Mathematics depends on the use of symbols for the expression and communication of mathematical ideas (Van Oers, 1992). However, school mathematics instruction over-emphasizes the ability to manipulate symbols and algorithms at the expense of conceptual understanding and communication of mathematical ideas. The subjects of this study were students in college math course for elementary education majors who had experienced an instrumental approach to mathematics. In this course, students were expected to explain and justify their answers. As they attempted to meet these expectations, the students perceived a disparity between their self-generated solutions and the answers they derived via standard algorithms or manipulation of conventional representations. This study considers the learning opportunities that arose as the students' cognitive struggle became the explicit focus of the subsequent mathematical activity. The analysis is grounded on the notion that individuals construct knowledge as they resolve problematic situations in interaction with others (Piaget, 1963; Labinowicz, 1985; Cobb, 1989). The data collection consisted mostly of field notes of students' activity during small group and whole class discussions; notes from their journal reflections; and informal interviews.

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Rational Numbers

USING MODELING ELICITING PROBLEMS TO EXAMINE STUDENTS' UNDERSTANDING OF EARLY FUNCTIONAL REASONING

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A model eliciting approach specifically engages students in the activities of creating meaningful symbolic, graphical, and numerical representations and descriptions of situations when solving non routine problems. The study reported in this paper focused on the modeling cycles that emerged as children solved one model eliciting problem. The modeling eliciting approach discussed in this paper provides learners with an opportunity to create and continually refine their early functional understanding by grappling with the problem of how to represent the relationship between two variables in such a way that decisions can be made. The characteristics of each stage of the model eliciting task are illustrated by collaborative small group examples. Student multiplicative reasoning about the relationships between and among quantities is discussed. The results of this classroom-based case study suggest that students were able to move from naïve additive understandings to more sophisticated multiplicative understandings and create generalizable systems (or models) for making decisions.

Introduction

This study traced the emergence of early functional reasoning and the modeling cycles which three students produced when they solved model eliciting problems (Lesh & Lamon, 1992). These problems are designed explicitly to focus on the activities of creating symbolic, graphical and numeric representations and descriptions of situations that are meaningful to the students. That is, to a large extent, the process is the product when solving model eliciting problems. Consequently the product explicitly reveals significant information about the reasoning processes that produced it. Thus, we focused on the explicit information that was provided about the kind of functional relationships the students were attending to, and how they represented the relationship between two variables.

Theoretical Background

Traditional word problems in textbooks and tests are intended to emphasize computational skills. However it is often the case that the main thing that students find difficult about such problems is that they are required to make meaning out of symbolically described situations. The solution process to typical word problems is basically applying the problem information onto an invarivant model using symbolic notation, usually numbers and arithmetic operations. This type of problem rarely allows for students to investigate the mathematical model in a creative manner that allows for creating a conjecture, experimentation, revision and refinement of their mathematical ideas.

In contrast to such problems, the model eliciting problems that we refer to in this study, are almost exactly the opposite kind of processes that tend to be problematic. Thus, just as in most real life situations in which mathematics is useful, students must try to make symbolic descriptions of meaningful situations. Students collaborating in this type of model building is not seen as finding a solution to a given problem rather as developing a tool that a learner can use and re-use to find solutions (Doerr, 1997). Each stage of the modeling cycles include different interpretations, descriptions, conjectures, explanations, that are constantly refined and reconstructed by the learner, usually interacting with the other learners.

Description of the study

This study is part of a larger research project that utilized a research design referred to as a multi-tiered teaching experiment (which is explicitly detailed in *The Handbook of Research Design in Mathematics and Science Education*, Kelly & Lesh, in press). This research design was chosen as it produces auditable trails of documentation that focuses on the development of deeper and higher-order understandings.

Within the multi-tiered teaching experiment, some of the most important key events focus on sequences of model eliciting problems in which participants are repeatedly challenged to reveal, test, and refine, or revise important aspects of their ways of thinking. In this way, these problems promote learning; yet, at the same time, a byproduct of learning is that auditable trails of documentation emerge that reveal important aspects about the nature of the construct being developed, in this case, early functional reasoning.

This study, the first tier of this project, examined three students from one middle school mathematics classroom in an urban school in the southeastern United States. The main goal of this study was to understand the collaborative chronological series of student's meaning, verbalizations, interactions, written models, and constructions of mathematical concepts and operations in regards to early functional reasoning. We used 4 one-hour small group sessions, to understand and document the students' collaborative models while solving one model eliciting problem per session. This paper reports one of those sessions.

Data Sources and Analysis

Each class session of the overall unit was audio-taped and video-taped. The audio-tapes were fully transcribed while the video-tapes of class sessions were reviewed and selected portions were transcribed for more detailed analysis. During small group work, the groups were observed by the researcher and extensive field notes were taken. Student reflections and all class work done by the students was made available to the researcher.

Description of the Model Eliciting Task

The investigation consisted of four problem situations (one of which is discussed in this paper) that focused on the core mathematical idea of representing situations and number patterns with tables and graphs as well as representing relationships between variables. The students had no formal instruction on these ideas before the onset of the unit. Rather the model eliciting unit was designed so that students could engage in meaningful situations and explore, use, and modify quantities in ways that would be meaningful to them as well as to the rest of their class and lastly to be re-used in new situations.

The Computer Game Problem was the third model eliciting problem that we used. During the first two sessions, the students solved their first two model eliciting problems. These problems introduced the students to the ideas of representing patterns with tables and graphs, representing the relationships between variables, and working with large data sets to ultimately make decisions. Thus, *The Computer Game Problem* was designed to extend and refine their initial ideas.

The students were given an article from a "mathematically rich student newspaper" which gave them background information on the problem situation. They were also given a complete data set comparing the retail stores where the computer game, *Space Fighters*, was sold, as shown in Table 1.

Table 1. Compilation of Data on Retail Stores Selling Space Fighters

Name of Retail Store	# of Sales People	#of Copies Sold	Hours Open Per Week	Price Per Copy
Mega Mall Computer Giant	28	1400	110	$49
Wheelock and Sons Software	5	385	90	$65
Valley West Mall Store	9	425	85	$58
Super Mega Mall Store	26	1055	100	$51
Clark's Computer Software	6	375	80	$77
Main Street Software	5	350	76	$72
Software America	4	248	48	$76
North Grand Mall	8	462	70	$53

The Computer Game Problem: Janet and Mitch own a software company that sells software games through a number of retail stores. In the last three months sales have boomed! They would like to reward the top three stores with a gift to show their appreciation. However, they have a problem. Mitch and Janet understand that just looking at total sales would not be fair because some of their stores are smaller "mom and pop" stores, whereas some of their other retailers are situated in large, mega mall centers with a large sales force.

You are a marketing and financial whiz. Your firm specializes in evaluating retail stores. Mitch and Janet have sought out your help to determine which three retail stores they should honor considering all of the factors in their data records. Please evaluate how well the different stores performed the last three months and decide which three they should reward. Write a letter to Mitch and Janet giving your results. In your letter describe how you evaluated the stores. Give details of how they can check your work, and give a clear explanation so they can decide whether your method is a good one for them to use in the future.

Results and Discussion

These results come from the analysis of small group interactions form the aforementioned model eliciting problem. All excerpts supporting the major findings are from three seventh grade students: Trent, Nolan, and Kelly.

Three major findings will be discussed. First, students constructed multiple modeling cycles, up to 10 cycles were documented. Second, these modeling cycles appeared to increase in stability and sophistication throughout the problem solving session. Lastly, results indicated that model eliciting problems produced valuable generalizable and re-usable systems (or models) for selecting, ranking and describing the relationship between variables.

The Computer Game Problem evoked several early and unstable interpretations that were revised and refined within the small group. However, after Trent, Kelly and Nolan came together, their next interpretation of the problem was to *focus on the sales for each store*. They first used a trial and error method and applied a simple formula: total sales = price (#of copies sold). They organized this information in a chart to easily compare total sales. At this point some of the other teams essentially quit working. However, Kelly influenced her group to further extend their thinking by expressing the need to consider all factors in the chart.

The next interpretation that Trent, Kelly and Nolan constructed was focused on *sales by number of employees from each different store*. They made necessary calculations, a chart, and a bar graph that illustrated their dollars made by employees. Further they ascertained that it would be *average sales by employees by store*. Again they realized they needed to address all of the variables which indicated a higher level of thinking.

Which led them to focus on *trends in rank across two variables, total sales and average sales by employees by store*. They went ahead and rearranged their previous charts so that they could rank from one to eight the top store by total sales earned and then by top employee average sale. They analyzed these trends and concluded that the rankings were quite different and a new option was needed that "would be more fair."

Thus, it is apparent that Trent, Kelly and Nolan have gone through at least four modeling cycles where they have refined their more naïve conceptualizations to gradual yet more sophisticated ways of thinking. They went on to explore the *hours during the three-month time period*. They were able to clarify that the "field wasn't even." Thus, they tried to use subtraction as a means of "evening out the hours" because then they "could fairly compare the different retail stores." However, at one point of frustration, Trent referred back to his own summer job where they always evaluated the workers on how much they sold each hour. The group agreed. Their next interpretation was to focus on *dollars per hour*. This indicated that the group was thinking in a more multiplicative manner as they had collapsed two variables into one variable to analyze the trends. This led to their conversation that led to their final interpretation that was *focused on trends across three variables*. They focused their letter to Mitch and Janet on a method that could be used every three months to verify the most productive retail store.

Conclusions

The students' solutions to the model eliciting problem illustrated that through the course of one hour, their early functional reasoning became more sophisticated as these ideas were continually identified, tested, and refined. This was documented by ten stages of interpretation or ten modeling cycles. The student's final interpretation was a method of determining the best stores using the given variables which was a re-usable product. These results were similar to those of all four of the model eliciting problems. It appeared that each time the students solved a different model eliciting problem they started with naïve interpretations even when the model eliciting problems were focused on the same kinds of core mathematical ideas. Thus, the students had previously made sense of a mathematical idea within one contextual situation but they were not able to re-apply this knowledge in another contextual situation. Thus one idea that permeated this investigation was that knowledge is organized around situations as opposed to knowledge being organized around abstractions. This ideas warrants further research.

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QUALITATIVE REASONING IN PROBLEM SOLVING RELATED TO RATIO, PROPORTION, AND PROPORTIONAL VARIATION CONCEPTS

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This paper reports on the findings of the first of three stages of a qualitative-oriented research in relation with the analysis of problems with non-numerical data, linked to the topics of ratio, proportion, and proportional variation. This study is being carried out with undergraduate students seeking a BS degree in Mathematics at the *Escuela Normal Superior de México* (Higher Teachers School, Mexico City). The findings touch upon diverse aspects, such as: a) a qualitative-type analysis of problems with non-numerical data, and the strategies that students bring into play in order to predict an answer; b) the construction of this kind of problems when starting from the analysis and interpretation of «school problems» that are studied in the arithmetic, algebra, and geometry courses given at the aforementioned Institution; and d) the way in which the researcher creates and uses various representations in order to analyze and interpret the students' qualitative-type discourse, as expressed in (the students') vernacular language.

The importance of a qualitative-type analysis as a basic component of proportional reasoning has been studied by several researchers. Tourniaire and Poulos (1985), through the perusal of works published between 1958 and 1983, report the contributions of researchers who have helped in the characterization of this kind of reasoning. From these studies, and others recently reported, we identify those that are related with the ones which have a bearing on the research purposes which have been an endeavour of the *Escuela Normal Superior de México* (ENSM) for the last two years. These research efforts have to do with the prediction and comparison of results, i.e., explicitly standing out in them are the qualitative-type relationships among non numerical quantities. Piaget and Inhelder (1969) hold that the notion of proportion arises, in a qualitative and logical manner, around ages eleven or twelve, before it achieves a quantitative structure. Such a notion is constructed in very different environments, and always in the same manner, which is initially a qualitative one (p. 141). Streefland

(1985) has suggested that the learning of ratio and proportion is a long-term process beginning with qualitative comparisons (as quoted in Hart, 1988, p. 202). Lesh, Post and Behr (1988) assert that proportional reasoning implies a form of mathematical reasoning which involves a sense of co-variation and multiple comparisons, as well as the ability to mentally accumulate and process various fragments of information. Proportional reasoning is related to actions implying inferences and predictions, underlying which, both forms of thought —qualitative and quantitative— (p. 93). Resnick and Singer (1993) argue that children avail themselves of a set of *protoquantitative* schemes which allow them to reason about ratio and proportion as relationships requiring no numbers. They affirm that protoquantitative reasoning is not limited to a pre-mathematical stage of development, but that it can go on being useful for a whole lifetime (p. 109). In other areas of knowledge it has also been recognized that a quantitative-type analysis of problems is a primary component for the understanding of the latter, and that qualitative predictions can be useful in the solving process of diverse qualitative tasks. To examine the role played by the qualitative-type analysis in understanding the solving processes of problems related with ratio, proportion, and the proportional variation is one of the central objectives of this research, which is being carried out with 20-26 year-old ENSM students; these are the future teachers of the Mexican secondary (basic) school system.

The Study and its method

This research is a qualitative-type study whose fundamental purposes are: a) to identify and characterize by means of a catalogue of strategies, those mechanisms employed by students for solving non-numerical problems, and b) to identify those qualitative-type mechanisms which favor the understanding of the solving processes for quantitative or other school problems. Research has been approached in three stages: 1) An exploratory study which permits to identify, by means of non-numerical problems, those solving strategies involving a qualitative-type analisis. 2) Individual interviews employing a methodology which combines a clinical-type inquiry and an experimentation through the simulation of didactical sequences. And 3) Case studies allowing to establish a relationship between qualitative-type strategies and the ways to solve school problems. This article presents results deriving from the first stage of the research.

The exploratory study is being carried out as per the following phases: 1) The perusal and analysis of textbooks that are recommended for the various courses on arithmetic, algebra, and geometry at the ENSM. The purpose of this enquiry was to establish a catalogue of problems related to ratio, proportion, and proportional variation which appear in textbooks and which, therefore, are a part of the didactic sequences which are being taught in this institution charged with the formation of teachers. These problems have been called *school problems*. 2) The selection of school problems related to the following topics: a) direct proportional variation; b) inverse proportional variation; c) compound proportional variation; d) variation with respect to time; and e) proportional sharing. 3) The design of problems in the statement of which no numerical data appear, and which in a certain sense were "equivalent" to the school problems selected. The former have been termed *non-numerical problems* (see Figure 1, where an example is given of a typical school problem). 4) The preparation of a written examination comprising eight problems, whose composition is shown in Figure 2.

School problem	**Non-numerical problem**
Thirty workers can perform a task in ten days. If the same task is performed in fifteen days, how many workers are required? a) 45 worker b) 20 workers c) 30 workers d) You cannot tell Argue your choice	A certain number of workers can perform a task in a certain number of days. If the same task is performed in more days, then the workers required are: a) More b) Less c) An equal number of worker d) You cannot tell Argue your choice

Figure 1. "Equivalence" between a school problem and a non-numerical one

Topic	Context	Quantity
Direct proportional variation	a) Triangles and b) Mixtures	3
Inverse proportional variation	a) Workers-days and b) Depreciation of the value of things	2
Compounded proportional variation	Workers-days-task	1
Variation with respect to time	Water pumps	1
Proportional sharing	Prize-Cost	1

Figure 2. Topics and contexts of the non-numerical problems

5) The application of this exam to two groups studying fourth and sixth semesters of Mathematics BS, as per the curriculum at the ENSM. On the whole, the examination was given to 65 students (36 in fourth semester, and 29 in sixth) and it lasted for approximately 1 1/2 hours. And 6) An analysis of the answers in each problem focusing on the category characterization that allowed grouping them as per common features.

Discussion

1. The design of equivalent statements. The translation of school problems into non-numerical ones (see Figure 1), starts with a number of assumptions: a) it is assumed that a procedure exists which will lead from a school problem to a non-numerical problem without any changes occurring in the structure of the statements in the former, except for the sense or meaning of the questions; in school problems one is interested in knowing how much or how many; in the latter, instead, what matters is to forecast results; and b) it is assumed, at the beginning of this research, that non-numerical problems permit the identification of a number of qualitative-type strategies which it is implicitly believed the students use for solving school problems. What happended during the course of this investigation? Well, insight was gained, furthermore, on some strategies that students use, such as similitudes and differences, concerning the construction of the «equivalence» of these two types of problems. It was noted that the relationships in non-numerical problems lead to more than one possible answer, which is something that does not usually happen with the quantitative relationships in school problems, for these promote the use of algorithms that have been learned, as for instance the rule of three and, consequently, they lead to a calculation whose result is one only possible answer. In the school problem of Figure 1, option (b) is the solution, the other choices being mere distractors. In the non-numerical problem, on the contrary, any of the options, except (a), may be the correct one, depending on the considerations one makes as regards the amount of days that the term is increased to do the task. Thus, the translation of a school problem into a non-numerical problem brought forth a generalization of the information, and it also shed light, after an analysis of the students' answers, *on a number of implicit aspects* which they do not question when solving school problems. In the latter, like the one in Figure 1, apparently nobody wonders or explicitly discusses whether the workers must or must not work at the same rate, or whether their work-loads are equivalent, nor is it a matter of discussion the way in which these assumptions affect, or do not affect the interpretation of the problems. With non-numerical problems, on the other hand, reflexions such as these arise in a spontaneous way. An examination will now follow on some of the considerations that students participating in this research took into account when answering the non-numerical problem in Figure 1. The distribution of answers to this problem, from the 65 students was as follows: option (a), 5; option (b), 47; option (c), 5; and option (d), 2. Of all these, it had been foreseen that students would privilege option (b), since it was considered that they would not have any difficulty as to identify and, in some way characterize the inverse relation existing in the problem data. What the research showed was that there were diverse arguments to favor this choice. Some of the response categories which were characterized in this option are described in Figure 3.

2. A first approach in the shaping of a catalogue of strategies. Non-numerical problems generate a variety of answers, such as those described in Figure 3; this puts a hindrance to the definition of "category." A first attempt to achieve such a characterization is to focus attention on the basic aspects relating this kind of problems with school problems. A preliminary separation which identifies common features in all non-numerical problems is as follows: a) the use of written arguments, b) the use of drawings and graphics, and c) the use of numerical examples. It should be mentioned that each category generates a number of subcategories which maintain a strong link with the context of the problem.

3. The use of various representations: a resource to interpret the students' answers. One technique which has permitted the analysis of the students' answers is to reconstruct or to try to complement their arguments by means of schemes where the qualitatve-type analyses are emphasized. The first category appearing in Figure 3, for instance, is analyzed and interpreted by means of a scheme of "scalar" analysis like the one shown in Figure 4.

It is assumed that students check (inverse process), but do not prove (direct process) their choice.

According to this scheme, the students' arguments could be thus translated: if the number of workers (W) diminishes (W-) then the work-load (Wl) increases (Wl+), but if the work-load (Wl) increases (Wl+), then the number of days (D) should increase (D+). Therefore, the number of days suffers an increase.

Concluding Remarks

1) In school problems, the modelled situations are simplified, and there is no evidence there of implicit considerations which in non-numerical problems arise in a very explicit manner. 2) The use of drawings, diagrams, and numerical data supplements the qualitative-type analysis of non-

Categories

By a sharing and redistribution of the work-load (10 students): this strategy arises from an effort to give an orientation, according to the context of the problem, to the meaning of the arguments: the *sharing* of the work-load among the total number of workers. This tactic allows the students to *test* their choice of option (b). It cannot be known, from the explanations they give, how they forecast the answer, but what is known is the way they validate it: if the number of workers decreases, then the work-load increases for each one of them and, therefore, an increase appears in the number of days to finish the task. As it can be seen, they check, or test (an inverse process), but they do not prove (a direct process) their choice.

The invariance of one item in the data (16 students): it is detected that one of the three data in the problem does not change (the same task). By a "logical" process, as the students state, if there is a change in one of the other two data (more days), then they predict that the other one must change in the inverse sense (less workers). Although they do not always state their detailed "logical" arguments, the print these leave allows us to assert that the identification of the invariance of the aforesaid item in the data helps them to organize their ideas, and to decide in favor of option (b).

The use of numerical quantities (4 students): this strategy, the use of numerical quantities, favors the understanding of the problem and the forecast of the answer. It goes from the particular to the general, and guides students in the determination, in non-numerical problems, of a change in direction between the quantities of one same measuring space and, therefore, it permits them to infer the directionality of the quantities in the other measuring space.

Figure 3. An example of categories belonging to option (b) of the non-numerical problem in Figure 1

Workers	Work-Load	Number of days
W	Wl	D
W-	Wl+	D+

Figure 4. This representation shows the "scalar" analysis of the first category in Figure 3.

numerical problems. And, 3) The qualitative-type analysis is a recourse that may contribute to the design of teaching sequencies, and may generate links between the components of the teaching models and those of a cognitive nature; and this turns the construction of a new local theoretical framework in the sense proposed by Filloy (1999) into a possibility.

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MULTIPLICATION AND DIVISION OF FRACTIONS: CONVERSATIONS WITH PRESERVICE TEACHERS

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This paper reports on exploratory research conducted jointly by a mathematician and a mathematics educator investigating the rational number concepts of preservice elementary school teachers and conditions that promote deeper understanding. In the part of our study reported here, we focus on interpretations of multiplication and division by whole numbers and fractions with two preservice teachers. We use mathematical conversations to examine and strengthen students' rational number concepts. The purpose of this work is to enhance both our own understanding and that of our students and to inform teaching at both levels.

We have undertaken a two-year research project investigating the rational number concepts of preservice elementary teachers and the potential strengthening of these concepts through extended clinical interviews of pairs of students. The work reported here involves a series of discussions about the meaning of multiplication and division of integers and rational numbers with two preservice elementary teachers who had recently completed a mathematics pedagogy course. Both of these students had participated in mathematical interviews during the course and were preparing for classroom work. They had found the interview process useful and wanted to continue it in order to deepen their understanding.

Perspectives

A major perspective informing our work is the empowerment of prospective elementary teachers as mathematical thinkers through in-depth conversations about mathematical concepts. The form of these conversations is based on the clinical interview, as described by Ginsburg et al. (1983) and Davis (1983) and extended by Duckworth (1987). In our work, students are interviewed in pairs so that they can observe and benefit from their differing strengths and approaches.

In our work with students on rational numbers and the operations of multiplication and division, we are influenced by Vygotsky's (1962) theory on the social construction of knowledge and the nature of the interrelationship between scientific concepts and spontaneous or everyday concepts, the notion of the multiplicative conceptual field discussed by

Vergnaud (1994) and related works on multiplicative reasoning (Harel and Confrey, 1994), including the splitting conjecture presented by Confrey (1994, 1998).

The results of recent studies indicate that many preservice and inservice elementary school teachers (Ball, 1990; Borko et al., 1992; Orton, 1988; Post et al. 1988) and college students in general (Silver, 1986) have a limited conceptual understanding of rational numbers. There is some evidence to suggest that prospective elementary school teachers' intuitive notions of multiplication and division may interfere with their understandings of these operations on decimals (Behr et al 1994; Tirosh & Graeber, 1989) and that their limited understanding of multiple meanings of division is related to their difficulty with division by a fraction (Ball, 1990; Simon, 1993). Fischbein et al (1985) argue that the operations of multiplication and division are attached to "primitive behavioral models" and that these models are insufficient and cause most of the difficulties encountered when one attempts to solve problems requiring these operations.

In our interviews we explore the ways in which two prospective elementary teachers make sense of the symbols for division of whole numbers and division of mixed numbers by a fraction. Hiebert and Carpenter (1992) argue that understanding of written symbols can develop by establishing connections within representations or by establishing connections with other forms of representation such as physical objects, pictures and the spoken language, where the source of meaning is presumed to be the "internal networks that already have been created for these forms." We agree with the suggestions of Vygotsky (1962), Freudenthal (1973), Fischbein (1987) and Hiebert and Carpenter (1992) that it becomes necessary for students to learn to operate with mathematical objects without insistence on attachment to real-world intuitions.

Research Methodology

Students in a mathematics pedagogy course for prospective elementary teachers were initially given a series of written questions of the form 'Can you find a number..?' They were asked to specify numbers with certain properties or to state that no such number exists and to justify their answers as fully as possible. They were then invited to participate in a series of interviews using their answers to the questions as a starting point. Students were interviewed in pairs and asked to explain and justify their answers to each other. Differences between the approaches and understanding of the two students contributed to the direction taken by each interview. Interviews were recorded on audiotape for later transcription and analysis by the researchers and the students themselves.

Two students, Carol and MaryAnn (not their real names), had each participated in a series of pair interviews focused on their rational numbers concepts with a different partner during their mathematics pedagogy course. These interviews were discussed in Clark and Lukas (1998). Because Carol and MaryAnn had found the interview process very helpful in deepening their understanding of mathematics and the teaching of mathematics, they asked if it would be possible to continue during the summer.

At the beginning of the second sequence of interviews, each student was asked to listen to the tapes of her earlier interviews, report on her observations, and suggest areas for further discussion. These observations and suggestions formed a context for the summer pair interviews. During the summer, Carol and MaryAnn participated in three 1 1/2 hour interviews together as well as an individual interview each. At the end of the summer, each student was again asked to reflect on the interviews and to provide observations.

Interviews with Carol and MaryAnn

Among the topics Carol and MaryAnn wished to discuss further were the meanings of arithmetic operations, primarily multiplication and division, when applied to fractions. We began with the multiplication of 2/3 by 1/2, considering Cuisenaire rods, paper strips, and the number line. Echoing the claim of Confrey (1994) and others, MaryAnn said that it is misleading to refer to multiplication as repeated addition, as this doesn't make sense when multiplying by a fraction.

We then moved on to division, asking them to explain the process of dividing 1 1/2 by 1/2. Carol did this quickly by referring to the number line and reasoning that the number of jumps of length 1/2 required to reach 1 1/2 is 3. MaryAnn used "invert and multiply" and also obtained the correct answer quickly but didn't trust it. When she tried to make sense of the problem, she expressed confusion, both between the notions of "dividing in half" and "dividing by a half" and by the use of division language for a problem in which the answer is larger than the original number.

> "Although it made sense to me because I knew just from doing it a million times, you know, when I was a kid that that was right. But my brain said 11/2 divided by 1/2, you know, how can you take 11/2 pieces of something and then you're trying to divide that in half and end up with 3. It just, you know for some reason that didn't make sense to me. So I got out the (Cuisenaire) rods and, let's see. It got a little complicated. But I, first I said to myself, okay, if you're going to divide 11/2 by 1/2 I asked how many units of 1/2 are there in 11/2".

MaryAnn seems to be wavering between the partitive and quotative senses of division here. The quotative or measurement sense ("how many units of 1/2.") enables her to answer the question correctly, but the partitive or sharing sense seems to contribute to her confusion between dividing "in half" and dividing "by a half". This confusion has also been shown in story problems for division by fractions composed by teacher candidates who were mathematics majors as reported by Ball (1990). When asked for a real situation represented by 1 3/4 ∏ 1/2, some of Ball's students gave correct answers using quotative division, while others used splitting language and, like MaryAnn, constructed problems which called for dividing in half rather than dividing by 1/2. We believe that this confusion in both of these cases is due to interference from the partitive model.

In support of the notion of primitive models (Fischbein et al. 1985), the data of Tirosh and Graeber (1989) relate the prevalent student misconception that "division makes smaller" to reliance on the partitive model of division. This relationship was also expressed in Carol and MaryAnn's work on division by 1/2.

> MaryAnn: "I think the word "divide" itself just is a confusing word... I always thought that division was making something smaller. Taking, you know, a unit and cutting it into a smaller piece."

> Carol: "And division is just separate into parts and to make small, you get a smaller number."

We asked them to discuss the meaning of division for integers; what is happening when 6 is divided by 2? Possibly because of influence from the fraction problem she had been working on earlier, MaryAnn responded only in terms of quotative division - that is, how many groups of 2 give a total of 6? Carol interpreted the problem in both quotative and partitive terms of division. Although MaryAnn had clearly understood and used the partitive model of division earlier, now when Carol offered the interpretation "If a total of 6 is divided into 2 equal groups, how many will be in each group?", she expressed surprise and asked Carol to repeat it several times.

While the commutativity of multiplication guarantees that either approach gives the same numerical answer, they represent different processes and do not translate with equal facility to problems in which the divisor is a fraction. Although it is possible to express division by a fraction in partitive terms (e.g., If a half of a collection is 1 1/2, what is the size of a full collection?), such an interpretation feels contrived. Thus Carol could easily state that 3 jumps of length 1/2 result in a total of 1 1/2, but had difficulty even stating the problem in partitive terms:

> "Okay, so I have 1 1/2, a group of 1 1/2 here. And I want to separate it into 3 equal parts. No, I don't. I want to separate it into half equal parts. ... Well, I'm going to look at it and see if I can cut that into 3 equal groups. Oh, I can't say 3 because that's the answer. All I know is a half. .... I couldn't even say it".

The two senses of division were made explicit through consideration of the different approaches to division by an integer used by Carol and MaryAnn. Once the two senses of division (partitive and quotative) had become explicit, both students were able to construct story problems reflecting either sense using integer divisors. Carol talked about dividing a 6-foot log into 2-foot sections or 6 people into 2 equal groups, while MaryAnn constructed a situation in which her son had 6 video games and either shared them equally with a friend or divided them into packages of 2 each. For division problems with rational divisors, both students constructed story problems using the quotative sense. Carol's problem stayed close to the number line model ("If I walk 1 1/2 blocks and pass a store at each 1/2 block, how many stores will I pass?"), while MaryAnn said "Cindy has 1 1/2 buckets of grain. How many horses can she feed if each horse eats 1/2 bucket?"

When asked for a story problem represented by 1 1/2 times 1/2, the students chose opposite roles for 1/2. In Carol's problem, 1/2 was the multiplicand ("How many miles do I walk if I walk 1 1/2 blocks and each block is 1/2 mile long?"), while in MaryAnn's, it was the multiplier ("Jimmy does his paper route in 1 1/2 hours on Sunday. On the other days he takes 1/2 as long. How long does he take on the other days?")

Conclusions

Our conversations with students provide evidence for the powerful nature of explicit reasoning in making connections between intuitive concepts and relationships within the symbol system. Alfred Manaster (1998) pointed out in his analysis of the TIMSS data that the United States eighth grade mathematics classes in the study exhibited an almost complete lack of explicit reasoning. Simon (1993) reports that "students were unable to think flexibly and consciously about division as partitive or quotative" (p. 247). Explicit recognition of the partitive and quotative meanings of division and the relationships between them enabled our students to develop a deeper and fuller understanding of numbers and operations. To make further progress with these concepts, students must be helped to develop intuitions about relationships within number systems apart from early concrete intuitions.

Mathematical conversations between pairs of students focused by instructors are an effective means to investigate and enhance mathematical understanding. This is a gradual process and gains made are often fragile. Our students were grappling with difficult questions and exhibited inconsistencies and confusions in their thinking. Nevertheless, each felt that she had benefited greatly from the process. Carol said that she had drawn new strength from the interviews, while MaryAnn felt that learning, both her own and that of children, is made possible by the use of concrete models and progression to abstract concepts through conversation.

Each student felt that it was very beneficial to be exposed to the other's approach to problems, both to enhance her own understanding and to prepare for the variety of approaches she expects to find among her own students in the elementary classroom. The conversations also serve to deepen our own understanding of the learning process. We believe that interviews such as those described here can serve as a vehicle for improvement of teaching in both elementary and college classrooms.

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DENSITY IDEAS IN PRIMARY SCHOOL

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From 1993 to 1996 a research project related to ratio and proportion was carried out in Mexico. A comparison study and further investigation has been implemented in Valencia, Spain since 1996. The purpose of this communication is to describe a scheme built up to analyze and classify primary school children's problem-solving strategies for tasks linked with the aforesaid concepts and to describe results obtained from qualitative analyses of the answers given by eighty students (from third to sixth grades) on four tasks concerning density ideas. Other researchers such as Cramer & Post (1993) and Fiol & Fortuny (1986, 1990) have included problems related to this topic in their studies.

The four tasks were designed taking into account two problems used in the Mexican study. Each of the activities were comprised of a pair of expositions and the relationship between them (Freudenthal, 1983); they are density comparison problems with data expressed in a pictorical form. Questions correspond to surface variation, which is represented as a discrete quantity and is related to a discrete quantity variation-area versus number of people.

The classification scheme defined in Mexico was restructured taking into account the results obtained in Spanish studies. Four categories were characterized: 1) relational perception of information elements; 2) isolate perception of information elements; 3) particular decodification of word problems; and 4 singular responses. Subcategories and classes were needed to include diverse modes of acting. The first two categories are relevant for identifying children's behavioral tendencies linked with relational thinking. The results obtained show evidence for considering the inclusion of density tasks in the primary school curriculum.

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Social and Cultural Factors

PLACING LINKED REPRESENTATIONS IN SOCIAL CONTEXT

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Over the last several years, there have been differing research findings pertaining to the efficacy of linked representation systems in supporting students' learning of mathematical relations. On the one hand, several authors (e.g., Yerushalmy & Schwartz, 1993) have suggested that technology with multiple linked representations showed promise in affecting students' understandings of mathematical symbols as they relate to concepts On the other hand, other researchers have raised questions as to whether students using such technology do develop deeper understandings of the links between representations (cf., Thompson, 1992). We address this disparity in findings by looking beyond the structure of the software and individual student learning to explore social considerations such as students' interpretations of the prevailing norms for argumentation. In this report, we describe the social concerns that may account for the differential success that we encountered in one classroom teaching experiment when implementing a microworld that featured linked representations.

There is a large variety of dynamically-linked software being developed by researchers to help students link meaning to graphical representations. Some of the many examples include SimCalc (Kaput & Roschelle, 1997), Function Supposer (Yerushalmy & Schwartz, 1993), Blocks Microworld (Thompson, 1992), and MacCandy Factory (Bowers, 1997), just to name a few. While some developers (e.g., Yerushalmy & Schwartz, 1993) have reported encouraging findings, others, such as Thompson (1992) and Bowers (1997) indicate that linked representations may not be a panacea for bringing meaning to mathematical symbolization. In this paper, we explore this issue from a socioconstructivist view of learning. Our goal is to illustrate the crucial role that social context plays when analyzing student learning with linked representations. In particular, we compare three computer-based activities that involved SimCalc, an exploratory microworld that contains linked position and velocity graphs which are both linked to a simulation world. The question we want to explore is: What aspects of the classroom microculture can support students' learning with linked representations?

Theoretical Framework

One of the reasons for assuming a socioconstructivist perspective is to challenge the purely cognitive assumption that students form internal representations that mirror external structures (cf. Cobb, Yackel, & Wood, 1992). We aim to challenge the notion that meaning can be built into the external structure of computer programs in which students are expected to "see" the one-to-one relationship between a particular mathematical relation and its accompanying algebraic symbolization.

A socioconstructivist alternative to this perspective is a view of tools as situative components of the social milieu. That is, symbols and tools are created in activity and meanings are constantly being reformulated as students reorganize their activities and goals. For example, Miera (1995) discusses the "interactive character of mathematical sense-making as materials and displays shape each other during activity" (p. 298). This relationship may be pictured as a cycle in which the students' initial mathematical goals influence the symbols and meanings they create.

In this research, we explore the possibility of extending this perspective to look at the role of norms for argumentation when working with linked representation systems in action. Our objective is to examine students' goal-directed activity with linked representation systems in social context, rather than looking solely at the software's potential from the developer's perspective or any one student's progress in isolation.

Methods and Data Source

The methodology of our research follows the design of a teaching experiment. Working within this qualitative paradigm, the research group collaborates closely with the teacher to define a learning trajectory. During each lesson, the teacher is fully in charge of the class flow and all in-class instructional decisions. The members of the research group are observers with whom the students feel comfortable talking during small-group work sessions. After each day's lesson, the teacher and the research team meet to triangulate data, discuss the outcome of the lesson, and consider how to revise the next day's lesson based on these observations. This methodology offers a unique opportunity to conduct scientific inquiry in that we are able to conjecture, refine, and re-test hypotheses as systematically as possible while still remaining situated in the "messy" real-world context of a classroom.

The teaching experiment was implemented over a three-week period in a seventh-grade classroom in an urban school in Southern California. The class contained 14 girls and 16 boys, all of whom were between the ages of 12 and 14. It is significant to note that approximately 65% of the class was classified as ESL (English as a Second Language) and therefore some had difficulties reading long scenarios and describing their mathematical reasoning in clear English. The main goal of the project was to design and research an activity sequence intended to support the students' emerging views of rate and early algebra concepts. Although they had not formally encountered line graphs, algebraic notation, or the concept of speed as a rate, we had hoped to build on their real-world experiences of speed to introduce conventional representations such as Cartesian graphs of position and velocity through the use of linked representation systems.

One of the tools we used to address this goal was the SimCalc program, a dynamic microworld for exploring one-dimensional motion in which any combination of three graphs (position vs. time, velocity vs. time, and acceleration vs. time) can all be linked to an animated simulation and to each other (Kaput & Roschelle, 1997). A student using the program can create or modify a position (or velocity) graph and view the character's corresponding movement. Because these representations are linked, changes in any one graph are reflected in the other graphs and in the movement of the character.

Results

This report focuses on differences among the ways in which three (of approximately 15) computer-based activities that involved the use of linked representations were interpreted by the students. The first activity involved four steps. In step 1, students were asked to read a story about the motion of two characters. In step 2, the students were asked to make a sketch of the position graph (using pencil and paper) to illustrate this scenario. In the third step, we intended that the students would re-create their "predicted" graphs on the computer. Then, in the final step, the students would run the simulation in order to check their predicted graphs against the original story. Our original design intent was that students would run the simulation and revise their sketches as needed. The underlying belief was that this activity would involve students' efforts to resolve any perturbation they encountered between what they expected and what they saw, and, in so-doing, make sense of the link between the position graph and animation representations. Our observations revealed two unanticipated interpretations. First, some students did not see a link between the computer-based graph and the simulation. In the second case, they saw the link between the computer-based graph and the simulation (i.e., the graph could control the character's movement), but they did not exploit this as a way to make sense of the graph as a representation of the story. That is, they used a "guess and check" process to accomplish the goal, i.e., to get the simulation to match the scenario but they did not reflection on their activities to construct mathematical meanings. In both cases, the social implication is that, for them, the goal of this activity was simply to get the correct answer in order to finish the "school task." Participation in the activity at this early stage did not involve developing explanations or arguments, it merely involved completing a worksheet. Moreover, the graphs the students were creating were not seen as means to an end; by virtue of being the "answer," they were ends in and of themselves.

To address these social concerns, we modified the way we had been using SimCalc so that the position graphs controlling the characters' motions were no longer visible. We also worked with the teacher to create ways in which she could support the evolution of sociomathematical norms for argumentation that focused on mathematical justifications rather than arguments based on what was seen in the simulation. The new activity involved again making predictions about the shape of a position graph, but the students were no longer able to simply rely on the linked representation system to find the answer. Students were asked to run a simulation and then choose one of four graphs from a multiple choice list printed on a worksheet that best depicted the simulation activity. Our observations revealed that as the group members discussed different possible graphs, they often developed a variety profound mathematical arguments to support their particular preferences. We further observed that the norms for argumentation that had emerged during whole-class discussions were now emerging during small-group discussions. For example, students began to reason mathematically by making mathematical justifications such as "It can't be that graph, because they didn't start at the same time." Our hypothesis for this shift is that the locus of authority moved from the computer to the students' own discussion group and the sociomathematical norms for argumentation shifted from checking with a computer to developing logical arguments based on known parameters.

One particular example from a group of three girls working together to solve a somewhat novel question illustrates the types of rich interactions that occurred. When watching a simulation containing four clowns that walk at the same speed for the same amount of time but start at different places, two of the students initially chose a graph from the multiple choice worksheet that illustrated the "graph as path" misconception. After a third student argued for the correct graph based on mathematical justifications such as the ending positions, time traveled, and starting positions, the two girls agreed that her nominated graph actually did depict the scenario more adequately. Moreover, they were able to reflect on why they had originally chosen the distracter graph. We claim that this was significant because the locus of authority remained within the group; they could not simply run the program to validate their hypotheses with a "guess and check" approach.

At this point, the reader may be thinking that the final message from this project is that linked representations may hinder group discussions. However, this is not the message we wish to convey. A final activity involved having students work in groups to plan, create, and present computer-based animations of their own design. The students created a variety of stories including cars racing, clowns meeting to tell secrets, aliens hovering and snatching baby ducks, and sports cars breaking out of traffic jams. Two critical elements of this assignment were: 1) students were required to write their stories before going to the lab, and 2) stories needed to include more than one character that changed rate and direction. The purpose of asking students to plan their activity prior to going to the computer lab was so that they would have a goal in mind, rather than just playing around with the graphs and then claiming that the end product was their original goal. As the students presented their simulations to the class, it became evident that their final projects were quite similar to their original plans, which indicated a strong effort on their part to use graphs to tell a particular story. More importantly, the locus of authority remained rooted within the group and they came to see the various graphs as means for achieving their ends rather than ends in and of themselves. Their decisions regarding how to achieve their stated goals were based on the norms for argumentation that had emerged in prior class discussions and small-group activities rather than their earlier "guess and check" activity.

In watching the students give their final presentations, it appeared that different groups came to view the value of the various tools slightly differently. For example, one spokesperson stated that her group used the velocity graph when they wanted to slow a character down. In contrast, another group reported that they made the position graph "less steep" to accomplish the same goal. Observations such as this serve as data for documenting the ways in which the students' mathematical understandings co-emerge with their goal-directed activities and situated discussions. We claim that working with the dynamically linked graphs and valuing graph-based mathematical justifications were necessary but not sufficient aspects of the activities for helping students see the connections intended. In fact, we found that even though the students did act with linked velocity and position graphs, they did *not* see any connections between velocity, as a measure of speed (as shown in the velocity graph) and distance traveled per unit time (as shown in a position graph).

Discussion and Conclusion

We briefly discuss two conclusions. First, we claim that the students who had developed an understanding of the links between the representations did so not from "playing" with the software or using "guess and check" strategies. Instead, once these meanings were formed through their efforts to create mathematically justifiable explanations, the meanings between the computer-based representation systems became self-evident. Second, as mathematics educators, we see connections between the concepts of slope, speed, and the value of m in the equation $y=mx+b$. In traditional parlance, these have been presented with graphic, tabular, and algebraic representations, respectively. Our results indicate that although the students developed increasingly sophisticated understandings of each of these concepts independently, most had yet to create connections between these various "representations" because such connections were not necessary to accomplish the goals of their activities.

As with many facets of educational research, it is difficult to judge the efficacy of linked representation systems without considering the social environment in which they are implemented and the students' goals that arise as they use the software. In other words, although it is critical to begin the design process by imagining what students might do with any piece of software, it is also critical to conduct situated research to investigate the ways in which the students interpreted the designer's intentions in the social context of different classrooms. Our research indicates that the collective orientation (Bowers & Nickerson, 1998) at the beginning of the teaching experiment could be characterized by a "guess and check" philosophy where the locus of proof resided with the computer, and the goal was simply to get the correct answer. At this point, graphs were viewed as ends, in and of themselves. However, as the class developed more robust ways of reasoning and arguing, the graphs became viewed as tools for argumentation and goal meeting. At the conclusion of the 3-week sequence, links between each of the graphs and the phenomena were seen as means for accomplishing goals and making argument. These findings indicate support the claim that the students engaged in mathematical explorations where mathematical understanding of concepts and notation co-emerged within the social context of the classroom. However, the links between the representations themselves (e.g., the way that velocity relates to the slope of a position graph, the fact that the coefficient of the m in $y=mx+b$ relates to the velocity and the slope, etc.) were not emphasized in either their activities or the whole class justifications. The result is that, although the students were *using* the links between the graphs and the phenomena, the links between the representations were not self-evident and did not support students who had yet to create these mathematical relations. This finding calls for a second iteration of our design-research developmental cycle.

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THE LITTLE TEACHER: A STUDENT ROLE IN MATHEMATICS SMALL GROUPS

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Teachers often assign students roles to assume while working in small groups. We are interested, however, in roles that students create for themselves. In particular, we document a role we call the *little teacher*. Students who assume this role take on many typical teacher behaviors including control of communication, direction of other students' work through task assignment, validation of other students' work, and being the authority for answers. We illustrate these behaviors with examples drawn from prior work. The *little teacher* role is difficult to spot in real time because of the group's desire to portray themselves as working well together. The net effect is to recreate a traditional classroom without the benefit of an experienced, knowledgeable teacher assuming the controlling role.

Teachers often assign students roles such as recorder or calculator operator to help facilitate small group work. But what of the roles that students choose for themselves?

Paul: OK, Ronnie, how do you get A?

Ronnie: Fill in another point?

Paul: Yeah, that's right. (Ivey, 1997)

In this brief excerpt, Paul initiates the interaction with a question directed at Ronnie, who responds, and then Paul evaluates his answer. This pattern of initiation, response, and evaluation is typically found in teacher-student interactions (Mehan, 1979), but Paul and Ronnie are both students. The following discussion seeks to document a specific role that some students take in small group interaction. We call this role the *little teacher*. Through comparison of examples, we reveal some commonalties of the behaviors that characterize this role. We also briefly examine the establishment and maintenance of this role. By identifying defining characteristics of this role, we reveal an important component of the structure of some group interactions.

Theoretical Underpinnings

Analysis of data presented here proceeds from a framework that assumes students bring many beliefs and attitudes into the classroom. (For a more complete exposition of this framework, see Williams, 1993.) This framework assumes that students' actions reflect the Heideggerian notion of being-in-the-world. They are thrown into a situation, and they act. How they judge a situation and act within that situation is often at a non-reflective level. As long as their assumptions do not come into conflict with something within the situation, they do not examine the basis of their actions nor the results. Precisely because these assumptions are not reflected upon, their existence has profound impact on the actions of individuals. This philosophical stance is further augmented by a consideration of culture. Following Weissglass (1992) we define culture as: "... the attitudes, beliefs, values and practices shared by a community of people which they often do not question, are often unstated and which they may not be consciously aware of" (p. 196). We will use this definition of culture and the idea of being-in-the-world to understand how and why the *little teacher* role developed in some small groups that we have studied.

Modes of Inquiry and Data Sources

Instances of students assuming the *little teacher* role are seen in several studies conducted by the authors. All of the studies used a variety of qualitative methodologies including participant observation, videotape analysis, audiotape analysis, student interviews, student questionnaires, examples of student work, student journals, teacher interviews, and classroom artifacts. The studies range from multiyear, multisite projects in middle grades (Ivey, 1996; Ivey, 1994; Ivey & Williams, 1993; Williams & Ivey, 1995) and high school mathematics classes (Walen, 1996; Walen, 1994; Walen 1993), to semester long studies in college classrooms (Ivey, 1997). These studies were conducted in diverse sites including rural public schools and regional comprehensive universities. For the purposes of this report, one specific example will be discussed in detail with reference to other instances not completely analyzed here. A longer paper with additional examples and analysis is available.

Defining the Role of Little Teacher

The *little teacher* role is not uncommon. It is characterized by behaviors that are commonly associated with teachers, such as controlling communication in the classroom by asking leading questions that require short, even one-word, answers from students. Other traditional teacher behaviors exhibited by *little teachers* include directing other students' work through task assignment, validating others' work, talking the most, asking questions to which they already know the answers, and serving as the authority for answers.

To illustrate some of these characteristics, we introduce Nancy, an eighth-grade beginning algebra student. In our talks with Nancy (Ivey, 1994), she described her role as a student in group work as follows: "In group work, you can kind of be a teacher to other people." What she means by teacher may be inferred from her actions in small groups. Consider the following episode in a group of four students—Nancy, Olivia, Edward, and Mel—working on the ideas of multiples and factors in beginning algebra (Williams & Ivey, 1994). The question that has been posed by the classroom teacher is followed by an excerpt from the group's discussion.

1. Write three statements using the given numbers or expressions, each using one of the following terms: factor, divisible, multiple.

   a. 5 25 b. 27 3 c. 7x 7

Nancy: Twenty-five is a multiple of five.

Olivia: What?

Edward: I don't understand this at all. Do you know what he is talking about?

Nancy: OK, I'll explain this. OK. You get five and twenty-five. How could you say that in factors?

Edward: Factors?

Nancy: What is a factor? Is five a factor of twenty-five or is twenty-five a factor of five?

Edward: Twenty-five is a factor of five.

Nancy: No, five is a factor of twenty-five.

Edward: Nuh uh. Twenty-five is a factor of five because five times five is twenty-five.

An interruption of 1 minute occurs while the teacher checks on the group's progress.

Nancy: OK. So. Five is a factor of twenty-five. What are the factors that are making up twenty-five? What makes up twenty-five?

Edward: OK. Like, five times five.

Nancy: Right. So five is a factor of twenty-five.

Edward: OK.

Nancy: So for number A, put five is a factor of twenty-five.

Note that Nancy exhibits several typical teacher actions in this interchange. First, she talks the most. Even though there is continual trading of speaking roles, Nancy says 84 words to Edward's 41 words and Olivia's one word. Mel doesn't speak at all. So approximately two-thirds of the words spoken are Nancy's. Furthermore, some of Edward's responses are repetitions of words or phrases first uttered by Nancy. Second, notice the direct nature of Nancy's questions. At one point, she asks the general, conceptual question: "What is a factor?" and immediately follows it with a more direct question that is a choice between two responses. Edward ignores the general question, and answers her specific query. Third, the nature of her questions are answer oriented, and when she gets the desired answer from Edward, she validates it by directing him to write down the response. This direction is repeated several times as the students continue to work through the assigned problems. When Nancy is satisfied with the answer to a problem, she directs Edward to "write that down," even sometimes telling him where to write it on his paper. As the teacher comes by, Edward volunteers the information that "They taught me how to do this." The teacher responds with praise for their group's work. This particular group's interaction continued over several days. By the third day, Nancy is directing Edward to "pay attention." She even uses a classic teacher question, "Edward, what did I just say? Please pay attention and work." At another point, she says "Does everyone understand that? You should be finished with it, because I just said it [the answer]." Through Nancy's assumption of some typical teacher practices, she creates a role for herself in the group. The other students' reactions to her help to maintain this role. The teacher also helps to maintain Nancy's role as a *little teacher* through his praise of the way the group is working together. This example illustrates some aspects of the role of *little teacher*, and it also indicates ways in which this role may develop and be maintained.

Discussion and Implications

Students come to class with strongly held beliefs about how school is supposed to be conducted. When teachers attempt to create classrooms with communication as a basis for developing understanding, some students have great difficulty finding appropriate roles (Ivey, 1994). It is possible for a teacher to prevent one student from dominating the discussion in whole class settings, but when students are placed into groups, the opportunity for a student to assume the *little teacher* role arises. The classroom teacher is not assuming the expected role, which creates a vacuum in some students' social scheme of school. The *little teacher* role is not taken because of some unreasonable desire for control. As McLaren (1991) points out, "...both teachers and students often come to believe and accept that the rules, regulations, systems of moral scruples, and social practices that undergird and inform everyday life in schools are necessary if learning is to be successfully accomplished" (p. 237). In this light, the development of the *little teacher* is not only reasonable, it is almost required. If the teacher is not going to fulfill the role that is believed necessary for learning to be accomplished, someone else must. In general, students react in ways appropriate to their knowledge of being-in-the classroom, so up steps the *little teacher*. The very fact that the underlying beliefs about the role of teachers are a part of the culture of the classroom makes them likely to be unstated and unquestioned and helps to make the *little teacher* role in small groups seem natural. The other students' responses to the *little teacher* also indicate that the role is natural and expected. The fact that students will accept this behavior from another student implies the expectedness of this role. Unfortunately, the *little teacher* does not necessarily share the goals and expectations of the classroom teacher.

Little teacher behavior can appear, on the surface, as helping behavior or scaffolding in a Vygotskian sense, but as we see in the example of Nancy and Edward, that is not necessarily the case. Nancy was not truly helping Edward to a higher level of understanding, as the teacher believed from his limited observation. In fact, she actually moved Edward from a tenuous understanding of the mathematics involved to no understanding, and worse, to no interest in understanding. (See Williams & Ivey, 1994 for a more complete discussion of this aspect of the episode given above.)

This episode also shows one difficulty of spotting a *little teacher* in normal classroom observations by teachers. The teacher told the students that he expected them to work together and to help each other, or their grades would suffer. This direction from the teacher is a common part of instructions for group work. In response, students repeatedly drew the teacher's attention to how well they were working together. In short, the students conspired to convince the teacher that they were "working well as a group." This ritualization of working together has been noted in other instances (Williams & Baxter, 1996). Students try to meet the expectations of teachers, but sometimes their methods for doing so have unexpected consequences. Ritualization of responses can lead a teacher to believe that a group is working as intended, when actually the group is functioning like a traditional classroom without the benefit of the teacher's expertise.

The role of *little teacher* is also problematic for the student who assumes the job. The *little teacher* ostensibly behaves like a teacher but does not have the knowledge or experience of a teacher upon which to act. Often, a *little teacher* must appear to know something that she is trying to learn. This lack of knowledge causes the *little teacher* great concern, especially when she realizes that she has "taught someone something wrong." *Little teachers* often assume the responsibility, as do many classroom teachers, for the lack of learning on the part of *her* students.

This paper helps to document one way in which efforts to encourage student communication in small groups can be undermined by student beliefs, expectations, and roles. The difficulty of identifying *little teachers* in classrooms in real time makes a detailed description of this role an important step to a deeper understanding of mathematics classrooms and the types of discourse that occur in them.

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VOICES OF WOMEN MATHEMATICIANS: GENDER AND THE ROLE IT PLAYS IN THEIR LIVES AS MATHEMATICIANS

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The findings discussed in this paper are part of a larger study that focused on examining the life stories of six women mathematicians and the reasons for their success in the mathematics. Guided by a feminist epistemological perspective, I describe and interpret the role that gender plays in the lives of these women as mathematicians. Findings suggest that gender played a variety roles in the women's lives as mathematicians. Of particular significance was the way the women *adjusted* to being a woman in a field primarily composed of men.

Teresa: The main point I want to get across is that being a woman in mathematics, means to me, not being a women in mathematics, you know what I mean, sort of being like everybody else and realizing that everyone is a person first and that's the only way I can survive.

Teresa's voice reflects how she frames the issue of gender in mathematics and the role it plays in her life as a mathematician. Teresa finds comfort in attempting to "just to fit in, and not really try to be very much of a woman." For her, comfort resides in making adjustments to fit in. As a feminist researcher in mathematics education, I was interested in exploring the issues facing women as mathematicians, particularly how they saw themselves in what is generally seen as a male-dominated career. Based on the narratives of six female mathematicians, this paper describes the role that gender plays in their lives as mathematicians and attempts to interpret their voices in light of the literature.

Theoretical Perspective

Fennema and Hart (1994) challenged the scholarly community to consider research in mathematics education that employs a feminist perspective. Heeding their appeal, my work is situated in and informed by feminist standpoint theory (Harstock, 1983). A major principle of feminist standpoint epistemology is its reliance on knowledge that is created by and situated within the viewpoint of women. Women construct their own knowledge and do so differently than men (Harding, 1991). This standpoint hinges on the lives, knowledge, experiences, and voices of women. Harding (1991) declared that "we must insist on an objective location—women's lives—as the place from which feminist research should begin" (p. 123).

This epistemology assumes women perceive their worlds from a subjugated position that arises from the patriarchal world in which we live. As a result of living in a male-dominated world, women are in a better place to examine the inequities of their position and evaluate those occupying the positions of power and dominance (Harding, 1991).

To fully engage in the lives of particular women, women's voices are paramount, not only in understanding their experiences, but also in achieving a feminist perspective. "Voice refers to the discourse that is created when people [women] define their own issues in their own ways, from their own perspectives, using their own terms—in a word they speak for themselves" (Secada, 1995, p. 156). By listening to women's voices we have an opportunity to glean insight into their world and to provide a forum for women to voice their thoughts.

Review of Related Literature

Using a feminist perspective on science, Maple (1994) explored the background, educational experiences, and career expectations of female doctoral students in mathematics and science. Maple found that support from parents and school officials was significant to her participants in helping them to pursue a degree in mathematics and science. The participants also expressed a sensitivity to an encouraging graduate school environment and mentioned that as they proceeded from undergraduate to graduate study, the school environment seemed to lack a support system.

Using the narratives of seven women who obtained a bachelor's degree in mathematics, Stage and Maple (1996) identified reasons why these women left the mathematics pipeline in order to pursue a doctorate in education. The reasons for leaving were negative experiences in their program, "the nature of the mathematician did not match their early perceptions," and "a perceived conflict between mathematics as a profession and other roles—such as, parent, community member, and significant other" (p. 38).

Methodology

Using a criterion-based selection approach (LeCompte & Preissle, 1993), I chose six women mathematicians to participate based on the following criteria. Participants had to (a) be female, and (b) have a Ph.D. in mathematics or were in the process of pursuing a doctorate in mathematics. The participants' status' and backgrounds are summarized in Table 1.

The study used a narrative inquiry approach (Rejssman, 1993) for data collection and analysis. Data included one semi-structured interview. In the interview, the women were asked to share their thoughts and feelings about various issues such as their personal background, self-identity, career,

Table 1

Status and Background of Participants

	Occupation	Age	Birthplace	Race	SES (childhood)	Occupation of mother	Occupation of father
Emmie	professor	40	Connecticut	Caucasian	Upper Middle	violinist	economics professor
Faith	professor**	28	South Carolina	Caucasian	Middle	homemaker, elementary teacher	professional
Iona	professor*	33	Iran	Caucasian	Middle	teacher	military
Teresa	doctoral student*	28	South Florida	Hispanic-American	Middle	writer, English professor	retail management, high school teacher
Marisa	professor*	late 30's	Croatia	Caucasian	Middle	elementary teacher	high school teacher
Sonia	doctoral student*	41	Kentucky	Caucasian	Middle	waitress, bartender	custom painter, locksmith, bartender

a powerful learning experience, and gender. To ensure credibility and trustworthiness of the data, I used member checks (Lincoln & Guba, 1985).

I began my data analysis by reading and rereading the transcribed interviews, making notes in the margin and searching for themes that the women perceived as significant in understanding themselves, their experiences, and their worlds, particularly how these related to gender. I grouped themes according to the role that gender played in the women's work as mathematicians and their concerns surrounding being a women in a field dominated by men. Throughout the analysis, I incorporated an emic viewpoint (Lancy, 1993, p. 30) which "represents the insider's perspective."

Results and Discussion

The various roles that gender played in the women's lives as mathematicians surfaced in their narratives. The women perceived gender as playing different roles that I have organized into *positive* (helpful), *neutral* (irrelevant), or *negative* (hindrance) roles. The women's perception of gender did not remain fixed; gender was perceived differently depending on the context or situation. Several of the women recalled instances where gender played a *positive* role, meaning gender facilitated their lives as mathematicians. Emmie, Teresa, and Sonia voiced their opinions that being a woman in the field of mathematics had actually helped them. Teresa and Sonia perceived the benefit in terms of affirmative action policies and opportunities for funding that were available to them as women for their advancement in mathematics. Emmie noted the peculiarity in her story as it related to being a woman in a male-dominated field:

To a certain extent I think it might have helped me because I think if I were in a department which was predominantly women, and if I asked to take a lot of time off to be with my kids, and to work part-time, I think there would be less receptivity towards that.

For Emmie, being in a department with few women allowed her to teach and also be with her children. This is an interesting contrast to findings of Stage and Maple (1996) who cited that several of their respondents left mathematics because of conflicting notions between their personal goals and the profession.

For Faith, Marisa, as well as Emmie, gender was perceived as a *neutral* influence, meaning gender held little, if any, precedence in their professional worlds. Faith perceived herself "as a mathematician that happens to be female" and viewed mathematics as "a gender blind subject." Marisa remarked, "I think it [being a woman] has become quite irrelevant to most people in the field." In contrast to Emmie's earlier comment, her thoughts paralleled Faith's. "I'm another person in math, I just happen to be a woman.... I'm not really super conscious of the gender issue." Based on her identity as a mathematician, Emmie's statement illustrates a shift (positive to neutral) in the way she thinks about the role gender plays in her work.

In contrast to Sonia's view of gender as helpful, she also affirmed the *negative* role that gender played, meaning gender was seen a hindrance or a road block in her work as a mathematician:

But the whole department seems to not want to recognize gender at all. They don't want to talk about women's issues because they are afraid that they have to recognize that there is gender.... They are very careful not to over advise their female students, be over supportive of the female students. They are afraid that would look like they are being sort of gender centric. What ends up happening is that the department seems to leave you out in the cold.

Sonia's statement documents the lack of support that Stage and Maple (1996) identified as cause for some women to leave mathematics. Iona also perceived gender as exhibiting a negative influence in her career as a mathematician:

I just get this feeling often that, just emotionally, it's harder because most of the things you do, there are so few women doing it.... People are less used to that so they don't make it easy for you, so it's like you have to make more of a push.

Iona's statement supports Stage and Maple's (1996) observation that several of their participants left the pipeline due to negative experiences that they endured in their graduate programs.

Apparent in the women's voices were descriptions of *adjustments* that *they made* in order to "fit in" a male-dominated field. Sonia offered the following view. "I ... never really thought about gender, or never really felt female until I got here [doctoral program] and then I felt like it was in my face all the time." When I asked if the situation had lessened in intensity, Sonia responded "I am not sure that it is any different now, but that I have adjusted to it." Marisa noticed that when she attended mathematics conferences, "there is[*sic*] very little women around." She commented that "I am so used to it that I probably would feel very strange if there were a room full of women." Marisa seems to have accepted the fact that there are fewer women in mathematics. Teresa's comment, "I think my voice [as a woman in mathematics] is that I don't try to be different," conveys her strategy for "survival" in the field. Teresa's statement provides insight on Stage and Maple's (1996) conjecture that women who remain in the pipeline are able to cope with the culture and the pressures of graduate mathematics study. Perhaps, for Teresa, Marisa, and Sonia, they have been able to persevere in a patriarchal playing field because they conformed to the existing social situation. Future research might seek to document the extent to which these adjustments have required women to compromise their goals, values, beliefs, or gender in order to fit in or survive.

Sonia's voice expresses the primary importance of being a mathematician to her identity and the less salient role of being a woman. In addition, Sonia's words capture the passion and persistence that are reflected in the narratives of the six women in this study:

I think of myself as a mathematician first because that is such a key part of who I am and I never thought of being female as being a key part of who I am. It just happens to be a biological fact. Whereas when I sat down and started doing those basic problems out of the college Algebra book, I had a really strong sense of "this is who I am." As I learned more and more mathematics, I kept getting a stronger and stronger sense that this is a big piece of me, and that I can't quit. I have to keep going as long as they will let me, as long as I can, as long as I can continue to learn, I have to keep going. I have a very strong sense of me being a mathematician. I can't stop.

Let's listen to Sonia's voice and examine the role gender should play, if any, in the lives of women mathematicians.

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CRITICALLY EXAMINING THE ROLE OF SOCIAL AND CULTURAL FACTORS IN AFRICAN AMERICAN STUDENTS SUCCEEDING IN MATHEMATICS

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This paper presents the results of a research study that critically examined the role of social and cultural factors in African American students succeeding in mathematics. The study investigated the mathematical experiences of two female college students. The study sought to determine these students' perceptions of how their ethnicity affected their mathematical experiences and, ultimately, their succeeding in mathematics.

As mathematics educators embrace the charge of reform documents (NCTM 1989, 1991, 1995) that accentuate "opportunity for all" and "mathematical literacy for all," it seems that a monumental challenge for mathematics educators is to question those schooling practices that undermine the charge and work to maintain the existing oppressive social structure. A fundamental responsibility for mathematics educators is to question the role that education and schooling play in perpetuating the inequalities and inequities that exist in the mathematics education of African American students.

This study depicts the stories of two African American mathematics students, Ashley and Sheilah, and denotes these students' perceptions of and responses to their experiences with school mathematics.

The objectives of this study were to: (a) identify the students' perceptions of how their ethnicity affected their mathematical experiences and (b) examine the role of social and cultural factors in their succeeding in mathematics.

Theoretical Framework

The ideas set forth by critical theorists are profound in the schooling and mathematical experiences of African American students. Social forces in schooling as described by critical theorists are potential barriers that African American students contend with in their schooling and mathematical experiences. Critical theory generally "rests on a critical view of the existing society, arguing that the society is both exploitative and oppressive, but also capable of being changed" (Weiler, 1988, p. 5). Critical theory is concerned with the role of schools in maintaining the existing social structure. Thus, critical theorists have asked "why, despite the meritocratic ideology of schooling, for certain groups—[African American] students, female students, and students of low social class—fundamental inequalities in school performance and societal position persist" (Reyes & Stanic, 1988, p. 27). Particularly, critical theorists bring into question social inequities and inequalities that exist in schooling based on race, class, and gender.

The critical theory lens was used in this study to frame and make sense of the participants' experiences of succeeding in mathematics. From the critical theory perspective, the participants' mathematical experiences were placed in a broader social context (Apple, 1992). The author was particularly concerned with the relationship between the participants' mathematical experiences and the existing oppressive social structure.

Methodology

This study employed a phenomenological research strategy. Phenomenological research describes subjective experiences of individuals (Tesch, 1987). It is aimed at interpretive understanding and describes individual experiences from the viewpoint of the individual (Tesch, 1987). Phenomenological research explores the personal construction of a person's world through in-depth, unstructured interviews and other data sources (Tesch, 1987). Data were collected in the form of initial surveys, autobiographies, and interviews (including a final interview that consisted of a member-check by the participants) to explore life histories of the participants in the context of their mathematical experiences.

Phenomenological research involves a back-and-forth movement between a phase of thinking and analyzing and a phase of data gathering, which is analogous to constant comparative analysis (Strauss, 1987). Surveys and autobiographies were used as data sources, were analyzed, and were then used as stimuli to gather more data during interviews. The first stage of the analysis occurred parallel with and informed subsequent data collection. During this first stage of analysis, particular themes were sought that were preeminent in the data sources.

The second stage of analysis received most attention at the end of data collection. The objectives of the study guided the search for invariant themes and patterns that emerged from the data. Particularly in this second stage, the author sought to explicate, interpret, and make sense of the invariant themes in terms of the critical theory perspective.

The selection of participants in this study was a criterion-based selection. "Criterion-based selection requires that the researcher establish in advance a set of criteria or a list of attributes that the units for study must possess" (LeCompte & Preissle, 1993, p. 69). The researcher "then searches for exemplars that match the specified array of characteristics" (LeCompte & Preissle, 1993, p. 69). The criteria for this study were college African American students who were near completion of a mathematics or mathematics education degree. Four (three females and one male) African American students were chosen to participate in the study. Two, Ashley and Sheilah, completed the study. Ashley was a junior in college pursuing a bachelor's degree in mathematics, and Sheilah was a graduate student in her final quarter of completing a master's degree in mathematics education.

Results—Listening to African American Students' Voices

Ashley's Voice

Considering the social constructs of Ashley's mathematical experiences, it is evident that Ashley believed she had to accept imposed ideologies of school culture to become successful with school mathematics. This notion is amplified in Ashley's description of wearing a mask that "symbolized my hidden feeling about being in a class where no one had my face, my thoughts, or my beliefs" (Autobiography, 7/11/96). From Ashley' perspective, her succeeding in mathematics and excelling in advanced mathematics courses compromised her identity as an African American. Specifically, Ashley *perceived* her thoughts and beliefs as being in conflict with that of school culture. For example, she said, "I think we, I think Black people look at a lot of things different" (Interview 1, 6/27/96). Ashley further stated that one had to think a certain way to succeed in school and "if you don't think of it a certain way, it is wrong" (Interview 1, 6/27/96). Ashley did not articulate what she meant by thinking a certain way, but she did articulate that "that way" was different from her own or from African Americans' ways of thinking. Further, she did articulate that she had to acquiesce to "that way of thinking" to become successful. This is evidenced by certain contentions she made such as "I am learning how you [Whites] think. . . but I know what I think" (Interview 1, 6/27/96) and "I guess I just did what I had to do to achieve my goals" (Interview 1, 6/27/96). Thus, these statements indicate that Ashley believed that she had to accept the conflicting thoughts and beliefs (i.e., ideologies) of school culture to become successful.

Moreover, Ashley believed that her being in classrooms where no one "looked like her" was the result of racial tracking. She asserted that the majority of African American students were put in low academic tracks because they were labeled as "underachievers, low. [School administrators believed] we didn't have any kind of education. We were all on welfare. . . . I just think that" (Interview 1, 6/27/96). From Ashley's perspective, African American students were in some sense not given a chance in schooling. She stated that most of her African American peers were in general mathematics courses while she was sometimes the only African American student in her advanced mathematics courses. Ashley did not know exactly how this took place, but she believed that racial tracking played a significant role. Consequently, Ashley was ridiculed by her African American peers (usually those in lower-level mathematics classes) and was accused of "acting White" and "selling out" when she continued in the advanced academic track and endeavored to succeed in mathematics.

Sheilah's Voice

By contrast, Sheilah believed that the ideologies of school culture were congruous with her own. She did not contend with being in classrooms where no one had her face until her graduate studies. Even then, Sheilah saw her being the only African American in higher-level mathematics courses as *not* her accepting ideologies characteristic of school culture or White culture for that matter. Rather, Sheilah perceived the situation as some type of "filtration" of African American students, a filtration that she could not explain. She believed that tracking was somehow the culprit. From Sheilah's perspective, she somehow survived the filtering process, whereas other African Americans did not. Moreover, for Sheilah, her survival was not contingent upon accepting thoughts or beliefs that were in conflict with her own or that were characteristic of White culture.

Sheilah stated that during her high school experience, she was surrounded by African American students who were in higher-level mathematics courses. She said that this was not the case during her student teaching experience which took place during the final quarter of her graduate studies. Sheilah asserted that she noticed that most of the African American students were in general mathematics or pre-algebra courses. She said that she did not realize until her student teaching experience that tracking, serving as a filter, played a significant role in African American students' mathematical experiences. Sheilah stated,

> I don't know what it is, but if you have one Black student out of 1500 students taking calculus, omething is wrong, and only five out of 1500, or maybe 10 out of 1500 taking trig, [in] two classes [combined]. Then the pre-algebra, just full of us [African Americans], you know, and the general math, just full of us, and maybe one or two Whites there [in general math or pre-algebra]. (Interview 2, 7/17/96)

Sheilah asserted that one reason she wants to teach mathematics is that she wants to help eradicate the problem of African American students being filtered into lower-level mathematics classes. She stated,

> That's probably the reason I want to go into mathematics education, you know, so it won't happen. . . . I mean somebody has got to look out for them [African American students], you know, somebody has, I mean, there are so many roles I want to play. I want to play the person who even if you [African American students] are not in my class, I can tell you, you can do this math, you are going to [do well] in that class, you know, you can do this and not let them just fall by the way-side. (Interview 1, 7/3/96)

Sheilah wanting to play these particular teaching roles parallels the underlying premise of critical theory that society is exploitative and oppressive but capable of being changed. Sheilah questioned certain aspects of schooling such as tracking that may be oppressive for African American students and that may contribute to African American students being filtered into lower-level mathematics classes. She believed that the educational practices she plans to use in her own teaching can play a significant role in countering this oppression and consequently help change the current status of African American students' mathematical experiences.

Concluding Remarks

Ashley's and Sheilah's voices suggest a need to evaluate tracking and its implications. In Ashley's high school experience, tracking served as a *separatist agent* in which Ashley was isolated from her African American peers who were in low tracks. Moreover, Ashley's peers associated being in high tracks with acting White, selling out, or giving up being African American. Perhaps the disproportionate number of African Americans in higher-level mathematics courses was a significant factor in shaping Ashley's peers' belief systems about who should be doing mathematics. If the norm is lower-level mathematics courses predominated by African American students, then it is conceivable, quite probable, that African American students would buy into the notion that mathematics is a discipline suited for Whites. This notion seems cyclic in nature and feeds on itself. That is, African American students' believing mathematics is suited for Whites is reinforced by lower-level mathematics courses mainly serving African Americans students, *and* lower-level mathematics courses mainly serving African American students is reinforced by African American students' believing mathematics is suited for Whites. Tracking seems to reinforce this notion rather than counteract it.

As Sheilah recognized during her student teaching experience, there is a need to question schooling practices that filter a vast majority of African American students in lower-level mathematics courses. Thus, it is essential to question the role of tracking or any schooling practice that works to undermine the notion that all students should have the opportunity to learn "important mathematics."

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MATHEMATICS TEACHING AND LEARNING: SOCIAL AND CULTURAL INFLUENCES

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This paper continues the case study of a Japanese teacher in American schools. Keiko's beliefs about mathematics teaching and learning were formed in Japanese culture and reinforced in that society. She believes that the emphasis in elementary mathematics is on developing mathematical thinking, and her teaching embodies the philosophy desired in mathematics reform. This paper discusses social and cultural factors related to her teaching mathematics in a midwestern private school.

As mathematics reform (NCTM, 1989, 1991, 1995, & 1998) is examined in American classrooms, it becomes apparent that cultural and societal factors influence the successfulness of students learning to think mathematically. The school mathematics curricula, teachers' attitudes and beliefs about mathematics, students' attitudes and achievements in mathematics, and the classroom environment (Reyes & Stanic, 1988) are embedded within cultural and societal norms. Relationships exist between mathematics beliefs (the nature of mathematics and mathematics pedagogy) and mathematics teaching practices (mathematical tasks, discourse, environment, and evaluation), and the teacher's past school experience strongly influences the consistency between these beliefs and practices (Raymond, 1997). Consequently, the examination of a teacher whose past school experiences were in Japan presents a cross-cultural perspective shedding light on the cultural and societal influences on mathematics teaching and learning. This case study continues an examination of Keiko's experiences in midwestern schools (Schmidt & Duncan, 1998) and provides a contrast for other research that examines social and cultural influences on mathematics teaching. This work emerges from research concerning social and cultural factors in teaching and learning, teacher and student beliefs, and school change.

Although Brown & Borko (1992) found that beginning elementary teachers who entered the teaching profession with nontraditional beliefs about how they should teach tended to implement more traditional classroom practices after they were faced with the constraints of actual teaching in the American society, this was not the case in Keiko's classroom. Studies concerning teachers' beliefs about mathematics and mathematics pedagogy found that teachers' beliefs are not always consistent with their teaching practices (Kaplan's study; Peterson, Femmema, Carpenter, & Loef's study, as cited in Raymond, 1997), and that these inconsistencies are deeply rooted in the differences between the cultural and societal expectations in education and mathematics. However, Keiko's beliefs about mathematics and mathematics pedagogy are deeply rooted in Japanese societal and cultural values which are consistent with mathematics reform expectations. Her preservice program in mathematics education was consistent with Japanese school-based beliefs, however, some education courses such as the general pedagogy course and field placement experiences were so inconsistent with her beliefs that she felt that she would have to give up her identity to teach in a midwestern public school (Schmidt & Duncan, 1999). Sugiyama (1993) noted that the higher achievement of Japanese students depends chiefly on the social conditions and the general educational environment in Japan and this, rather than excellent mathematics education, accounts for the higher achievement of Japanese students on international studies.

Method

The primary data source, Keiko, is a thirty-seven year old first year teacher who has lived in the midwest for thirteen years, speaks English fluently, and has an American undergraduate and graduate education. Other data sources include principal, students, and parents. Data collection included: e-mail messages; field notes taken at formal and informal meetings; student related artifacts such as portfolios, homework, and projects; Keiko's daily written reflections and summary reflection of the school year; school program evaluators' field notes; and interview notes collected throughout a nine month period. Pseudonyms are used in the paper. The sources for data interpretation were triangulated.

Background

Keiko's philosophy of education was developed in Japanese culture and reinforced by Japanese societal values. She learned mathematics thinking and reasoning in Japanese schools, and although similar goals are espoused in U.S. mathematics education reform (NCTM, 1989, 1991, 1995, & 1998) and cognitive learning theory, they were not valued in the traditional midwestern school communities she taught in during field placements. Keiko accepted a teaching position at a private school that seemed to have a progressive philosophy. She immediately developed a rapport with the principal who was hired at the same time and who valued and supported her ideas about teaching and learning. However, traditional practices dominated the school environment. The principal's past experiences included work at the Principal Center at Harvard University, and his leadership style was foreign to most of the school community. He described the school environment as a controlled environment where teachers were not open to new ideas or ways of teaching and learning. He believed that the lack of professional development kept them from accepting new ideas and resulted in a focus on testing and competition. For example, the fourth grade classroom was described as "controlled system" where there was "no recognition of individual strengths or weaknesses."

Keiko taught mathematics to both her fifth grade homeroom students and to the fourth grade students (while the fourth grade teacher taught language arts to both groups). The fourth grade class (n=18) was made up of students whose families were originally from the regional midwestern area, while more than half of the fifth grade class (n=14) was made up of students who were born in other countries or who were originally from different areas in the US. In both classes some parents were professionals, but those who were recognized as leaders in the local community were originally from the area. These local families tended to hold on to traditional beliefs about education that characterized both the public and private regional schools. On the other hand, the parents from diverse backgrounds tended to accept and support Keiko's teaching practices, seemed to be aware of different ways to learn about mathematics, and were pleased that their children were challenged to think and reason about mathematics.

Keiko's students began the school year "doing mathematics" because she wanted them to develop different expectations about what mathematics was in her classes. In previous years, students developed the notion that mathematics was independently completing textbook exercises and worksheets. She was familiar with the state curriculum goals, and she planned authentic mathematics tasks that were connected to appropriate grade level goals and objectives. As an illustration of Keiko's mathematics teaching, teaching episodes focused on understanding circles are presented in Appendix A and will be presented at the session.

Discussion

Teacher planning skills seemed to come naturally to Keiko. However, she was not aware that her skills were highly developed and seemed to think that all teachers should be able to approach mathematics in the same way she did. After reviewing our observations at a Japanese Saturday school mathematics classes, analyzing Japanese student and teacher textbooks, discussing our personal mathematics histories in school, and discussing literature we had read about Japanese culture, she realized that her personal experience learning mathematics in Japan was *very* different from the typical American's experience. When planning, she naturally repeats the mathematics learning cycle she experienced in Japan by teaching the way she was taught. The students: 1) are presented with a challenging, non-routine problems in a familiar context; 2) are expected to work together to find solutions for the problems; 3) discuss multiple solution strategies; 4) generalize, and 5) apply their mathematics knowledge by solving additional problems (Jones, 1998).

In the Japanese culture, it is thought that effort rather than innate ability determines success (Tsang, 1983). Japanese teachers put forth considerable effort in planning lessons by perfecting the content rather than the delivery performance of lessons (Nagasaki and Becker, 1993); students put forth effort in solving challenging problem and articulating their thinking. Teachers focus on intellectual discussions about lesson content and students' thinking during their planning time. Thus, the teachers are as immersed in thinking about the problem as the students are.

Keiko implemented the Japanese teaching sequence naturally. She engaged in productive 'teacher talk' with the principal and a colleague outside of the school after she found that local teachers did not share her interest in doing this. To her, it was *not* how the classroom lesson looked (students in their seats, quiet, intently working on a paper, etc.) but how engaged the students were in the quest i.e. discussing ideas, creating representations of their ideas, etc.

The educational community reacted to Keiko's methods with varying degrees of acceptance. The principal supported Keiko's mathematics teaching, and encouraged other teachers to develop a student-centered classroom environment like hers. However, most teachers did not understand that mathematics learning was taking place in Keiko's non-traditional classroom setting; they thought that 'seat-work' was the best way to teach mathematics and control student behavior. Some teachers appeared threatened by Keiko's teaching methods and disturbed by student activity.

The support of the school principal was essential to Keiko's teaching mathematics in a way that differed from the school community's view of mathematics teaching. He fostered an educational environment in the school that encouraged teachers to think and reason about professional matters including curriculum, teaching and learning, assessment, and most importantly, about how to provide an environment for students in which they could construct knowledge. When speaking about Keiko, he said:

She has the rare ability to encourage children to think, and to have them deeply engaged in activities. Keiko finds their minds, values their thinking, and sets the stage well for problem solving. She captures the children's imagination and talents, and takes them further in breadth and depth. Moreover, she creates a climate of belonging and safety, and the children like the way math becomes something else that fosters long-term understanding. She has an ability to come back and revisit issues in depth — to cycle and continue to build on prior experience and knowledge.

Keiko assessed student learning in various ways by applying what she had learned about assessment in mathematics methods courses. Projects and activities were holistically and analytically scored. The student selected items from the activities in their math working folders and prepared portfolios at the end of each marking period. They expressed their learning in written form and orally shared with their parents what they learned at parent conferences. Keiko monitored student progress by listing the goals and objectives for each grade level, and noting when a student demonstrated mastery of a concept, skill and/or problem solving objective. This information was shared with parents at conferences. This type of assessment was as new to Keiko as it was other teachers, but she was able to prepare personal profiles for her students because of her effort in designing a system that would work in her classroom. The effort to complete a challenging task is expected in Japanese culture.

In Japanese classrooms, students are expected to work as a classroom community when solving problems and appropriate student interaction is a part of the school culture. In Keiko's classroom, "cooperative learning" was not explicitly taught but developed by her implicit messages and high expectations. It must be noted that the degree to which this occurred varied between the fourth and fifth grade classes. In Keiko's class, classroom expectation remained consistent throughout the entire day while the fourth grade students returned to a traditional classroom environment after mathematics class. Over time Keiko found that her homeroom class demonstrated more growth and independence than the fourth grade students who continued to expect explicit direction in mathematics.

Conclusion and Implication

There is evidence that new teachers' beliefs and practices are rapidly absorbed by the dominant traditional school ideology and *change* is invalidated (Borko et al's study, Ensor's study, Schmidt & Duncan's study, as cited in Baldino, 1999). However, this study presents evidence about a teacher whose beliefs and practices are rooted in her cultural identity and were not absorbed in the traditional school ideology. Keiko personifies the beliefs and practices of the reform agenda, yet she faced conflict within the school community because of the differing societal and cultural values. The complexity of mathematics reform encompass far more than a teacher's beliefs and teaching practices. The issues involved mathematics education reform are impacted by *our* cultural and societal norms.

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APPENDIX A

Teaching Episodes

Learning about circles, a curriculum goal in fourth and fifth grade mathematics, emerged from and was connected with students' earlier investigations about planning a new school garden. The garden problem emerged from a real school situation. Writing, language arts, and science were integrated with mathematics project. The garden project included activities and discussions about: measuring the present rectangular shaped garden; calculating the area needed to plant various flowers; constructing garden plans on graph paper and poster board; presenting characteristics of new plans to other classes and school personnel. The students explored rectangular shapes and their relationships, by considering questions such as, "How will the area change when the length and/or width of the shape is changed?" Relating new inquiry to past learning about area and perimeter, Keiko posed questions about circles, "Will the same changes happen to the area if the garden shape is a circle? What do you need to know to determine the area?"

While some students responded that they 'knew' the formulas and terms to talk about circles, she encouraged them to generate ideas about measuring circles. The students suggested collecting circular objects from various places in the school and measuring them. As the students examined areas of the different shapes they found in the school, they employed different strategies: some used grid paper to draw circles; some cut a circle into many "wedges" and pasted the wedges on a paper to create a rectangle; some used grid paper to help orient the circle's center, and then approximated the number of square units by counting; others, cut out the circles and folded them to determine the diameter and radius, and then applied the formula. Discussions occurred on a regular basis during the exploration phase, and student generated 'discoveries' were posted in the classroom, referred to, and revised as new information was found. After these investigations, the students studied menus from local pizza restaurants to determine "Which pizza is the best deal for the money?", using what they knew about area and the various pizza prices. During review the following week, students generated a list of the new terms, their operational meanings, and relationships. Then Keiko used that information to prepare a handout for the class that listed the generalizations about circles, and she created several multi-step problems for the students to work on as a follow-up exercise.

SCHOOL BACKGROUND, MOTIVATIONAL BELIEFS AND ACHIEVEMENT IN MATHEMATICS

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This paper examines the influence of students' prior exposure to mathematics on the beliefs about self as learner of mathematics at entering college level. Students' perceptions about the ability to succeed in mathematics are related to achievement task values and to achievement. We find that the perceived self-efficacy beliefs act as important mediators of past experiences on academic motivation and academic outcomes. The structural models are examined for invariance across female and male groups.

The boundary conditions for the teaching of mathematics at entering university level are changing. It is a growing awareness that we as teachers of mathematics have to pay a broader attention to the background and prerequisites of the new group of students going into math courses in higher education. Over the years we have noticed that students level of success is closely related to own affective and motivational aspects for working with mathematics as well as to their prior background in mathematics. The starting point for our project was an interest to learn more about how these motivational aspects were related to prior experiences in mathematics and to their learning of mathematics at entering college level.

Theoretical framework

Students' ideas and thoughts about mathematics are considered to be key components to understand students' learning and behavior in mathematics (Pehkonen & Törner, 1996). The central theme in our discussion is the study of relationships between students' prior exposure to mathematics, their personal beliefs about themselves as learners of mathematics and their performance in achievement situations. We base our presentation within the tradition of expectancy-value theories of motivation with the major focus on the expectancy judgments concerning anticipated outcomes and the perceived value of these outcomes. The expectancy component may be considered to include students' beliefs about their ability to perform the task and are conceptually related to estimates of self-confidence and self-efficacy in achievement situations. Self-confidence is related to the term self-concept considered being associated with rather general judgment of self-worth or self-esteem. Perceived self-efficacy as used in social cognitive theories is concerned with more task specific judgments of personal capabilities to organize and execute the courses of action required to manage prospective situations (Bandura 1986, 1997). Eccles and her colleagues have in their model for achievement motivation introduced four aspects of achievement task values that can influence achievement behavior: intrinsic value, utility value, attainment value and cost. (Eccles (Parsons) et al., 1983).

The purpose of this paper is to examine the structure of beliefs dealing with expectancy judgments and achievement task values and how these beliefs relate to students' prior exposure in mathematics and to achievement. We also study the potential influence of gender on these relationships.

Method

The sample in our study consisted of 233 business college students, 155 males and 78 females, following a compulsory course in calculus. In the first week of the fall semester of 1996 the students were asked in an ordinary large lecture class session to fill in a self-report questionnaire. The questionnaire included items relating to students' beliefs and math courses in upper secondary school giving type of study program (academic or vocational), number of years taking math and the final grade in math. We also included a test developed by the Norwegian Mathematical Society to assess students' qualifications at entering university level. The test consisted of 38 items in the fields arithmetic, everyday life numbers problems, algebra, geometry and problem solving. The items on students' mathematical beliefs covered variables related to expectancy judgments and task values adapted from the literature. In this study we concentrate on the following six belief constructs. Regulation (REG) measures *self-efficacy for self-regulated learning* adapted from Zimmerman et al. (1992). Sample item: "How well can you concentrate on school subjects?" Motivation (MOTIV) measures *self-efficacy as part of motivational beliefs* adapted from Pintrich & DeGroot (1990). Sample item: "I expect to do well in this course". Ability (ABIL), variable named *self-perceived ability to learn* in Skaalvik & Rankin (1995). This variable measures self-concept of mathematics ability and is analog to confidence in learning math in the Fennema-Sherman scales. Sample item: "I can learn mathematics if I work hard". Interest (INTER), variable for mathematics as an *interesting* and enjoyable subject. Sample item: "I like mathematics". Useful (USE), variable for mathematics as a *useful* subject stressing the utility aspect. Sample item: "Mathematics is a worthwhile and useful subject". Anxiety (ANX) measures mathematics *test anxiety*. Sample item: "I feel anxious at math tests".

For the analysis we use structural equation modeling (SEM) in the LISREL8.14 implementation (Jöreskog & Sörbom, 1996). The belief constructs are treated as latent variables each loading on three observed items and the latent test construct is loading on the five observed scales from the test. The observed school variables program, year and grade are treated as fixed variables. Structural models related to the models examined in this paper have been studied in Pajares & Miller (1994), Stage & Kloosterman (1995), Meece et al. (1990), Risnes (1998).

Results

We start by formulating a measurement model, model 1, for the six latent belief constructs each loading on three indicators. Estimation by maximum likelihood gives a chi-square of 211.93 with 120 degrees of freedom and fit statistics like RMSEA=.057 and CFI=.95, indicating an acceptable fit to our data. To study the relationships between school background, beliefs and the achievement test, we hypothesize that the basic influence of the school variables program, year and grade on the belief constructs is mediated through the variable for self-efficacy of motivation (Bandura, 1997). Following Eccles et al. (1983) we introduce paths from the expectancy beliefs to the task value beliefs. We also include paths from the school variables to the test result and from the task values to test. Model 2 presented in figure 1 gives the result of a model generating approach based on our theoretical hypothesis searching for a model fitting our observed data well from a statistical point of view as well as giving substantively meaningful interpretation of the parameters introduced. All the included paths are significantly different from zero. For selection between competing nested models we use the chi-square difference test. Estimation of model 2 gives a chi-square of 423.15 with 283 degrees of freedom and fit statistics like RMSEA=.046 and CFI=.94 indicating an acceptable fit to our data.

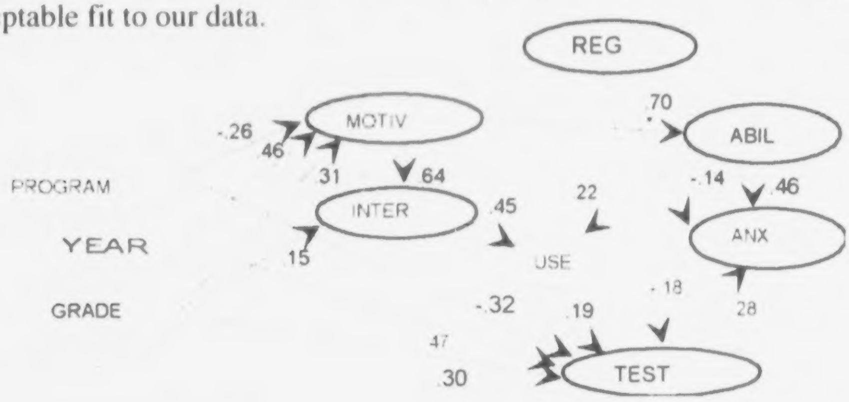

Figure 1. Structural part of model 2 with standardized path coefficients

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The question of possible differences by gender in the structure of mathematical beliefs is a recurring theme in math education (Leder, 1992). The SEM methodology gives a powerful tool to examine if the presented models are equivalent across the male and female groups. Following the Jöreskog tradition we start by testing the hypothesis that the measurement model 1 is invariant across gender groups. We conclude that model 1 with invariant factor loadings, invariant covariance matrices and invariant error terms gives an acceptable fit to the observed data with chi-square=414.88, df=291, RMSEA=.061 and CFI=.94. Extending the formulation of the measurement model to allow for possible differences in the mean values of the latent variables across gender groups, the estimation shows statistically significant differences in the mean values for the variables regulation, motivation and interest. Estimating the structural model 2 for invariance across gender groups we find an acceptable fit to our data. A closer examination of model 2 shows that it would be possible to improve model fit by allowing some of the paths to change by gender. For the female group we introduce paths from school program to anxiety and from number of years taking math to interest, as well as setting the paths from regulation to anxiety and to test equal to zero. For males we remove the paths from regulation to anxiety and from grade to interest. The resulting model gives fit statistics like chi-square=763.82, df=607, RMSEA=.047 and CFI= .94. The number of years taking math in school has a strong total effect on self-efficacy of motivation and on the test result for both gender groups. For the female group we find strong total effects from years and grade to interest and from program to anxiety. The variable self-efficacy of motivation has a strong influence on self-perception of ability and on interest for both females and males.

Conclusion

The results from the confirmatory factor analysis in model 1 indicate that the identified six belief constructs give an adequate description on aspects of students' beliefs. Our analysis shows that it is possible to discriminate between constructs related to expectancy judgments like self-efficacy of regulation, self-efficacy of motivation and self-perception of ability and between the achievement task values related to interest, useful and anxiety. We find positive associations between the competence expectancies self-efficacy of motivation and self-perception of ability and the achievement task values as predicted in Wigfield & Eccles (1992).

The analysis on gender differences indicates that the structure of beliefs is basically invariant across the male and female groups. We find that females are expressing a more positive view on self-efficacy of regulation and on interest, and males are scoring more positive on self-efficacy of motivation. We do not find significant differences by gender in the estimated mean values of the latent variables measuring self-perception of ability, anxiety and useful or achievement test.

Prior school experiences have a strong influence on students' beliefs and achievement with more years of math and better grades being associated with more positive beliefs and better test results. Students from the vocational oriented program are showing less positive beliefs and have a lower test result. Students expectations about doing well in the course as measured by self-efficacy of motivation, are strongly influenced by the three school variables with the strongest influence from the number of years taking math in school. The school background seems to be especially strongly related to the belief variables for the female group. This may be interpreted to indicate that the decision for females to choose math courses in upper secondary school is more related to their beliefs about self as learner of math compared to males. The findings confirm the hypothesis of students' performance histories as strong antecedents of current beliefs (Wigfield & Eccles, 1992). Following social cognitive theories (Bandura, 1997) we may say that the school functions as the primary setting for the cultivation and social validation of cognitive efficacy and these judgment of capability influence academic motivation and are powerful predictors of academic outcomes.

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DEVELOPING STATE MATHEMATICS STANDARDS:
A CASE STUDY

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The development of Florida's new mathematics standards was followed over a two-year period to better understand how mathematics standards are conceptualized and developed within a political context. This paper covers the first year of development [March 1994 - March 1995].

Introduction

The publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) set off a flurry of activity in the development of standards around the United States not only in other subject areas, but in the revision of curriculum frameworks and standards among the states (Ravitch, 1995). In many states, these standards are the center-piece of systemic reform packages. However, little is known how standards are conceptualized, developed, and debated within political contexts. The development of Florida's new mathematics standards was followed over a two-year period to better understand how mathematics standards are conceptualized and developed within a political context. This paper covers the first year of development [March 1994 - March 1995].

Research Focus

New mathematics standards were developed in Florida between March 1994 - May 1996. The development of these mathematics standards was the subject of this study. This paper is a subset of a much larger research project (dissertation) which explored two primary areas: (1) How the participants originally conceptualized the task of writing mathematics standards (covers the time period March 1994 - March 1995); and (2) the reasons why the standards underwent revisions as well as the issues over which disputes arose, why they arose, and how they were resolved during the revision process (covers March 1995 - May 1996). This paper deals only with the first issue and covers the time period March 1994 - March 1995 when a first draft of the mathematics standards was initially completed.

Methodology

Data collection included both formal interviews and informal conversations; document analysis (documents analyzed included memos / letters between participants (openly shared with me), early drafts of the frameworks, draft reviews throughout the process, meeting notes / artifacts (such as overheads, agenda sheets, handouts), and official state documents). Data were also collected from field observations such as writing / editorial meetings. For this phase of the research, the participants included the 24 members of the writing team which was composed mostly of classroom teachers and district curriculum specialist chosen by the editor, the state's curriculum specialist. Both authors participated in the development of the standards. [Thompson participated in the development of the mathematics standards as an assistant editor from March 1994 to March 1995 with the responsibility to facilitate reviews and to edit the standards as directed by the writing team; Jakubowski was a member of the writing team].

Data were analyzed inductively (i.e., patterns, themes, and categories of analysis came from the data rather than being imposed on them prior to data collection and analysis.) In all cases, assertions were tested by continuing conversations / interviews with the participants, and by looking for additional evidence in data already collected such as other interviews and documents. Data were searched not only to find evidence supporting the assertion, but for evidence against the assertion as well. To help ensure the integrity of the research, the following strategies were used:

- *Triangulation* - multiple sources of data were used to support emerging patterns;
- *Prolonged submersion /engagement at the site of inquiry* - data were collected over a long period of time (2 years) to gain an in-depth understanding of the phenomenon;
- *Negative case analysis* - attempts were made to look for confirming and disconfirming evidence to either support or reject interpretations constructed in the research process;
- *Member checks* - research findings were shared with key participants in the research and asked if the interpretations were reasonable.

Theoretical Framework

Using a symbolic interactionist theoretical framework, the focus was on understanding the meanings generated by the writers as they developed the new mathematics standards. This theoretical framework helped guide the interpretation of events, the focus of observations, and the development of assertions. In general, a symbolic interactionist theoretical framework helps focus attention on how individuals interpret and give meaning to their experiences, to other people, and to "objects" in their lives (in this case, standards), and to explore how this process of interpretation leads to particular behaviors. From a symbolic interactonist perspective, mathematics standards are social constructions and result from the interplay of diverse political interests. This perspective encourages the collection of evidence that helps reveal the differing intents / interests of the participants (Hall, 1995), and helps focus on the language and the politics of meaning in standards development (Placier, 1998).

Major Findings of the Study

Introduction: The "Medium is the Message"

This paper covers the time period March 1994 - March 1995 when a first draft of the mathematics standards was initially completed. The primary theme arising from this part of the research is that "the medium is the message." That is, the language and structure of the standards were "symbolic" of the writing team's larger reform agenda and carried "deeper meanings" other than the literal interpretation of the standards. The writers developed a set of strategies for writing standards based on their beliefs about how teachers would interpret / use them in their classrooms. The writers attempted to use the standards to reform mathematics education by (a) encouraging teachers to reconsider their views on mathematics and how it should be taught, and (b) influencing the types of assessments used at both the state and local levels.

Strategies Used to Convey Messages

In writing mathematics standards, the strategies used by the writers to achieve these goals included: (a) using "dynamic" verbs to encourage a more active pedagogy and "authentic assessments" (e.g., explore, investigate, analyze rather than verbs such as solve, compute, or factor), (b) integrating problem solving, technology, manipulatives, and communication in the benchmarks to encourage their integration into the curriculum, (c) integrating a variety of activities and content into single benchmarks in order to encourage a more holistic and integrated view of mathematics, (for example, *recognize, describe, extend, estimate, analyze, generalize, transform, and create a wide variety of mathematical relationships by using models such as tables, graphs (both one- and two-dimensional), matrices, verbal rules, expressions, equations, and inequalities* — September 1994 Working Draft); (d) writing broad benchmarks (lacking specifics) in order to provide both flexibility in classroom practice and to encourage teachers to engage in curriculum development at the local level (i.e., by requiring them to fill in details), (e) avoiding words which the writers felt carried "negative connotations" that could dissuade "teacher change" (e.g., lessening the prominence of "red flag" words relating to computation, statistics, and algebra), and (f) putting more detail and emphasis into topics that the writers felt should receive more attention and fewer details and less emphasis on topics they felt should receive decreased attention.

Additional Strategies

These strategies were not limited solely to writing the standards. Similar strategies were used by the writers in their development of the curriculum framework of which the standards were only one chapter. In a broader sense, the writing team felt that the presentation and organization of ideas in the framework (such as problem solving, use of manipulatives, or ESOL teaching strategies [English for Speakers of Other Languages]), could encourage or imply the literal application of the arrangement of these ideas into mathematics classrooms. For example, the writers believed that if problem solving is separated from the content and written as separate benchmarks, then teachers might teach it separately. As a result, process strands such as problem solving (and communication) were integrated into the standards and not given their own strand. Similarly, if strategies for ESOL students are singled out for the encouragement of manipulatives and cooperative learning, then the writers feared that teachers might use manipulatives and cooperative learning only for ESOL students; thus, strategies for "special interest groups" were not singled out in the frameworks for fear of "pedagogical stereotyping" (writing team characterizations).

Assessment and Writing the Standards

In the early phases of framework development, the benchmarks were not written with state assessment in mind. From the writing team's perspective, the primary intent of the benchmarks was to reform teaching and to change teacher's views of mathematics. The mathematics writing team was not opposed to using the benchmarks to influence assessment, but the primary intent was to encourage the use of "alternative assessments" at the *local* level only. After a new Education Commissioner took office in early 1995 calling for new state standards and assessments, the writing team was concerned that local control might be usurped if the state began assessing specific content. The writers believed that " ...the districts will just end up writing their own specific benchmarks to match the state assessment" thus, ruining the intent of the benchmarks as designed by the writing team.

The writing team had the concern that using the benchmarks in their current form as a basis for restructuring state assessment could send the "wrong message" or could be used in ways inconsistent with their goals,for reform. Therefore, at the last writing team meeting in March 1995, approximately 35% of the benchmarks were re-written to take into account the possibility of state assessment. Changes were primarily made to the verbs. (e.g., in two benchmarks, the verb "compare" was changed to "communicate an understanding of" and "investigate"; the phrase "use and describe the concepts of ..." was changed to "construct meaning for ...", and the phrase "use real-life experiences and physical materials to describe, classify, compare, sort, model, draw and construct ..." was changed to "explore.") Although the benchmarks contained similar verbs in earlier drafts, the writing team sought to expand their use to discourage traditional state assessments such as multiple choice questions.

Summary

As the first year of the standard setting process unfolded, strategies for writing standards emerged where almost every aspect regarding the language and structure of the standards was used to convey the writers message of reform. Later in the process, additional strategies were also developed by the writers to keep the standards from being "mis-used" by other political forces (e.g., assessment). Similar strategies were also used in other parts of the curriculum framework (e.g., to avoid "pedagogical stereotyping" by "special interest groups.")

In late 1994 and early 1995, the political context changed and new participants and interests came into play that would cause the new standards to undergo major revisions. However, the writing team would endeavor throughout these revisions to maintain as much of their original goals for the standards as possible — despite the changes in the political context. Exploring the larger political context, understanding why this resulted in the standards undergoing revisions, and the resulting turmoil in revising the mathematics standards will be the subject of future research papers.

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Note: A more detailed version of this and related papers are available upon request.

A MATTER OF TIME: EMOTION AND PERFORMANCE ON MATHEMATICS TESTS

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This study brings to the forefront how individuals' views of *time* influence their responses to different classroom settings. The focus of this paper's discussion is two students' experiences during traditional one-hour exams and how, through discussion of their life stories, they both overcame the panic they felt during testing.

Theoretical Framework and Objectives

McLeod's (1992) review of the research on affect provides an excellent summary of the current work central to mathematics education. His analysis indicates that, although critical, students' emotional responses have not been a major component of the research on issues in mathematics education. Those few researchers who have studied emotional responses have focused on students' responses to mathematical topics (Buxton, 1981), algebra problems (Wagner, Rachlin, and Jensen, 1984), conjectures (Brown and Walter, 1983), and completing problems (Mason, Burton, and Stacey, 1982). Bassarear (1989) also documented the significant role that emotional responses play in learning. This paper extends this discussion by illustrating a particularly salient source of debilitating emotion felt during mathematics tests, the sources of which evolve around students' perception of time and is based in part on Mandler's (1989) constuctivist view of emotion and motivated by McLeod's suggestion that careful observation, along with detailed interviews, should help in the analysis of emotional states of learners.

Mandler's (1989) approach to affective issues considers emotional experience (and behavior) to be the result of cognitive analyses and physiological responses. His position supports "the notion that emotions express some aspect of value, and the assertion that emotions are hot, implying a gut reaction or a visceral response" (p. 6). Particularly important to this study is Mandler's belief that "stress tends to decrease attention to peripheral events and to focus attentional conscious capacity on those aspects of the situation that the individual considers important" (p. 10). We will see this connection when the two women in this study respond to stress and focus on *time* and are subsequently unable to perform on a mathematics exam.

We will also show that differences in the perceived importance of time influence students' performance as they complete group activities or respond to traditional exams. Not all students in this study were bound by their traditional cultural views, however, many of them held perspectives of time that contrasted similarly to those documented between cultures. It is accepted that the Western, Newtonian view of time is a comparatively recent cultural construction. Slife (1993) notes,

> Ancient peoples did not view time as an objective frame of reference for marking events. They relativized time by making it conform *to* events, rather than events conform to time. . . .Time was a dynamic and adjustable organization tailored to fit our world experiences. (pp. 14-15)

Slife goes on to establish aspects of the Newtonian view of time that have been widely accepted in our Western culture: time is seen as objective, continuous, universal, linear, and infinitely reducible. By contrast, many cultures reject all or part of these aspects of time and view time as more subjective, subject to breaks, speeding up, slowing down, and so forth. For them, time is not something life is measured against, but rather something that conforms to life.

Examples of differing views of time are easy to find. Many cultures can be described as *polychronic*—they value attention to several activities simultaneously, rather than focusing on one. This focus de-emphasizes deadlines and, from a Western perspective, a *timely* completion of tasks. In such cultures, people and relationships are more important than deadlines.

It is with these issues in mind that we present our goals for the paper: 1. Illustrate the contrast between the evaluation of students' performances during working sessions and performances on more traditional exams; 2. Discuss how the telling of life-stories suggests that *time* alone prevents some students from successfully performing on exams; and 3. Use an interpretive theoretical model to analyze the underlying reasons for students' emotional responses and changes in those responses.

Methods and Data Sources

Data for this story of two women, pre-service elementary teachers, are a subset of a larger set of data gathered during a year-long study. The data set consists of participant observation notes, classroom artifacts, mathematical autobiographies, and interviews–informal, post-exam, and structured. Data were analyzed for salient categories and confirmed with participants.

Although the mathematics content in this course for pre-service elementary teachers is representative of classes taught across the nation, the material was delivered in a reform context. Students worked in groups as they solved problems, gathered data, or used manipulatives to develop insight into mathematical ideas. The structure of this class provided the instructor with the daily luxury of observing and interacting with individuals and small groups. Traditional lecture was minimized. However, several traditional teaching components remained to *standardize* the multiple section course: a textbook, homework, quizzes, and hour exams.

Two students' experiences during traditional testing and how they came to overcome the panic felt during exams is the focus of the following discussion. It is important to note that these students reported that this panic was felt *only during mathematics tests*. The two persons who tell their stories were non-traditional students, Jo and Pat. Although both women were members of minority groups, the stories they tell were not unique but magnify what other minority and non-minority students have validated in subsequent classes.

As one of their first tasks, students in this class write a mathematical autobiography. Pat began her autobiography by writing that she was "anxious and insecure about doing mathematics. It's not that I don't know what to do. It's related to past experiences of failure. I actually enjoy math." I was unsure of what she meant, and I made a note to observe her classroom interaction more closely. In working with Pat's group, she did not appear anxious or insecure. She often presented good ideas but could be easily talked out of them. However, just as often, Pat's ideas were accepted by the group.

Pat progressed through the course–successfully, until the first hour exam. As I evaluated her test, I was dismayed; she had failed. I *knew* she knew the material. I had observed her working on problems similar to those on the exam earlier in the week. She should not have had a problem with any of the questions, but her exam was almost blank. When I returned the exam to Pat, she shook her head as though it was what she expected.

While returning exams, it is my practice to assign a group activity. While the groups work, I interview each individual about the exam and the class. At this time Pat said,

> The test was fair, but I just couldn't think. I started to work and got involved with one problem. When I looked at the clock, there was

> only twenty minutes left. I knew I would never finish. I panicked. My heart pounded. I saw spots. For me, the test was over. I turned in the exam and left.

Mandler (1992) describes Pat's response as *hot*. Much research has examined negative attitudes towards mathematics, however, Pat did not have a negative attitude towards mathematics. She liked mathematics. *Time* was Pat's enemy, *not* mathematics.

Course policy included exams. However, the traditional exam was not equitable for Pat. It did not even allow her to completely answer any of the questions; questions for which I knew she had answers. During an informal interview I suggested that Pat could take as much time as was necessary to complete her next exam. She responded incredulously, "Really! I can really have all the time that I need, really?" On the next exam, Pat required an extra thirty minutes to successfully complete the questions. On the remaining exams, Pat did not use additional time. She told me, "When I have all the time I need, I don't even need the whole hour."

Later Pat shared that she had first felt panic on math tests during the second grade while trying to memorize addition facts. She said, "I just couldn't seem to get fast enough to pass the tests. I knew the facts, but I was not fast enough. It was my mom. She called the school. If I didn't pass, I would get beaten." Working with Pat allowed her to acknowledge her competence in mathematics and to contrast this with her test performance. She recognized the root of the problems she faced in taking tests, confronted, and overcame the fear she had first faced in second grade and had repeatedly faced since then–the fear of running out of time.

The following semester, Jo enrolled. Like Pat, Jo talked in her mathematical autobiography about feeling anxious during math tests. Jo also appeared to understand in class but failed the first exam. Jo described what happened, "I panicked when I read the first question." I had noticed Jo's reaction. She had palled, then flushed, then broke out with beads of sweat. It appeared as though she was ill. When asked if she felt OK, she said, "I'm fine." It was clear, however, that she was not fine.

Jo responded instantly when asked about her anxiety, "It was those tests...those tests that we took on multiplication. I'm so slow. I just couldn't do it. I understand, but I just can't do it when there's *time*." In the next few weeks, Jo began to express the opinion that what had happened as a child really did not have much to do or say about her mathematical ability. In scheduling the next hour exam, Jo arranged to have additional time. On the day of the exam, with five minutes remaining in the period, Jo stood and walked over to where I was standing. I thought she needed to have a question on the exam clarified. I was wrong. Jo had completed her exam and was turning it in. She smiled and said, "I did it. I did it in the hour. I don't need more time. Because I could have more, I didn't need it. I didn't even feel a bit scared."

Jo's and Pat's stories of success both focus on time. When time became a moot point, they perform as well during an exam as they did in their small groups. After experiencing success on an exam, time was no longer an issue. They could focus on the exam and not on *time*, allowing their test to reflect the mathematical insight they had demonstrated working in groups.

Conclusions and Points of View

How can something that simple dramatically change a student's ability to perform on an exam? Perhaps taking what they had always perceived as a problem, *time*, out of the way let these two women concentrate on the task at hand. This would support Mandler's (1989) suggestion that stress focuses attention both appropriately and inappropriately. Perhaps the validation that the individuals found in discussing their life histories allowed them to place the test in perspective and to re-think the validity of their emotional reaction. Carter and Doyle (1996) strongly support this interpretation in their literature review of personal histories.

Although both Pat and Jo were familiar with and able to function easily within the dominant culture, it is certainly possible that they retained, or constructed on their own, aspects of a temporal viewpoint that was at odds with traditional Western temporality and with *timed* tests of knowledge. Others have suggested cultural differences can clash with the timed tests. Knutson and McCarthy (1993), for example, suggest that the timed tests given to locate students for gifted and talented programs discriminate against American Indians because the tests reward quick answers while American Indian cultures reward careful, slow, well-considered responses.

In Pat's and Jo's cases, the initial imposition of a temporal structure on their mathematical knowledge was an affront to their cultural, or perhaps personal, views of time. It was not the mathematics, but rather the imposition of a foreign temporal structure over their mathematical knowing, that lead to their discomfort and eventual paralysis in mathematical testing situations. We do not wish to suggest that eliminating time from testing situations solves all problems, however, we do have other evidence to include in the final paper from minority and non-minority students that support the statement that time can be an issue. This issue is particularly significant in the light of the goals of PME to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics.

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LEARNING FROM CALIFORNIA'S EXPERIENCES TO MOVE FORWARD THE MATHEMATICS EDUCATION REFORM AGENDA

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The reforms in mathematics education delineated in National Council of Teachers of Mathematics documents (1989; 1991; 1995) are meeting with increased resistance throughout the United States. In this article, two influential leaders in mathematics education discuss resistance to reforms that they have encountered in California and throughout the country. They also identify strategies to advance reforms at a time when reaction to mathematics education reforms is dramatically shaping the curriculum for the next millennium.

Ms. Long was a high school mathematics teacher for many years before assuming leadership of an important statewide association in California for mathematics teachers in the mid-1990s. She linked the backlash against mathematics education reforms in California to the negative attention that education in general was receiving. Mr. Gilmore was an author for one of the largest and most successful mathematics textbook publishers in the country. He discussed how many parents and teachers value "naked practice" and repetition. He also provided examples demonstrating that reforms in the early 1990s in California proceeded too rashly, fomenting the backlash of the later part of the decade. Ms. Long's experiences demonstrated the need for the mathematics education community to politically mobilize and seek colleagues to advance the reform effort. To alleviate potential difficulties with teachers and parents, Mr. Gilmore believed that curricular materials should include not only rich mathematical problems, but also a skills practice section and a resource section that includes definitions and examples. Ms. Long remained optimistic that the pendulum will swing back in favor of the mathematics education reform movement in California.

PEDAGOGICAL PARALLELS OF EXPERT MATHEMATICS TEACHERS TO THE IDEOLOGIES OF THE RESEARCH PARADIGMS: POSITIVISM, INTERPRETIVISM, AND CRITICAL THEORY

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In this study, the verbal responses and observed practices of three expert mathematics teachers were analyzed with possible alliances to three research paradigms; positivism, interpretivism, and critical theory. What worldview do the experts act from and do these worldviews remain constant or vary for each situation and/or each expert?

The experts in this study were selected through recommendations from the administrators, counselors, and mathematics teachers from three demographically different schools (rural, parochial, and suburban). The purpose of this recommendation process was to allow for a measurement of expertness that emerged from the qualities of expert unique to the various schools' culture and the individuals recommending; referred to in this work as emergent measure. These experts varied in pedagogical style, content and methodological knowledge and usage, and managerial practices. This difference referred to in this work as situated effectiveness.

The responses of interviews (2) and observations (4) of each expert were analyzed using a matrix of the ontological, epistemological, and methodological principles of positivism, interpretivism, and critical theory in terms of teacher pedagogy. The data was interpreted in terms of group and individual alliance to the paradigms.

When looking at the data as a group there is a relatively equal balance of ideologies shown from each of the worldviews. On an individual basis, each expert's recorded data showed an underlying alliance with one of the paradigm principles while revealing situationally-determined alliances with paradigms other than their dominant. Each expert in this study had a different underlying worldview parallel.

The final work raises questions about educational philosophy and differences in pedagogy.

AFFECTIVE AND COGNITIVE METAREPRESENTATIONS

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Affect has an important role in learning and doing mathematics. Emotional experiences have a central role in the changing of mathematics related attitudes and beliefs. Affect can be regarded as another kind of representational system that is intertwined with cognitive representational systems. With respect to emotions around mathematical problem solving Goldin and DeBellis have been doing important work (e.g. Goldin, 1988; DeBellis & Goldin, 1997).

Metacognition is a relatively recent concept, but it has rapidly been recognised as a central research object. Several new concepts (e.g. meta-affect, meta-emotion, self-awareness, metamood, cognitive emotion) have extended similar idea into the emotional sphere of human mind. In this poster these concepts are discussed, and the metalevel of representations is conceptualised in a structured manner. It is divided into four domains depending on the representational system (cognitive or emotional) and the content to be represented (cognition or emotion). The four domains are: 1) metacognition (cognitions about cognitions), 2) emotional cognition (cognitions about emotions), 3) cognitive emotions (emotions about cognitions), and 4) meta-emotions (emotions about emotions).

The author has been doing a three-year (grades 7 to 9) ethnographic study with one mathematics classroom. The role of cognitive emotions in doing and learning mathematics and in the changing of attitudes and beliefs is illustrated by case studies of individual students.

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Teacher Beliefs

NARRATION AS A TOOL FOR ANALYZING BELIEFS ON CALCULUS – A CASE STUDY

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Abstract: In the following research report the question is pursued concerning which specific beliefs are present in the field of calculus in the university course for prospective mathematics teachers. Narration is methodically employed as a research tool, allowing answers to a number of research questions to be expected. On the one hand they entail content-specific beliefs on calculus which are researched in view of their development in the sense of when they were developed. Due to the open method of data collection we receive, on the other hand, information on which dimensions calculus is perceived by prospective teachers. Finally, we obtain information to what extent permanent temporal lines of consistent views can be determined. This allows conclusions to be drawn on the efficiency of university courses for prospective mathematics teachers.

1. The context of the case study

In the German educational system calculus plays a central role in the secondary grades (age groups 16-19) (see details of the German system in [Robitaille 97]). This circumstance dates back to the initiative of the famous mathematician Felix Klein at the beginning of this century. His concepts are still markedly present in the German mathematics curriculum of today. This results in the fact that calculus contents are always compulsory parts in the *Abitur* examination (the final school-leaving examination). The demands in the mathematics *Leistungskurse* in Germany (equivalent to A-Level courses in Great Britain) are greater compared to equivalent examinations in the United States (NCTM 1991). Calculus also already plays a dominant role in the first semester of university prospective teacher courses for the *Sekundarstufe II*. These students visit the same lectures and seminars as those studying for a purely academic degree.

However, it can not be ignored that calculus lessons in schools are increasingly slipping into an orientation crisis. The unfulfilled, exaggerated expectations towards New Maths can be held partially responsible: it views calculus primarily as a propaedeutic field of science. After all, the present discussion is also being pushed forward by the availability of CAS-Tools in pocket calculators featuring graphics display facilities (DERIVE). These calculators downgrade the usual traditional curve discussions and calculus extrema tasks (suitable for schema-oriented lessons) to literally trivial example applications. This view amongst teachers, however, has at present not yet affected the traditional-oriented calculus canon at university. In contrast to the United States, a calculus debate has not yet got underway in Germany.

Nonetheless, new demands are being discussed in occasionally offered lectures on mathematics education for prospective teachers. However, such seminars do not predominantly serve the acquisition of methodical aspects of calculus but intend to exemplary introduce general and specific didactic aspects related to mathematics lessons. The subject material employed in this paper originated from such a seminar.

Reflection on mathematics lessons can itself be carried out on different levels and be subject to widely differing objectives. In this paper the question is discussed as to what extent self-reflection in the form of narration elements can offer highly differing information on system-internal states which might reveal concealed parameter. The innate ideas of the learner are also articulated here. The subject material employed for this question were topics of papers by 7th-semester students who were requested to write essays on this in a seminar on mathematics education focussing in particular calculus.

In the literature, reflection is emphasized as having a central role as a structuring facility. Both for oneself and for the external observer, reflections reveal beliefs and also conceptual systems expressing teachers' thoughts and actions. Reflection has, therefore, gained significant acceptance as a basis of prospective teacher education. Chapman (1998) (cf. the literature quoted there) emphasizes that, in general, preservice teachers tend to rely on their personal experience as learners in constructing meaning for classroom events. Self-reflection is also employed as a means to accompany and support processes of change. Self-reflection, therefore, is also a basis for a prospective teacher´s professional progress. Different possibilities arise for the initiation of self-reflection. The use of narration in our study is based on the view that narration is a way in which one makes sense of the world (Bruner, 1986, see Chapman, 1998) by incorporating in particular one's own experiences.

2. The research questions

According to Schonfeld (98), the interdependency of teacher's goals, beliefs and knowledge is decisive for the teaching process. From this we derive the following questions for our case study:

2.1 Which beliefs on mathematics, in particular on calculus, are characteristic for prospective teachers? When were they developed?

2.2 In which dimensions are calculus lessons perceived by prospective teachers: (A) the self-experienced lessons (in school, at universities)? (B) lessons considered worth having according to one's own experiences at university and on the basis of knowledge gained in seminars on mathematics education? (C) lessons given in the future as a qualified teacher?

2.3 To what extent can permanent temporal lines of consistent views be determined? To what extent are self-experienced teaching situations criteria for self-planned lessons?

3. Method and procedure

Within the framework of a seminar on calculus the students ($n = 10$) were asked to write essays (1 – 2 pages) on the topics: (A) Calculus and I: how I (have) experienced calculus at school and university. (B) How I would have liked to have learned calculus. (C) How I would like to teach calculus.

These essay topics were subsequently handed out at intervals of three weeks. The students knew nothing of the actual case study approach of the author. Altogether six students handed in contributions on all three topics. Their essays are used here for this case study.

In the first viewing of the essays the given topics can be categorized as being similar. The research approach realized here is classified as triangulation (Cohen; Manion 1994). This procedure can be justified through the following basic assumptions: Learning and teaching are dual processes that can be individually considered as linked together. Possibly experienced deficits are categorized - when viewed positively - as points of emphasis of one's own responsibly conducted lessons. Positive experiences lead to reinforcement of one's own actions towards others. In this respect a temporal invariant consistency in the evaluation of one's own teaching and learning processes is presumed, whereby one must bear in mind that repeated mentioning of its aspects can lead to its confirmation. Due to the limitation of a maximum of 2 pages (per essay topic) an exhausting presentation of the three topics by the students cannot be expected here. No content-related expectations were placed on the students so that freely written essays were ensured. The location and topic change, induced by the respective question formulation, enables new reflection impulses and recapitulates new aspects of a topic from the viewpoint of the other students. As the three topics are intended to illuminate different time concepts (A – past, B – present, C – future) it is to be expected that through the essays it will be possible to determine time-invariant lines.

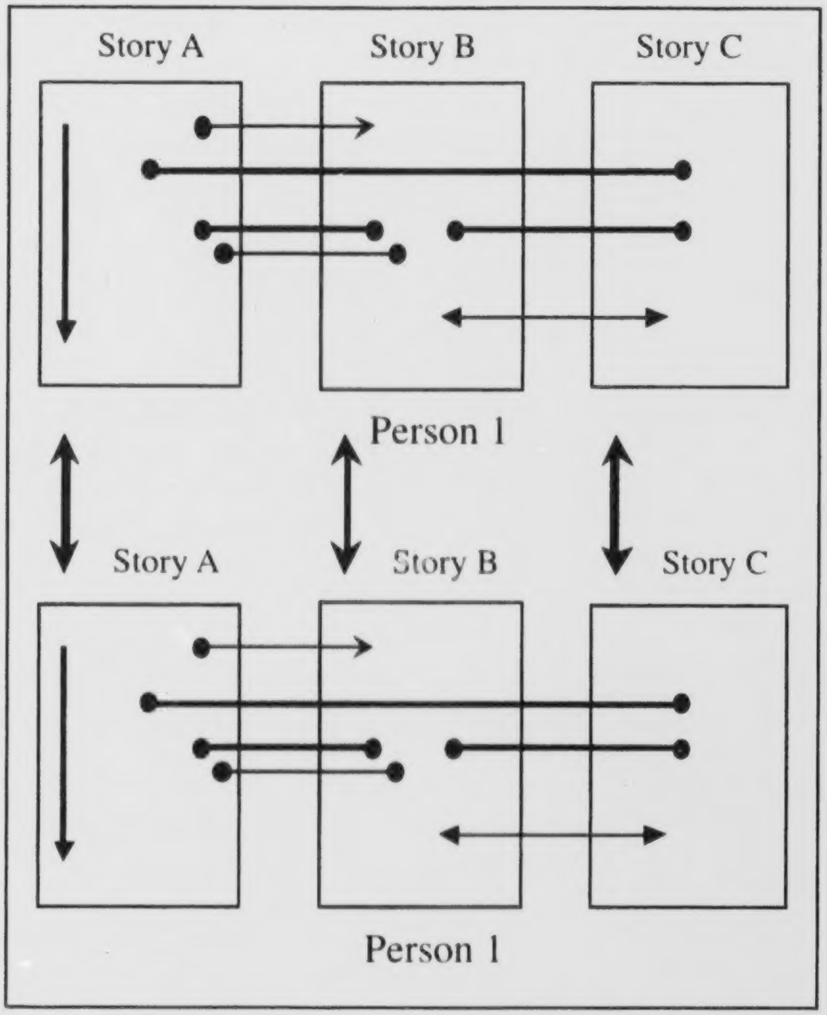

Figure 1. Possible clusters of evaluation

4. Results and evaluation

With respect to research question 2.2 the lines given by the three topics ((A), (B), (C)) offer text contributions which can be assigned to the following fields: (1) objectives of mathematics lessons and calculus, (2) views on mathematics, in particular calculus, (3) calculus and formal elements of mathematics, (4) calculus and learning / teaching mathematics, (5) calculus and demonstration, (6) calculus and the emotional dimensions of mathematics, (7) personal specifica. Not all essays contain explicit contributions to each of these fields. All contributions to these fields cover aspects on cognition, beliefs and goals (see Schoenfeld [98]). For reasons of concise presentation only a few general results can be presented here, reported rather as a tendency.

Concerning the beliefs (research question 2.1) implicit in the essays of the students, these beliefs are in particular found in the text when the essay writers explicitly remark their contents as self-estimations being subjected to alterations in the course of time (from the perspective of the sectional levels (A), (B) and (C). They concern, e.g. calculus as a field of mathematics in view of its place within mathematics: Calculus in the upper secondary grades is perceived by pupils as an independent and new mathematics field, whereas on the lower secondary grades (age groups 10-16) mathematics is viewed rather as a whole. Possible „colourations“ of this mathematics field are also to be classed as beliefs. University mathematics reinforces the impression of calculus being an independent field of mathematics, in particular as the student becomes familiar with further fields of mathematics. While the school view of calculus is schema-oriented, university students possess primarily a formal orientation. It seems that stimulation towards a stronger problem-oriented viewpoint is hardly given. In this respect their views towards calculus and mathematics in general present on the whole a one-sided picture. Also the role that may be placed in calculus by aspects of logics or application can be understood as subjective beliefs. Formulations such as „calculus training“, „breaking the code of inequations“, „differentiation as handicraft“, or „integration as an art“ reveal something about calculus learning picture of the individual essay writer.

Beliefs are expressed more clearly when evaluations in the form of described emotions are explicitly mentioned. Their views on ideal calculus lessons seem to be nurtured from their own positive experiences in lessons or to be complemented through self-experienced deficits in school and at university. Within this framework, calculus lessons are experienced at university as being not very constructive. Admittedly, calculus seminars serve a universal function in academic mathematics, whereby aspects of educating prospective teachers how to teach mathematics only plays a subordinate role. On the other hand, the few chances for illuminating the relevance of academic mathematics for school teaching are apparently hardly made use of.

In all the six essays emotional value judgments are given towards calculus, whereby in particular negative statements stand out towards calculus experienced at university. The scope lies between, 'interesting but difficult', 'found it very hard to', 'more brutal than any other mathematics seminar'.

One must hereby note that in particular those students who successfully completed calculus under considerable effort have the tendency to exaggerate the formal aspect (for instance the logical structures of calculus terms) as the most important characteristic of calculus. Is such a formalistic view of calculus for prospective teachers, however, possibly more than slightly responsible for learning difficulties of their future pupils?

Conclusions

It hardly needs mentioning that these results cannot be considered representative either for calculus lessons in the upper secondary grades in Germany, or in respect to statements of prospective teachers in general.

Nonetheless, the evaluations offer the author a number of valuable insights, in particular it provides a deeper and better understanding of the psychological aspects of teaching and learning topics around calculus.

Calculus is understood as a central part of mathematics; insofar beliefs may be conveyed onto mathematics *pars pro toto*. Thus it would be complementary, however, in the interest of the students, to go beyond the individually specialized fields and instead enable (again) an integral all-encompassing view of mathematics to be experienced. Here deficits were clearly articulated. Whereas the literature offers an abundance of papers on teachers' and pupils' beliefs, descriptions of specifically related beliefs are, however not abundant at all. In comparison, papers on calculus (in the upper secondary grades) are hard to find (see e.g. Fox 1998).

More concern must be given to the circumstance that calculus in the evaluation of future mathematics teachers is emotionally highly loaded - however not primarily in a positive manner. Finally, action is required when one feels compelled to note that university seminars on calculus do not seem to offer a convincing contribution towards the professionalization of future teachers. It appears that the well-known quotation of F. Klein (1908) on the so-called *'two-fold discontinuity'* still has its relevance after almost one hundred years:

> 'The young student sees himself at the beginning of his university course confronted with problems which in no point remind him of things he was concerned with at school; of course this is why he forgets all these things rapidly and thoroughly. However, when he enters a teaching position after completion of study, he is expected to teach traditional elementary mathematics in the traditional school manner; as he can hardly relate this to his university mathematics, he will in most cases embrace traditional teaching within short time, and the university course will remain to him only a more or less pleasant memory that has no influence on his lessons.'

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Teacher Education

THE EVOLUTION OF PRESERVICE MATHEMATICS TEACHERS' REPRESENTATIONS DURING TRAINING: A CASE STUDY

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Abstract: Previous studies relating to preservice teacher education show that, generally, students' representations of mathematics learning and teaching are well established when they begin teacher training. These representations act as filters through which training situations are viewed, and appear to be decisive in terms of how their future professional practices will develop. To better understand the influence of interventions developed by a team of mathematics educators in preservice teacher education over a four-year period, individual interviews were conducted at important stages of this training, with four students being chosen on the basis of their entry profile in the program. The results bring out how the preservice teachers' representations became complexified over the course of training, and provide useful insight into the elements of this training that contributed to this evolution.

A recent review of research on learning to teach (Wideen, Mayer-Smith, Moon, 1998) has confirmed the importance of the representations of teaching held by beginning teachers prior to entering programs of preservice teacher education. These prior representations are developed through personal experience and instruction (Richardson, 1996), through what Britzman (1986) has called "implicit instructional biographies"—i.e., the cumulative experience of school which contributes to a certain image of a teacher's work. In the case of mathematics in particular, several studies have shown that these representations, which have been constructed over the course of 12 or 13 years of prior schooling, stem from a mechanistic view of mathematics and of the way this subject should be taught (Bednarz, Gattuso, Mary, 1996, Kagan, 1992). These representations constitute the grid through which preservice teachers give meaning to the situations which emerge during their training (Weinstein, 1990), and appear to play a decisive role in terms of their future professional practices (Kagan, 1992, Reynolds, 1992). Thus, after having become teachers themselves, as soon as they encounter a problem, they will revert to and adopt models of intervention that implicitly refer to their own experience as students during the previous 13 or more years; they will, at that point, fall back into a certain habitus (Bourdieu, 1980) that will work as an unconscious principle of action. If teacher training curriculum is to effectively counterbalance student teachers' previously formed views of mathematics and teaching, it will have to take into account the full weight of these students' own past classroom experience. We developed a series of different interventions aimed at having preservice teachers achieve this developmental goal over 4 years of training (Bednarz, Gattuso, Mary, 1995, Bednarz, Gattuso, 1998).

Overall features and underlying foundations of this training model

This training was beginning during the first session with a course in problem-solving. The primary objective of this course was to throw into question the ideas that students have developed concerning mathematics and mathematics learning and teaching throughout their entire previous schooling. As well, it was designed to make them receptive to the didactic type of questioning that would subsequently occur during their training. In this course, they were confronted with various problem situations and were required to articulate their solutions to other people, argue over the validity of the solutions that either they or the others had proposed, and so on. The issue was of

> sensitizing students to the hidden structures of this complex field of activity, in particular by dwelling on their own experiences as a school student (in retrospect) and the limitations inherent to these experiences; and indirectly contributing to the development of an alternative school mathematical *habitus*. (Bauersfeld, 1994, p. 182).

This initial contact with a different way of viewing learning and teaching mathematics would be returned to subsequently during various classes in mathematics education. (see Bednarz, Gattuso, Mary, 1995). The following competencies were targeted under this approach: learning to observe; formulating relevant questions; making appropriate use of the interactions and productions (i.e., answers, comments, etc.) of pupils (the focus of this attention remains the pupils with their difficulties, manners of reasoning, conceptions); conceiving of a variety of ways of putting into practice situations related to precise concepts (situations are proposed, used, analyzed; objectives are thrown into question); using previously developed analyses (conceptual analysis including anticipation of pupil's difficulties, errors, conceptions, modes of reasoning) to design a teaching scenario, successive lessons on a given subject, or a learning sequence covering a longer period. Great stress was laid on: continual verbalization by preservice teachers of reasonings and mathematical ideas; and the use of materials, representations, and contexts to support this verbalization and the construction of meaning in mathematics. The various activities that were worked up as part of this training were based on a certain form of teaching advocated by the preservice educators that: 1) was consistent with the stated goals (Janvier, 1996, Bednarz, Gattuso, 1998); 2) was linked to the types of reasoning and the ideas developed by preservice teachers in context; and 3) attempted to make these ideas and reasonings evolve. The present research has aimed at achieving better understanding of the conditions of this teacher training program that contributed to eventual changes in student teachers' views of mathematics teaching.

Objectives of the Research

To clarify what representations preservice teachers have of learning/teaching mathematics when they first enter the program, and how these representations are actualized in practices aimed at the learning of mathematical knowledge? and to better understand how these representations evolve during their training?

Methods

Four students previously selected on the basis of their entry profile in the program were followed throughout all four years of their training: Vincent (20 years old; engineering), Stéphane (19 years old, science), Sylvie (20 years old, humanities); Louise (24 years old, optometry). They represented the different categories of program clientele. Most of the students entering university had studied Science (50%), Humanities or Literature (32%), or had come from other faculties (18%).

Data Sources

- *Individual interviews* were conducted at major stages of training: *1st semester*—i.e., when entering the program; *4th semester;* after three courses in mathematics teaching and a first teaching practicum*; last semester*, at the end of their training, following completion of all courses in mathematics teaching and a final practicum during which students were fully in charge of the classroom. During these interviews, preservice teachers were put into situations related to different components of their future profession, where they had to react:

  How do preservice teachers view the pupils? How do they account for them in their practice?

- Reviewing a homework assignment in class (for example, using pupils' answers to a problem figuring in this assignment)
- Using answers given by pupils in class during a lesson (in relation to a problem, for example); or taking advantage of a generalization context

situation with pupils and constructing formulas, taking advantage of pupils' solutions;

- Intervening on an error made by several pupils.
- Validating a statement (many solutions are given in class: responding to these various pupils' solutions to validate them)

How do preservice teachers view their own teaching?

- Drawing up a lesson (for example on the multiplication of fractions)
- Reflecting on classroom intervention by teachers using videos (for example, a lesson as presented by two different teachers)
- *More general questions* regarding the reasons they enrolled in the program, their motivation (at the beginning), how their ideas on teaching have changed, decisive experiences in this change of points of view, etc.

Results

The analysis of the first interviews (conducted when they entered the training program) shows different ways of approaching the teaching of mathematics and interactions with pupils. This becomes particularly apparent in the way preservice teachers actualize their ideas when asked to react to different situations (see table 1)

Evolution over the course of training

We will show how the representations of preservice teachers become complexified thanks to the action and reflection of these students on action at various stages of their training. To aid in this analysis, we will draw on one case (that of Sylvie), in connection with several situation scenarios that have been excerpted from interviews we conducted with her.

At the outset of training, this preservice teacher showed she could draw on *only rather limited resources* to deal with the mistake of one of her students (involving the addition of fractions).

> "*I would remind him that you can't add fractions the way you add 8+2; it's not something you do by adding the numerators and the denominators together. I'd ask him to try and remember just a bit what we said about adding fractions...*" In this example, her intervention is limited to redelivering the algorithm disguised as a question put to the child: "*Does the common denominator ring a bell? At that point, I would take the tack of having to find a denominator that is common to them both...* (and further on) *now that you know you have to put both of them over 16, what do you do with the numerators?*"

This intervention implicitly gives evidence of *a certain vision of learning (imitation of a model) and teaching mathematics* (which in this case consists in making sure that the pupil recalls the model that he/she previously learned).

During the 2nd interview (at the end of the 3rd university session), the same question was put to Sylvie. This excerpt now gives evidence of the *wealth of resources* which this preservice teacher was able to draw on in dealing with a pupil's error.

> The first idea she evidences consists in *confronting the pupil, and triggering some form of self-questioning with respect to the answer offered*: "*Maybe ask the child if his answer makes sense, his answer—that is, less than a whole number.*" This confrontation is centered on reflection over the size of the fraction, as part of an attempt at *constructing a meaning for the concept*. "*We'll take a look at each of the fractions that are added. 7/8 is practically a whole number, it's just a wee bit less. Then you add 1/2 to it...Now, 1/2 is smaller than 7/8 ... 7/8 is almost a whole number that you're going to add 1/2 to: does it make sense that when you've added them together, you still get less than a whole number? Because when you add something to it and you were already close to having a whole number...*"

Sylvie then takes a different tack, and now attempts to *construct a meaning to the addition of fractions* in the pupil. To do so, she will draw on various supports: an *illustration* (she makes drawings on her sheet of paper as she goes further into her explanation), which she combines with the *use of a context* in order to give meaning to both the fraction and the operation:

> "*I'd still go with representations. I mean, we have 7/8, a chocolate bar that I divide into 8. I made pieces of chocolate, my bar is divided into eight pieces. My friend got hold of another chocolate bar, but she took only half of it. She took half of the chocolate bar. Together, how many pieces of chocolate do we have? How much chocolate do we have?*"

Thus, she began to solve this problem in *context*; to aid her, she used not only an *illustration* but also a verbalization of the key stages of this problem-solving process. So doing, she attempted to *construct a meaning* for this solution in the pupil: in other words, she was attempting to show the necessity of using a common unit—the idea behind the common denominator.

Table 1

	Vincent	Sylvie	Stéphane	Louise
Review of a homework assignment (in the first of the solutions developed by the pupils, there was an error)	Centered on his own solution to the problem proposed; does not give consideration to student's suggested solutions	Tries to understand student's solutions, stresses the importance of language used (language becomes for her the object of teaching)	Is not able to "enter" into students' solutions and understand them, but would use them in class (to show various solutions); is open-minded regarding pupil's error	Analyses each solution and sees links between them; recognizes the value of the initial form of reasoning, even if it is mistaken
Reaction toward an error (ex. 7/8 +1/2 = 8/10)	Presents the algorithm	Presents the algorithm	Uses an example and a context	Intervenes on the size of the number in order to get the pupil to question his own answer
Lesson planning (ex. multiplication of fractions: 1/3 x 1/4)	Presents of the algorithm with examples and exercises with which to apply it	Uses a simpler example and a drawing to quickly introduce the algorithm	Uses an example and a context (however a complex choice here, related to probability)	Refers to the meaning of the fraction and its size
Reflection on a classroom intervention (lessons by 2 teachers), points seen as positive	Discipline: the teacher does not let the pupils say whatever they feel like right away	Frequently interacts with pupils (seen as positive), but not offer much positive feedback (seen as negative)	Frequently interacts with pupils good control of the class; openness to pupils' ideas; uses examples	Asks many questions; calls on pupils; encourages different positions; uses errors

> *"I have 7 pieces once my bar has been divided into 8 equal pieces. Now I'd like to know how many pieces I have, total; thus to be able to compare them, I'll have to make them into the same size."* Following this, she showed how to bring out this common unit. *"Thus, if I want to make her piece the same size as mine, I'm going to have to divide her chocolate bar into 8 pieces too, the way I did with my own bar. She already had 2 pieces. I'm going to divide each of her pieces into 4 equivalent portions, in 4 equal pieces, so that her chocolate bar is also divided into 8 pieces."* Then she verbalizes the equivalence: *"...Thus, the only piece that she took has now been divided into 4 because I separated both pieces into 4, which means that she now has 4 pieces."* Finally, she attempts to re-express the sum in relation to the initial total: *"she now has 4 pieces and I have 7. If we put them together, we'll have 11 pieces. But what does that work out to in relation to our chocolate bars? I mean, what size pieces do we have? Each of our pieces represents 1/8 of my chocolate bar, and we now have 11 of these pieces. That means we have 11 times one eighth of a chocolate bar, which works out to 11 eighths."*

At this stage of training, it is apparent this verbalization is still somewhat awkward.[1] But it is equally clear that this preservice teacher is able to rely on a wealth of resources to structure her intervention with the pupil; such supports include: illustration; use of a context; and verbalization in context, based on illustration. These same resources will be actively drawn on when this preservice student: structured a lesson (interview 2); planned a lesson during the first course in mathematics education or a sequence during the 3rd year of training. These same resources will all emerge out of her interventions among pupils when she was performing her practicum. Each case gave evidence of a conceptual analysis that was implicitly at work (in the case outlined above, analysis of the concept of fraction and addition—and of the key modes of reasoning to be developed—underlies her verbalization). The preceding analysis also brings out how this preservice teacher gradually worked toward a new way of viewing mathematics

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[1] For example, the reference to the whole is somewhat lost in the verbalization concerning the fraction (she said, "I have 7 pieces.." rather than, for example "I have a chocolate bar that I split into 8 equal pieces and from which I kept 7 pieces"). The verbalization of the equivalence is also awkward (she said, "I no longer have 1 piece but 4 pieces..." rather than "If I split my chocolate bar into pieces that are four times smaller, I will need four times more chocolate to come up with the same amount of chocolate").

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teaching and learning: it is no longer a question of imitating and applying a previously presented algorithm, but rather of constructing a meaning for concepts, that originates among the pupils.

Conclusion

As has been seen above, analysis of the interviews and other data obtained during training give evidence of a process of complexification occurring among student teachers' representations of the learning and teaching of mathematics. A didactic type of questioning appears to have led them to look at mathematics and learning differently and to find another way of addressing mathematics teaching. These preservice teachers appear to have developed a series of interventions that derive from articulated structuring resources (Lave, 1988), such as contextualization, representation, verbalization, conceptual analysis—all elements which were present throughout their training and which play a central role central in fostering an alternative teaching habitus among these future teachers.

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THE FORMATION OF EDUCATIONAL PSYCHOLOGISTS THROUGH A PROGRAM FOR HELPING CHILDREN WITH A LOW SCHOOL PERFORMANCE IN ARITHMETIC

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In this paper, the components of a teaching model are described which take up, as a starting point, the problem of a low performance in mathematics during the first and second years of grade-school, and the need is therefore seen to prepare psychology students to face such a problem in their future professional career. Some considerations are brought forth about the low grade-school performance and the role the educational psychologist may play in this respect; the research characteristics are described, as well as the teaching model; a mention is made of the procedures to evaluate the model, and some of the results obtained through various didactical experiences are also described.

The central purpose of the teaching model is to provide a group of psychology students at the *Facultad de Estudios Superiores Zaragoza, Universidad Nacional Autónoma de México,* with the theoretical and practical elements which may allow them to: a) understand the construction processes of arithmetical knowledge among children in the first and second grades of primary school, which their teachers consider to be low in school performance; b) to evaluate such processes, and also to identify the difficulties arising from the learning of certain arithmetical contents; and c) to design and apply didactical activities which might encourage a competent use of the children's arithmetical knowledge.

Theoretical perspectives

In the Mexican grade-school, low school performance shows a doubly selective status: it affects more severely those students from low-income families, and it shows a higher incidence in the first two school grades.

Unfortunately, an explanation still prevails concerning low performance which is based on certain pathological models focusing on supposed behavioral, linguistic or cultural shortcomings on the part of the pupil, and this determines not only that failure be attributed to him/her and his/her family, but also that instruction be more oriented to compensate for the alleged behavioral deficiencies (Khisty, 1995). Opposed to this, there is another position —with which we agree— where more emphasis is placed on the pupils' abilities than on their possible deficiencies; it focuses more on promoting their advantages rather than in trying to remedy any deficits, and its principles propose an effort to understand cultural patterns in order to avoid that differences become disadvantages. From this perspective, various studies have been carried out which have proved that a change in the conception held on low school performance, together with a change in instruction, result in an enrichment of the students' arithmetical knowledge (Fuson, Smith & Lo Cicero, 1997; Resnick, Bill, Lesgold & Leer, 1991).

Among the professionals working on the problem of low school performance we find school psychologists, who "can be of enormous assistance in the design of developmentally appropriate school programs for all children who are at-risk for school failure" (The University of California, 1995-99). Foremost among the functions of these specialists is the collaboration with teachers, parents and school principals in order to seek solutions to problems which might arise in relation with the children's learning process; they are also directly involved with the students or their relatives. To the extent that the teaching programs for psychology students includes information on the nature of arithmetic, the child's mathematical thinking, and the appropriate ways to evaluate and favor this thinking, psychologists will be better equipped to face the problem of low school performance. The model we now present falls within this perspective.

Characteristics of the research and of the teaching model

This research is part of a wider research project which centers on the curricular structure of psychology studies at the *Facultad de Estudios Superiores Zaragoza*, and is articulated around three axes: teaching, service, and research. The first axis aims at providing the students with the theoretical and practical elements which permit them to understand the acquisition processes of the required arithmetical knowledge for school success during the first two grades of primary school. By means of the service axis, help is given to children at-risk of school failure, and to their families, the latter being low-income households settled on a marginated urban area in Mexico City. Through the research axis, various studies are promoted —to be carried out by psychology students— with an aim to gain further insights on the problem of arithmetic learning and teaching in the first two school grades. Participating in the present research are a researcher, who is also the Professor in educational psychology, a group of ten students from fourth and fifth semesters in psychology, a group of ten children attending the first or second primary grades who are considered to be low-performance pupils, and these children's mothers (from here on, the researcher will be called the Professor, and the psychology students will be called the students).

Research activities are carried out at a multidisciplinary clinic in the *Facultad de Estudios Superiores Zaragoza*, where the children and their mothers go some hours after the children have attended their regular classes at the grade school. The construction and refinement of the teaching model have been achieved through the implementation of successive studies of a qualitative nature and a pedagogical intervention, within an action-research line where research is conceived as a cyclical process in which, starting from a given educational situation, a plan of action is designed; this is carried out, and a reflection is made on the intervention performed, to then initiate a new cycle which leads to a more solid posture than the preceding one (Doig & Hunting, 1995; McKernan, 1991). The teaching model comprises five components, which are summarized in Figure 1.

Component	Activities	Participants
Formation	-Workshops on the teaching and learning of arithmetic	Students and the Professor
Design	-The design of initial and final evaluations of children's arithmetical knowledge -The design of instruction activities addressed to the children and as a support for their mothers	Students and the Professor
Application	-The application of evaluations to the children -Application of instruction activities to the children	Students and children
Advisory	-Advice to students by team of two -Group advice to students -Group advice to mothers	Students, mothers, and the Professor
Report	-Preparing individual reports on the children -Preparing group reports on the children	Students and the Professor

Figure 1. Components of the teaching model

Six topics are considered in the formation component: a numerical series, counting, numerical relationships, a system of decimal numeration and positional value, verbal additive problems, and calculus. In each of these, the following aspects are taken into account: a) their arithmetical characteristics; b) the children's behaviors; c) the evaluation methods; and d) the didactical activities which are required to encourage certain strategies. As an example, in the case of verbal additive problems various types of problems are revised, as well as the strategies the children use to solve them, the different ways of evaluating such strategies, and the didactic activities aimed at promoting more advanced strategies. In the design component, the desired result is that the students prepare, or adapt an evaluation instrument to assess the children's arithmetical knowledge, as well as the teaching activities they will employ with these pupils; at the same time, they prepare diverse back-up activities for the children's mothers. Later on, within the application component, the students give instruction to their pupils by means of two types of sessions: a) individual, where one child and two students participate, and b) group-sessions, where both students and children participate together. The duration of these sessions is approximately one hour and they are held twice a week for about four months. Through the advisory component, the Professor guides the students in three central aspects: the analysis of information derived from the application of the evaluating instrument; the adequacy of both the chosen teaching activities and of actions undertaken by the students during the work-sessions with the children, and of the preparation of reports on the children's performance; also, the students advice the mothers regarding the necessary actions to improve their children's school performance. By means of the report component, the students submit a written account of the children's' changes in arithmetical knowledge as a result of the instruction they imparted.

Procedures employed to evaluate the model

With the support of a position which conceives the researcher as the main instrument in the collection of data and its analysis, and whose task "is to seek out the meanings in a situation with reference to declared interests or goals" (Jaworski, 1998, p. 117), the Professor selects various significant events according to his experience, then embodies them within his theoretical perspective and, through an analysis of them, assesses the efficiency of the different components of the model. The sources of information have been: the students' written work, video recordings of the work sessions, various evaluation tasks applied to the children, the children's school grades obtained, and the researcher's notes on different events that took place during the implementation of the model.

Results and discussion

There are various ways of approaching the achievements of the teaching model. Some reflections follow about four of these: the way the model was prepared, the effect on the formation of students, the change in arithmetical knowledge and the children's school performance, and the impact the model has had on the community to which it renders a service. To begin with, it can be asserted that one result of the research process has been the very construction of the model; this has permitted to prepare, organize, and refine the different activities and pedagogical materials that have been used both with the students and with the children. At the present time, various computer-based educational programs and video recordings of children's performances are available, which are used as pedagogical activities with the students. There is also available a bank of children-oriented activities which makes it possible to face the most common situations arising in the learning of the arithmetical contents of the primary-school early grades, and an evaluating instrument has been designed by means of which the children's initial state, their degree of advance, and the final state of arithmetical competences is detected. Secondly, a change has been observed among the students as to the concepts they held on the children's arithmetical behavior; this change is moving from a vision which explained it through individual features and manifest deficiencies, towards the appraisement of strategies that the child possesses, and the most convenient ways of encouraging the use of more elaborate strategies. This transition has made itself evident through the reports that the students deliver, where they describe the work-sessions with the children, and also thanks to the remarks made by the Professor concerning those sessions. An analysis of such reports and remarks allows us to appreciate the way in which a "training of the gaze" is being constructed, by means of which students are beginning to pay attention to aspects which heretofore they did not see or did not consider relevant; at the same time, a change is perceived in the language employed, in such a way that it becomes a tool favoring the children's learning. Thirdly, it is convenient to make it clear that a majority of the children who have participated in the activities of this model have improved their school grades and have been promoted to the next school year. In the last didactic experience, out of fourteen children whom their teachers had reported for being at risk of failing the school year, twelve were able to be promoted to the next grade. On the other hand, the application of the instrument used for the evaluation of the children's arithmetical knowledge allows us to assert that the majority of children have registered an advance at various levels. It has also been possible to determine, precisely, what are some aspects of arithmetic in which most of these children face difficulties. Such is the case of the positional value, and the posing of comparison verbal additive problems. Last, but not least, it should be pointed out that the model has had some effects in the surrounding community, which is reflected in the high demand of it coming both from family mothers and from school teachers working in the area which houses the facilities where the teaching model is applied.

Conclusion

Although it is still convenient to consolidate several aspects of the model, this can be considered to be a viable pedagogical strategy to make psychology students, within their professional functions, capable of promoting the arithmetical knowledge of children who have traditionally been considered as low school performance pupils.

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TECHNOLOGY, TOOLS, AND MULTIPLE REPRESENTATIONS: PRE-SERVICE TEACHERS' UNDERSTANDING OF FUNCTIONS AND MODELING

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Abstract: Using technology to improve student outcomes in mathematics is a challenge facing all pre-service teacher programs. In this paper we make a preliminary report on a new pre-service teacher program. Specifically we analyze an item from a pre/post-test assessment given in a course on functions and modeling designed in line with the generative domain knowledge framework.

Introduction

The inadequate and ineffective use of technology in the classroom is well documented. The President's Committee on Advisors on Science and Technology (PCAST), Report (PCAST, 1997), states "...new teachers typically graduate with no experience in using computers to teach and little knowledge of available software and content." The Office of Technology Assessment Report (OTA, 1995) describes the situation as "Overall teacher education programs in the United States do not prepare graduates to use technology as a teaching tool." A recent study by the ETS Policy Information Center (Wenglinsky, 1998) found that the way in which technology was used in the classroom made a significant difference on student achievement. The more time students spent using computers to perform lower order tasks, the lower their outcomes in mathematics.

The University of Texas at Austin has responded to this situation by designing a new teacher preparation program through a close collaboration between the College of Education and the College of Natural Sciences. An important part of this program is the design and development of exemplary domain courses. The theoretical framework underlying these courses is that of *generative domain knowledge* as introduced by Confrey (1999). The first such course to be implemented is on "Functions and Modeling". This course is designed to focus on topics typically found in algebra, trigonometry, pre-calculus, and calculus. One of the major themes is that of modeling and having the mathematical structures emerge from the phenomena. Active engagement in mathematical modeling plays a crucial role in framing and mediating the students' experiences. These pre-service teachers investigate a wide variety of problems ranging from sequences of integers to computer simulations of population dynamics to analyzing bouncing balls with motion detectors. Specific examples from this course are discussed relative to the generative domain knowledge framework in Callahan & Confrey (1999).

This study will evaluate the impact of this new course on pre-service teachers' understanding of functions and modeling. We designed and administered a pre-test/post-test based on the ideas and framework developed and used by Monk (1989) and Carlson (1998) to assess students' understanding a functions in a variety of contexts. In this paper we will focus on the added dimension of technology and its impact on the students of the course of the semester. We will look at the interaction between the mathematics content and the use of tools and technologies in the classroom. We will also discuss how the tools and representations shape the mathematics itself.

The Class

The "Functions and Modeling" course was piloted in the fall semester of 1998. This paper will study the spring 1999 offering of the course. This class had 30 students, of which 21 were part of the pre-service teacher program. The other nine included undergraduate mathematics majors not interested in pursuing a career in teaching as well as several graduate students in mathematics education and one graduate student in mathematics. The course is required for students in the pre-service teacher program. Other students took the course as an elective. The only prerequisite for the course was to have had a calculus course or be currently enrolled in calculus.

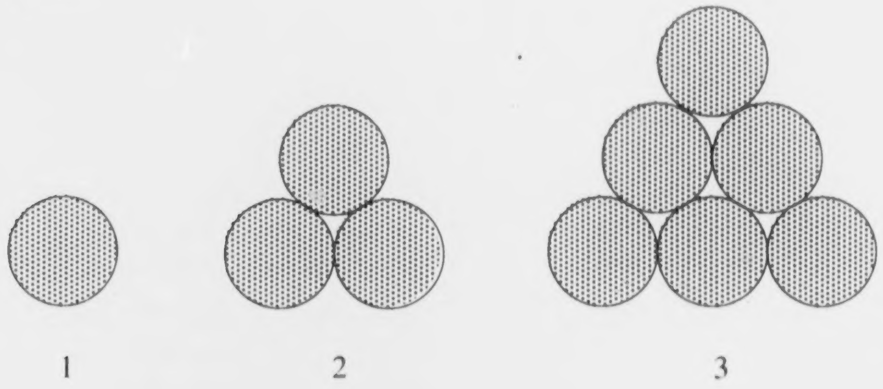
The class met twice a week. All classes were held in a computer lab. The computer lab was equipped with about 15 Power Macintosh computers. Some students brought their own laptop computers or graphing calculators. An important factor in the design of this course was to create a climate where the technology served as a tool for inquiry. The students would typically work on activities and hold discussions in groups away from the computers. The students would go to the computers when they wanted to test conjectures, analyze data, or explore certain situations.

Results

We will focus on one item of the pre/post-test relevant to qualitative graphing, pattern structure, and exponential functions.

This item involved finding the terms in a sequence of integers. We were interested in the students' ability to recognize and extrapolate a pattern as well as formulate an expression for the general term of the sequence. Below is the problem as it appeared in the pre-test.

Problem 1. Consider the sequence generated by the following pattern of coins:



a) How many coins are in the fourth term of this sequence?
b) How many coins are in the tenth term?
c) How many coins are in the nth term?

A similar problem involving pentagonal numbers was given on the post-test. In both cases we used a simple 5 point rubric similar in style to that used by Carlson (1998). For parts a) and b) 5 points were given for a correct solution: 10 and 55 coins respectively. Students who demonstrated some understanding of the pattern, e.g. drew correct pictures, but made minor errors in counting or adding were given 3 points. For part c) students were given 5 points for giving a function equivalent to $\frac{1}{2}n(n+1$. Students who arrived at quadratic solution, but made arithmetic or algebraic errors in their formulation were given 3 points. Students who expressed the solution recursively or as a sum $1+2+3+\ldots+n$ were also given 3 points. Table 1 summarizes the results.

Table 1

Item	Pre-test Mean	Post-test Mean	Percent Change
1a	4.7	4.9	+4%
1b	4.1	4.5	+8%
1c	2.5	4.1	+32%

The small changes in the first two parts are mainly due to the fact that most of the students answered these correctly both times. It is interesting to note that several students who successfully gave a general answer in part c) failed to give a correct specific answer in part b) because of arithmetic and algebraic errors. For the most part, students were able to recognize the patterns in the sequences. It was the ability to express the pattern symbolically as a closed function that presented a great deal more difficulty. Many of the students during the pre-test used various ad hoc approaches, including nonsystematic guessing. For the post-test, however, there was evidence of a more structural approach.

The structural ideas used here have their genesis in the use of tables, including tables of differences and tables of ratios. See Dennis and Confrey (1993) for a discussion of the historical uses of tables and their current applications to mathematics education. We used this research to inform the design of the course. In particular, we emphasized the use of the table features of *Function Probe* (Confrey and Maloney, 1998). It here that it became clear how the tool and technology was actually shaping the mathematical content. *Function Probe* allows for the user to interact dynamically with a table of finite differences. The sequence and the associated sequences of differences and higher order differences are represented a columns in a table. In turn, these columns become objects that the students act upon and manipulate. For example, in the sequence in the item above, students note that the second differences are all equal to one. Knowing that the prototypical function has constant second differences equal to two, the students conclude that the sequence is quadratic with leading coefficient equal to one half. Subtracting the degree two term from the unknown sequence yields a linear sequence that can then be handled in the same fashion. These are interesting structural actions facilitated and shaped by the medium of analysis, in this case, *Function Probe*. The question of what is the "content" of Problem 1 is not straightforward. The content depends on the tools and the technology.

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MOTION DETECTORS IN THE SCHOOL: TEACHERS MAKING SENSE OF RATE OF CHANGE AND MOTION[*]

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This study discusses how the use of computer-based motion detectors helped one teacher develop his own ideas about rate of change relative to velocity and position concepts. One teacher was interviewed before he taught a lesson on qualitative graphing using motion detectors. We discuss how the technology and activities helped the teacher develop a deeper notion about velocity and position. We conclude that the innovative approach created learning opportunities for this teacher.

Introduction

The use of rate of change is prevalent in secondary mathematics and includes a wide variety of subjects and applications. It is also the precursor to more complex problems that deal directly with differential and integral calculus. Current research defends the earlier introduction of rate of change in the curriculum, such as an entry point to think about functions (Confrey & Smith, 1994).

Some studies on students' understanding of rate of change have shown that by using computer-based motion detectors, the ideas of distance, speed, and acceleration can be experienced through modeling motion (Nemirovsky, Tierney, and Wright, 1998). Before one can fully understand how teachers' actions affect students' learning, one must first investigate the depth of conceptual understanding in the teachers themselves. Therefore, we need to study what conceptions teachers have about rate of change. In this paper, we briefly explore the development of a teacher's understanding of rate of change while implementing a lesson on qualitative graphing using computerized motion detectors. A longer version will be available at the conference.

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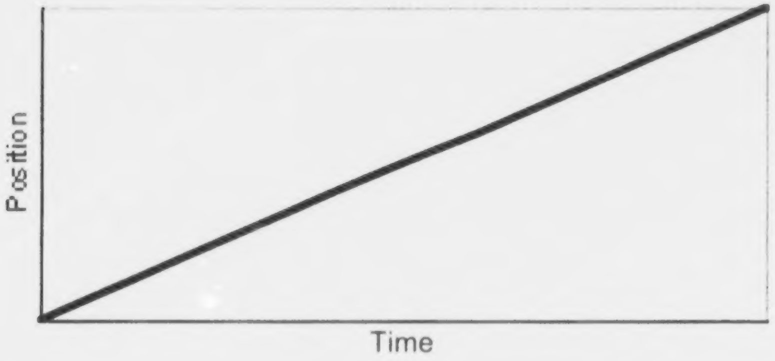
Methods

The study was conducted at Tree High School, an urban school in central Texas with a majority population of Hispanic students. Eight math teachers were implementing a replacement unit in the Algebra I curriculum, spanning from qualitative graphing to linear functions and simultaneous equations. Motion detectors were used to introduce the idea of slope as a constant rate of change. A handout was provided consisting of a series of qualitative graphing activities designed to investigate situations involving motion. The situations included walking at different rates and in different directions.

In this paper, we will focus on how one teacher, Felipe, developed his concepts of rate of change relative to velocity and position during an interview conducted by the first author. The interview intended to probe teacher's ideas about the activities and his plans for the lesson[1].

Graph 1



Results

This interview was conducted before motion detectors were used in the classroom. We started by looking at the handout and asking for his ideas about the questions. The first question was "describe how one would walk to produce the following position versus time graph":

Felipe accurately described the walking, and he also demonstrated it using the motion detector. However, he became confused on the third question which asked to predict the corresponding velocity versus time graph. He explained that a velocity versus time graph means a graph produced by a motion, and that a graph of position versus time means a graph produced by a series of positions. Felipe seemed to think that the position versus time graph is formed by a series of discrete points produced by multiple people or objects where no motion occurs, and that the velocity versus time graph is a continuous graph where motion does occur. The interviewer probed him on this matter, asking him to produce what he was calling the position versus time graph using the motion detector. In response, he lined up a series of mouse pads in front of the motion detector and dropped each one at different intervals of time, creating a step graph.[2] He seemed to think that the position versus time graph is only a series of discrete points and that the velocity versus time graph is that same position versus time graph, but made smooth or continuous. This interpretation is consistent with his idea that the motion detector works by taking pictures of individual positions, but those points are so close that they look continuous on the computer screen.

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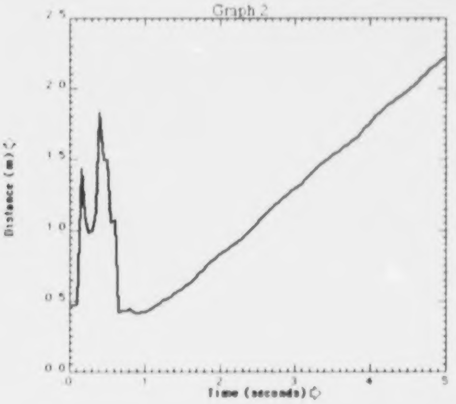
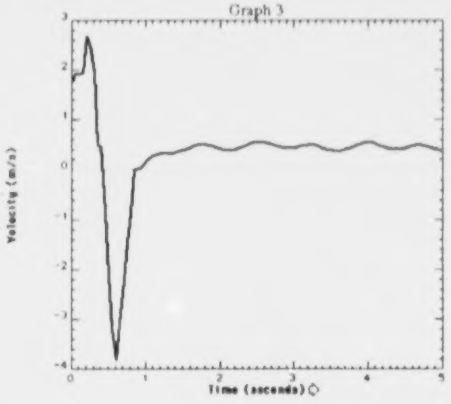
[1] A variety of data was collected as part of the first author's dissertation study.

The interviewer then suggested that Felipe walk away from the motion detector at a constant velocity while both position versus time and velocity versus time graphs were being displayed on the computer screen (see graphs 2 and 3). Felipe was able to explain what the graphs were showing:





> I can see why the distance [graph 2] is gonna keep [going] on, the further I get, the higher is gonna go...And this [graph 3] I understand too because it is a constant, I was going at a constant velocity so its gonna be the same.

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[2] In a previous workshop, Felipe had participated in producing a step graph through lining up people or objects at different positions in front of the motion detector.

However, Felipe was confused with the idea that one motion could produce two separate graphs: "See, I need to think why is that we are gonna do the same walk, one to measure position versus time, then do the same walk to measure velocity versus time". He understood that velocity could be obtained from a position versus time graph, by finding its slope. In fact, he thought that this made more sense than to have a separate velocity versus time graph. He explained how one can find the velocity in a position versus time graph and began to understand that there are two different quantities (position and velocity) which are represented in two different graphs for the same motion.

> For every meter, you go one second. Or other meter, you go another second. And then, when you put a line to it, that's [it]. This is gonna be that graph. And you are saying distance to time, which is speed[3] ...And that's why I expected, when they say, well, graph the speed. Well, that one right there [graph 2] is a speed graph...The bottom [graph 3], just show you, you are comparing the velocity from one second to the next. Cause it stays constant... And I understand that's what it tells. But when it asks where is the velocity versus time, I might be inclined to say, oh, where is the velocity graph? That's it right there [pointing to graph 2]. And I guess you have to say, where is the velocity versus time graph. That's this one [graph 3].

The development of Felipe's understanding can be summarized as follows. He initially thought that a graph of position versus time could only be produced by lining up people or objects in front of the motion detector, but with no motion. He also thought that motion was only necessary to produce a graph of velocity versus time. After visualizing both graphs on the computer screen for one single motion, he realized that the graphs actually represented different comparisons. One graph was comparing position to time and the other graph velocity to time. However, he was still intrigued about why one would need two graphs when velocity is constant. In this case, he argued that only the position versus time graph would be necessary.

Discussion

A superficial interpretation at Felipe's ideas could be that he had misconceptions that needed to be corrected. An alternative interpretation is that his ideas were in development and needed more reorganization and refinement, rather than replacement (Smith III, diSessa, and Roschelle, 1993). To examine this alternative, we need to discuss the possible sources of Felipe's ideas.

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[3] Felipe at this time is making no distinction between speed and velocity, or distance and position.

One possible source comes from Felipe's awareness of the dynamics of motion detectors. Felipe understood that the motion detector takes discrete data of distances away from the motion detector and then displays a continuous graph. Even though that created some difficulties for his interpretations, it suggests a deeper understanding of the dynamics involved in the use of motion detectors. Another possible source may be the fact that the activities used the linear case, that is, walking with a constant velocity to produce the position versus time graph[4]. Felipe's questioning of the need to use two different graphs to represent the same action is valid in the case of linear graphs. Felipe was able to find the velocity from a position versus time graph, therefore his conception was not a limitation but a reflection of his deeper understanding of constant slope. Given that the slope of a linear graph is constant, one could argue that the position versus time graph also shows the velocity, and therefore a separate graph of velocity versus time is not necessary. (Stroup, in progress). Bowers and Doerr (1998) found a similar interpretation in a student drawing of graphs of position versus time and velocity versus time in a constant velocity context.

We discussed Felipe's ideas and how they evolved during his experience with motion detectors. To conclude we want to point to some important aspects which helped Felipe develop his understanding. Evidence was shown that the materials and the technology aided Felipe's understanding. In fact, the real time graphing was vital to his development since observing the results of his actions allowed him to test his conjectures. Seeing both graphs of position versus time and velocity versus time was so important to him, that he argued for using both graphs together in the same lesson.

Our study points, not only, to the need of developing teachers' knowledge but also to some potential solutions. We explored how innovative approaches and technology can help to develop teachers' knowledge. We also discussed how apparently limited ideas can reflect a deeper understanding of other issues, and how those ideas can be developed. We explored the concept of rate of change as a particular example, but we argue that similar results could be found in other content areas.

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[4] In another interview after the meeting, Felipe pointed out that most of the walking done was with a constant speed.

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LIVING MATH HISTORIES: THE INFLUENCE OF TEACHERS' PRIOR MATH EXPERIENCES ON THEIR IMPLEMENTATION OF A REFORM MATH CURRICULUM

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This paper utilizes elementary teachers' narrative constructions of their prior experiences learning and teaching math (their math stories) to understand their implementation of a reform math curriculum. Each of the teachers' stories has one or two prevalent themes, such as "math has meaning" or "math is about tricks and rules" that frames the way they think about and teach mathematics. These themes help explain the ways in which the teachers interpret and implement the reform curriculum. This research has implications for both the design and the implementation of math reform curricula and policies.

Introduction

This study investigates the connections between teachers' prior experiences with math and their approaches to using a reform math curriculum. It builds upon earlier research which indicates that teachers' beliefs affect their practices (Spillane, 1995), though often in ways that are not readily apparent (Cooney, 1985). Using case studies of several urban teachers implementing a new elementary math curriculum, we found strong connections among teachers' stories of their previous experiences learning and teaching math, their current teaching practices, and their stated goals for their students in math. The teachers had coherent stories to tell about math which stretched from their pasts to their presents, with implications for their futures. These stories resemble the memory structures described by Lofgren (1989):

> The cultural organization of memory has to do with forms of remembering, forgetting, blocking, selecting, but also with ways of condensing or crystallizing certain events and situations, which become charged with strong symbolic content. Seemingly trivial events or everyday details of the past may come to represent turning-points or decisive moments, important victories or failures. They become in a manner key symbols or stand for central values of ambitions (Lofgren, 1989, p.147).

As a result of this coherence among teachers' stories, goals, and practices, these stories can add substantially to our understanding of each teacher's interactions with the curriculum and of the unique implementation processes observed in their classrooms.

Methods

The data for this study were collected at two urban elementary schools, one small and one large. Both schools serve primarily Latino populations of mostly English-speaking students. Intensive case studies of eight teachers were undertaken. One was teaching Grade 1, three were teaching Grade 2, three were teaching Grade 3, and the other Grade 4. The teachers varied with respect to gender, culture, and years of teaching experience. All eight teachers were implementing a research-based curriculum, *Children's Math Worlds* (CMW) (Fuson, 1998), designed to support children's and teachers' mathematical understandings. The teaching of math was observed one or two times per week for the entire school year. Extensive notes were taken, and most sessions were videotaped. Teachers were also interviewed frequently about what they were learning from their experiences implementing the curriculum. Selected videotape portions and all teacher interviews were transcribed. In addition, each teacher was interviewed about his or her previous experiences learning and teaching math (their math stories). Our math story interview protocol was adapted from Dan P. McAdams (1993) Life Story Interview.

The data were analyzed using an iterative process by which the math story interviews were analyzed for evidence of significant themes and plots and relevant classroom evidence was examined in light of the issues raised by the interviews. As clear themes, such as "math has meaning" or "math is about tricks," began to emerge, the individual data sources (math stories, classroom observations, and post-observation interviews) were analyzed a second time to either support or disconfirm the validity of the themes. This process was repeated for each teacher.

Results

Several of the teachers' stories are described below. They illustrate three ways in which teachers' personal life stories relate to their implementation of reform curriculum. The first example illustrates a teacher whose experiences with learning math impressed in her the value of making sense of math and using everyday situations to reinforce this sensemaking. The second example depicts a teacher who struggled to build his own self-confidence in math and as a result of his success believes that all of his students can be successful as well. The third story illustrates the influence of a teacher's understanding of math as a series of tricks and rules on both her interpretation and her implementation of the reform curriculum.

Math Has Meaning

Important in the development of Ms. Martinez's beliefs was a particularly successful experience in high school geometry. In this geometry class she was asked to defend her thinking about math for the first time. She believes that having to defend her thinking helped her learn math in a deeper way. These experiences affected Martinez's core convictions about the way people learn math:

> Like, if they use word problems - just bring together the word problem and the math, it just brings everything together in the picture. It involves all sorts of thinking skills, and I think that it will trigger something in the kids so they'll have a better understanding of math. Like, with geometry, we had to draw the picture and show what we did, and just that in itself, I think, was very helpful.

This experience has led her to value the process of sensemaking in mathematics. As a result, she has been eager to implement that core aspect of the CMW curriculum whenever it is appropriate.

A later experience teaching adults for the Mayor's Office of Unemployment and Training solidified in Ms. Martinez's mind the benefit of using situations from everyday life to illuminate math principles:

> They were like, "well, what does this have to do with this?" I'm like, "you have to look at it in an everyday type of situation." So we had to come up with scenarios that had to do with everyday life experiences which they could apply math to. 'Cause - at first - they didn't really get it. So many people - at the end, "oh, I finally got it!"

Martinez's personal experiences with math helped her to formulate a personal philosophy about what is involved in meaningful math teaching. She believes it is important to elicit students' thinking about math and feels that sensemaking can be scaffolded by connecting math to her students' worlds. This philosophy is consistent with CMW and this consistency allows her to implement the curriculum in the spirit in which it was designed. Her own math-content background is somewhat weak. However, since the stance she takes in her classroom is one of a sensemaker herself, she has been able to learn about math and about CMW throughout the implementation process.

Math In Life

Learning math has never been easy for Mr. Gomez, but he has worked hard to conquer his fears and perceived inadequacies. While living in Mexico, he studied to become a veterinarian. His first job after college involved preparing the diets for animals on a large farm:

> So I needed to do a lot of math. When I got the job ... "this is what you got to do." I was ... "oh no." But, little by little I learned how to do it. . . The level of protein you've got to keep in this diet, so you need to put more something from animals, or some units - very complicated to do that. In the beginning, I was afraid. I didn't know what I was going to do, but I'll do it. . . Because I wanted to do the job right, so I took my time. . . I went back to the library and I was reading, and studying in the night. I was trying to do my best.

The fact that Mr. Gomez has been able to overcome obstacles in his pursuit of understanding math has led to his belief that all of his students can be successful with math.

Mr. Gomez also firmly believes in tying math concepts to the "real world." He attributes his learning of math as a child to experiences he had outside of school. Mr. Gomez grew up as the eldest son of a rancher in Mexico:

> O.K.... "we are going to sell this cow and then we can buy this and this and this." They used to do that all the time. So I would say, "we sold 2 cows and they paid us 200 - we can do this and this and that". . . it was my life. . . And at that time we bought animals - to buy and sell to make a profit, and it was important for me to know. We'd have to sell 1,200 in order to make a profit and it's what my father was doing so, for me, it was important to learn how to do all this. They say "you know how to do all these numbers". . . My father was sharing that with me because I am the oldest brother, so I need to know what's going on 'cause I have to help him to manage affairs.

One of the key teaching strategies in the CMW curriculum is to have students make up math stories from their lives and then to have the class solve the problems that result. Gomez has fully embraced this pedagogy and believes that it is the best way to teach math to his students. He often tells his students stories about the ranch back in Mexico and he and his students create problems from these stories, as well as from stories about the children's lives. Because Mr. Gomez's personal story and beliefs line up closely with the foundational goals and activities of the curriculum, implementation of the curriculum in his classroom has been very successful.

Math Tricks Up My Sleeve

Ms. Gordon, a third-grade teacher, consistently refers to her underlying belief that math is best understood as a series of tricks and rules. This belief is apparent in many of her memories of learning and teaching math and also in her descriptions of her strengths as a teacher:

> My strengths as a math teacher, I would say, I do have tricks up my sleeve I guess, I do have alternative ways of teaching, you know.

For Ms. Gordon, the alternative ways of teaching are simply means to achieving the end of having students learn whatever rule, trick, or skill is being taught. In fact, it seems that this teacher views the curriculum as simply one more of these alternative means. In several observed lessons, Ms. Gordon would teach the lesson more or less as written until the end of each lesson when she summarized the day's activities with a "rule" to help the students remember and apply what they had learned. This "rule" was always developed by the teacher and never by the curriculum. For instance, one lesson was concluded with the rule, "Always borrow when you have a zero on top," which, though not entirely correct and certainly not the way the concept is presented in the curriculum, is clearly the way this concept is understood by the teacher.

In subtle ways, this understanding of math is also apparent in this teacher's descriptions of her previous experiences with math. She reports having been very good at math in grade school, but then hating math in high school:

> ...once the algebra hit, that was a bad experience for me. It just threw me, you know, the formulas, formulas are a nightmare...geometry formulas, I hated. I hated, I feared it, I feared tests, it was horrible.

Clearly, math became too difficult for her when the "rules" became too complex. When asked to describe a memorable experience in teaching or learning math, this teacher again refers to the need to know "tricks" in order to teach math:

> I've had a couple of experiences in the past where, uh, especially last year, just none of them were getting it. And that's a scary feeling. 'Cause you always have to have back-up tricks up your sleeve.

Ms. Gordon was distressed when she felt her students were not prepared with enough tricks or rules to pass the standardized tests; however, she resolved this by spending several days instructing them in "tricks," such as looking for key words when solving word problems. This was not part of the curriculum; in many ways, it is antithetical to the philosophy of the curriculum. However, this practice is quite understandable when it is viewed as an integration of the curriculum's directives with Ms. Gordon's own personal, and deep-seated, philosophy and beliefs about math.

Conclusions and Implications

This study was designed to provide a better understanding of how and why individual teachers interact with curricula and teach their math classes in different and unique ways. Our results indicate that one way to understand the teachers' different patterns of curriculum use is to understand each teacher's past experiences and current beliefs about teaching and learning math. Knowledge of these experiences and beliefs helps us gain insight into the particular choices and adaptations made by these teachers as they implemented the CMW curriculum. It is also clear that certain math stories may support teachers' attempts to reform their math teaching and teach for understanding, while others may hinder these efforts. At the same time, other research with these math stories has shown that they may change dramatically over time as the result of teachers' having new experiences and arriving at new understandings (Drake, Spillane, and Hufferd-Ackles, 1999). Finally, this research supports the conjecture that the math story interview may be a particularly useful method for understanding teachers' belief systems, one which reduces the frequently cited disparity between teachers' professed and attributed beliefs (Cooney, 1985).

These results raise a number of important issues for further research and consideration. First, although this paper suggests some clear linkages between teachers' past experiences with math learning and teaching and their current math beliefs and teaching practices, these linkages need to be studied much more closely by extending this research to teachers in other contexts. The linkages may be quite different for teachers in other school settings, at different grade levels, and teaching different subjects. Furthermore, the long-term implications and potential for effecting change in these stories requires in-depth investigation. Finally, the implications of these results for the design of math curricula warrant further study, as it is clear that teachers with different math stories have markedly different approaches to curriculum interpretation and implementation.

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ALTERING ELEMENTARY EDUCATION STUDENTS' CONCEPTIONS OF MATHEMATICS

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The vision of school mathematics driving current reforms will require teachers who possess quite different world views about mathematics and its learning and teaching from those common in the past. Fundamental to such a change in viewpoint is a deeper understanding of mathematics than most teachers currently possess. The researcher studied elementary education majors in two mathematics content courses he teaches in order to learn whether students' course experiences affected their views of mathematics. Data were drawn from two sources, students' evaluations of the course and their course portfolios, and were analyzed to determine the extent to which students' views of mathematics became less dualistic and more experimentalist. The data analysis reveals that some students are moving toward a more dynamic, problem-solving view of mathematics. Moreover, there is also evidence that some students are beginning to see as important connections between conceptual and procedural knowledge.

The *Standards* documents published by the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 1995) articulate a vision of school mathematics education which "represents a radical departure from traditional mathematics classes" (Simon, 1994, p. 72). The *Professional Standards for Teaching Mathematics* (NCTM, 1991, p. 3) presents five major areas of change in instructional practice that are necessary to bring this vision to life in classrooms and notes that each of these areas of change will require teachers who possess quite different world views about mathematics and its learning and teaching from those common in the past. Fundamental to such a world view is "a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess" (Schifter, 1998, p. 57).

Cooney (1994) discusses reform in mathematics teaching in terms of adaptation from what one is able to do to what one wants to do. After noting a connection between what a teacher is able to do in a mathematics classroom and the teacher's knowledge of mathematics, he asserts that "in the absence of such knowledge, the process of adaptation is severely limited, if not impossible" (p. 10).

The researcher teaches in a teacher education department in a large university in the midwestern United States. He often teaches mathematics content courses for students preparing for K-6 teacher certification. In addition, he has developed and teaches a content course for students who intend to extend their K-6 certificate by adding a mathematics K-8 endorsement. In order to help these students begin to develop views of mathematics that are less dualistic and more experimentalist (Wilson & Goldenberg, 1998), while simultaneously coming to a richer understanding of some of the conceptual complexity of the mathematics underlying the elementary curriculum, the researcher featured student mathematical explorations (Schifter, 1998) as the primary mode of instruction in these courses. The purpose of this study is to investigate the degree to which such an instructional approach was successful in achieving these goals.

Wilson and Goldenberg (1998) propose a model of intellectual development as a tool for analyzing inservice teachers' struggles as they attempt to align their practices with current reforms. Their framework, which builds on and extends Perry's scheme, uses four broad categories of intellectual development: dualism, pluralism, extreme relativism, and experimentalism. They describe dualism as essentially a world view that sees everything as right or wrong, with no meaningful middle ground between the extremes. From a dualistic perspective, there is an absolute authority which resides outside the individual. Pluralism is defined as a world view which begins to see some "gray areas," but within an essentially dualistic framework. However, pluralists still tend to believe that from amongst a number of possible choices, there must be one best choice. Pluralists also tend to hold the dualists' belief in an external source of absolute authority. Wilson and Goldenberg characterize extreme relativism as "pluralism shifted radically away from dualism" (p. 272). The extreme relativist also sees many possible viewpoints, but views them as equally legitimate. Although the extreme relativist sees authority residing within the individual, making effective choices from among myriad possibilities may be quite difficult. Finally, experimentalism is described as a world view which accepts that, while there are many possible viewpoints in every situation, some viewpoints may be ineffective, and others are best avoided. From this perspective, the authority for making choices is internal to the individual, and an experimentalist will continue to maintain a particular position only so long as it remains useful and effective.

Ernest (1989) suggests three qualitatively different philosophical views of mathematics that are observed in the teaching of mathematics: a platonist view, an instrumentalist view, and a problem-solving view. In Ernest's scheme, a platonist views mathematics as a static, but unified body of knowledge consisting of interrelated structures and truths. An instrumentalist views mathematics as a useful collection of essentially unrelated facts, rules, and skills which can be brought to bear on a wide range of human endeavors. From both of these perspectives, mathematics is typically thought to be discovered, and teachers holding such views may insist on "one right way." On the other hand, from a dynamic, problem-solving viewpoint, mathematics is seen as a continually expanding field of human inquiry that is not a finished product, but open to revision. From this perspective, mathematics will typically be seen as created, rather than discovered, and a teacher viewing mathematics from this perspective is more likely to accept students' methods and approaches as valid.

The foregoing theoretical perspectives on intellectual development and the nature of mathematics are useful in analyzing students' views of mathematics as well as the depth of their understandings of the content they are studying. In particular, platonist and instrumentalist views of mathematics seem to mesh with dualist or pluralist categories of intellectual development, while a problem-solving view of mathematics seems a good fit with the experimentalist category of intellectual development.

The subjects in this study were all elementary education students in one of two mathematics content classes taught by the researcher: a standard "math for elementary teachers" course, and "number theory and algebra for middle school teachers." In both of these courses, students spend a large proportion of the available instructional time in small group investigations of rich mathematical situations. For example, in the first course, students investigate the infinitude of the primes, the countability of the rationals, and the uncountability of the reals, while in the second course, students investigate the relationship between the division algorithm and the Euclidean algorithm, completely characterize primitive Pythagorean triples, and develop solutions in integers for linear Diophantine equations and connect those solutions to finite continued fractions.

Approximately 75% of the subjects in the study were undergraduates at they time they were students in these courses. The rest were beginning master's students in elementary education. Most of the subjects are female, and a fair number of both the undergraduate and graduate students are "non-traditional" students.

Data were drawn from two sources: student portfolios and student responses to open-ended items on university-wide Student Evaluation of Teaching (SET) forms. The student portfolios were required as part of the course assessments and constituted 15-18% of their grades. In the portfolios, students self-selected the three items which best demonstrated their growth in learning and provided a written rationale for each selection. In selecting items, students could choose from among classroom investigations, homework items, questions on tests or quizzes, or "extra credit" challenges.

The data were analyzed using the method of constant comparison (Glaser & Strauss, 1967). In particular, student comments from the SET forms and the portfolio rationales were searched for evidence of movement toward a problem-solving view of mathematics (Ernest, 1989). In addition, the intellectual development reflected in these comments was classified using the categories proposed by Wilson and Goldenberg (1998).

The analysis of these data appears to provide evidence that the goals of helping elementary education students develop more experimentalist views of mathematics, while at the same time gaining an appreciation of the conceptual complexity that underlies much of elementary school mathematics are being at least partially met in the two courses. Some students seem to have progressed toward a problem-solving view of mathematics and the experimentalist category of intellectual development.

Analysis of comments on the SET forms from the two courses indicates that a number of students are beginning to move from platonist or instrumentalist views of mathematics toward a dynamic, problem-solving view. In doing so, they seem also to have developed a deeper understanding of the content they studied, as well as movement toward the experimentalist category of intellectual development. For example, students wrote of learning to seek patterns and connections, asking themselves "why" questions for the first time, and considering more than one solution strategy. Typical comments included:

- "This course has really caused me to look at math as a series of patterns and now I ask why a lot more."
- "I learned to consider more than one strategy to accomplish a problem."
- "The way topics lead so easily into the next helped me to understand."

Analysis of student portfolios from the two courses provides confirming evidence. For example, in constructing their rationales for the inclusion of items in the portfolio, students wrote things like:

- "I started looking for some meaning and patterns."
- "It made me realize and think about how there can be more than one way to solve a problem."
- "I made a connection between Diophantine equations and continued fractions."
- "This problem helped me to understand the importance of asking myself, 'Why does this work?'"

Both sources of data also provide direct evidence that some students may be changing their view of mathematics. For example, on the SET form, one student wrote:

- "The course brought a new way of looking at math. It taught me to think in a logical and systematic manner, rather than compute things by rules that made no sense."

The SET forms contain a number of similar comments, and the student portfolios again provide confirming evidence:

- "I learned a new way to look at numbers."
- "I learned how to increase my understanding of mathematics by prodding myself to look deeper at a rather simple problem."

Finally, there was some evidence in the portfolios that some students have begun to understand an important connection between conceptual and procedural knowledge:

- "Once you understand how and why something works, you no longer have to memorize the formula."
- "To better understand the algorithm, I had to also study the proof."

Due to the nature of the data used in this study, it proved difficult to classify students into one of Wilson and Goldenberg's four categories of intellectual development or as holding one of Ernest's three views of mathematics. What was clear, however, is that a fairly large number of students are offering unsolicited comments which indicate that they are beginning to "see" more than one right way. This implies that these students are beginning to move to a more experimentalist and less dualistic category of intellectual development, in Wilson and Goldenberg's scheme, and toward a more dynamic, problem-driven view of mathematics in Ernest's scheme. In fact, there may be a number of student's who now share the dynamic, exploratory viewpoint offered by one student on the SET form: "To me, our class really focuses on the roots of mathematics. Why does it work? How does it work? If we manipulate this or that will it still work? If we try this, will we discover something new?" Furthermore, the connections that some students are making between conceptual and procedural knowledge are noteworthy. Hiebert and Carpenter (1992) have posited the importance of such connections to the understanding of mathematics.

In the best of all worlds, most students would be making such comments and writing such rationales, but, of course, that is not the case. Comments such as the following also appeared on the SET forms:

- "This math class was too intense."
- "I do not see how these concepts can be used to teach."

- "We need more examples and solutions."
- "The textbook questions were worded differently."
- "I do not like writing papers for math class."

A natural extension of this research will be to push the analysis in an attempt to learn what is different about those students who are apparently changing their views of mathematics and gaining a deeper understanding of the content and those who are apparently not. Perhaps conducting interviews with some of those students who do seem to be changing their views of mathematics after the completion of the course would shed some light on this question. Moreover, interviews might provide the additional data needed to classify students using either Wilson and Goldenberg's or Ernest's theoretical schemes. The reason some students begin to change their views of mathematics may be tied to students' categories of intellectual development. It may be that as students become less dualistic and more experimentalistic, they begin to alter their views of mathematics toward a more dynamic, problem-solving view of mathematics that enables and enriches a deeper conceptual understanding.

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SUPPORTING TEACHER LEARNING VIA CURRICULUM MATERIALS

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This paper examines how curriculum materials can support teacher learning. Three teachers were observed and interviewed throughout one school year as they implemented a new reform-based mathematics curriculum. Our analysis revealed three areas in which teachers learned while implementing the new curriculum: (a) content, (b) classroom processes, and (c) interpreting curriculum. All of the teachers developed new understandings in each of these three areas. However, the area in which initial learning took place differed among the three teachers. Furthermore, the teachers cycled through learning in the subsequent areas in different ways. Based on these findings, we suggest that curricula be designed to provide different entry points for teacher learning. In addition, however this initial learning takes place, we find that teacher learning is an ongoing process. Therefore, we recommend that curriculum materials seek to make connections across learning in different areas explicit for the teacher.

Much of the reform effort in mathematics education is focused on helping students build flexible and deep understandings of mathematics. Teachers are seen as a critical agent in this process, for it is they who bring reform to the classroom, interpret and implement new materials, and direct students' learning. Yet, while reform measures emphasize the constructivist nature of students' learning, the same issues are not always addressed for teachers. This paradox is particularly evident in the development of curriculum materials.

Reform-based curricula are generally designed to support the idea that students learn by actively constructing their knowledge. That is, students cannot simply be told something and then be supposed to understand it. On the other hand, teachers are often expected to be able to use new materials that they are given, with minimal or no preparation. Even though these new materials require that teachers consider new concepts in new ways, the requisite learning for teachers is rarely taken into account.

In contrast, we argue, that for mathematics education reform to be successful, the need for teacher learning must also be addressed in the design of curricula. Here, we begin to examine the ways in which curriculum materials can support teacher learning. In particular, we pose two questions: (1) What do teachers learn about mathematics and about mathematics teaching as they engage with a new curriculum? (2) What is the process through which this learning occurs?

Teacher Learning in the Context of Reform

In the past decade, teacher learning has become an important goal of mathematics education reform with particular attention on the need for teachers to develop new understandings of mathematics. First, researchers find that reform requires teachers to have deep and flexible understandings of the subject matter per se, that is, the facts and concepts within the domain. In addition, teachers need an understanding of what Shulman (1986) calls *pedagogical content knowledge* — subject matter knowledge that is specialized for teaching. This involves selecting appropriate representations and explanations, and anticipating and interpreting students' methods.

In light of such research, there have been numerous attempts to design contexts that support teacher learning. In some cases, teacher learning is supported through intensive collaborations between researchers and teachers (e.g. Wood, Cobb, & Yackel, 1991). In other cases, teachers participate in a series of on-going inservices or summer workshops (e.g. Schifter & Bastable, 1995). As these examples show, teacher learning is often supported using extensive resources, both financial and interpersonal. While this is extremely valuable, it is not feasible to implement on a large scale — only a fraction of the teaching population can participate in such programs. In contrast, if we can find ways of supporting the need for teacher learning via curriculum materials, we have the potential to reach a great number of teachers. Our research takes a first step in exploring how curriculum materials can be an effective site for professional development.

Prior research on teachers' use of curriculum materials supports the notion that there is no such thing as a "teacher-proof" curriculum. While most classroom teachers are guided by a set of published curriculum materials, these materials are not used blindly. Particularly in the context of reform, teachers adapt and revise materials in light of their students, their school, and their own teaching style and goals (Ball & Cohen, 1996; Ben-Peretz, 1990). Researchers find that at times, teachers may transform a novel lesson into a lesson with which they are more familiar, even if in doing so they bypass the intended purpose of the new lesson (Brown & Campione, 1996; Sherin, 1996). However, in other cases, the process of implementing a new lesson or unit provides valuable learning opportunities for the teacher, particularly as students offer novel explanations (Hufferd-Ackles, 1998). The goal of this research is to extend prior studies and examine more closely why and how curricula provide opportunities for teacher learning. As Ball and Cohen (1996) explain, "[M]ore research on teachers' knowledge and learning would be required to design curricula as a resource for teachers' own understanding of the content… We know far too little about how written materials might support teachers' learning," (p.8).

Research Design

For the past two years, our research team has been studying the relationship between curriculum design and teacher learning. Our work takes place in the context of the development and implementation of the reform-based curriculum, *Children's Math Worlds* (CMW) (Fuson et al, in press). CMW lessons are designed to facilitate students' understanding of sophisticated mathematical concepts while also supporting efficient problem solving. Furthermore, students are encouraged to develop strategies that are personally meaningful as well as mathematically powerful. A central goal of the curriculum is the creation of a community of learners in the classroom. Teachers are expected to develop a "helping and explaining culture" that facilitates learning on the part of both teachers and students. For this reason, we expected the implementation of CMW to provide a valuable context in which to explore how curriculum materials can support teacher learning.

The data for this study involve three elementary school teachers who implemented CMW during the 1997-98 school year. The teachers work at a Catholic school in a working-class neighborhood of a large city and teach grades 2 - 4. On average, each of the teachers was observed once a week, and most observations were videotaped. Following the observations, the teachers were interviewed to elicit their thinking about particular aspects of the lesson and to probe areas in which learning may have occurred. These interviews were audiotaped and later transcribed. A sizeable portion of the classroom observations were transcribed as well.

As part of a larger project, both the classroom and the interview data were analyzed in detail in order to examine the discourse that developed in the classrooms and the teacher learning that occurred during this process (Hufferd-Ackles, 1999). A central purpose of this prior work was to examine the sources of the teachers' learning and the different kinds of learning that occurred. In the work presented here, we looked for patterns and changes over time concerning the types of teacher learning that occurred over the course of school year.

Results

Based on our work thus far, we have identified three areas in which teachers learned while implementing CMW: (a) content, (b) classroom processes, and (c) interpreting curriculum. By *content*, we mean that the teachers developed new understandings of mathematics. For example, the teachers learned new ways of thinking about specific mathematical concepts and new strategies for solving particular types of problems. We also include here new ways to think about students' understandings of mathematical ideas. Learning in the second area, *classroom processes*, involved learning how to structure classroom interactions to facilitate the development of a community of learners. This included developing techniques for facilitating discussion and having students share strategies, as well as finding ways to encourage students to respond and to build on each others' ideas. Finally, the teachers also learned effective ways of *interpreting curriculum* for their own needs. The teachers not only came to better understand the goals of the CMW curriculum and the intentions of particular lessons, they also developed ways to adapt a lesson while maintaining the stated goals. Furthermore, they learned to interpret lesson plans in terms of their own students' learning, adjusting lessons as needed for the context in which they taught.

The interaction among these three areas looked very different for the different teachers. All of the teachers we studied developed new understandings in each of the three areas outlined above. However, interestingly, the teachers cycled through learning in these areas in different ways. For example, in the case of Ms. Martinez, teacher learning began in the area of *classroom processes*. The emphasis that CMW placed on building a community of learners appealed to Ms. Martinez, and she worked hard in the first few months of school to develop classroom processes that would facilitate such a community. As students began to share their ideas and talk about mathematics, this created opportunities for Ms. Martinez herself to learn mathematics, and she did. Moreover, as she became more comfortable with the mathematical content that her students explored, Ms. Martinez became better able to interpret the curriculum materials in light of her students' abilities. For example, one lesson late in the year called for the teacher to solicit a variety of student methods related to multiplication by six. In an interview, Ms. Martinez explained that she now felt able to anticipate which methods students were likely to suggest. In addition, she recognized which of the strategies described by the curriculum were particularly important for her to elaborate if they were not raised by students. Thus, Ms. Martinez's learning moved from classroom processes to content and finally to learning how to interpret the curriculum materials (Figure 1).

In contrast to Ms. Martinez, Mr. Samson's learning began in the area of *content*. As he reviewed the CMW curriculum, Mr. Samson was concerned about his lack of conceptual understanding of the mathematics that he would

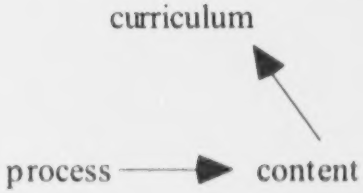

Figure 1. Ms. Martinez's Learning Path

teach. Therefore, he spent time reading the curriculum and talking about mathematics with colleagues and this helped him to develop deeper understandings of the content. Mr. Samson then used his content understanding to adapt and implement CMW lessons. In particular, he applied his own understanding of mathematics in order to interpret how his students would respond to the materials provided. Through the course of implementing these lessons, Mr. Samson also learned about the importance of classroom processes. Specifically, his students began to invent their own novel strategies, and Mr. Samson saw the need to discuss these strategies as a class. In this way, Mr. Samson was introduced to and began to thoughtfully implement new classroom processes. In summary, Mr. Samson's learning path moved from content to interpretation of curriculum to classroom processes (Figure 2).

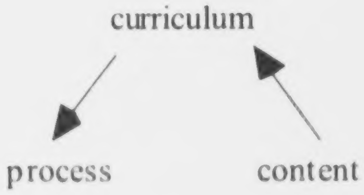

Figure 2. Mr. Samson's Learning Path

Ms. Nelson's learning process was different from Ms. Martinez's and Mr. Samson's. Ms. Nelson's initial learning concerned being able to *interpret the curriculum* appropriately. She spent countless hours pouring through curriculum materials as she planned and prepared for class. In this process, she carefully studied how to carry out each day's lesson and planned to implement the lesson with attention to the detail and intricacies explained in the curriculum. Despite her comfort interpreting the curriculum, Ms. Nelson was not always sure how her students would respond to the mathematics of any given lesson. It was only during implementation that she began to learn specifically about her students' understandings of mathematics — and in turn, her own understandings of mathematics increased as well. Finally, as her students' ideas about mathematics became the focus of class, Ms. Nelson came to learn the value of developing classroom processes that supported having students' sharing their thinking with the class. In this way, Ms. Nelson's learning moved from interpreting the curriculum to understanding content, and then to classroom processes (Figure 3).

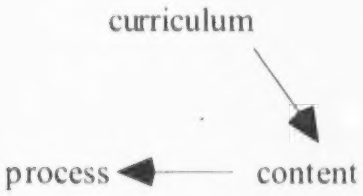

Figure 3. Ms. Nelson's Learning Path

Conclusions

What does this tell us about developing curricula that can be educative for teachers? This research illustrates that teachers can learn in the context of implementing a novel curriculum. Furthermore, we describe three specific areas in which teachers learn and we look at the process through which teacher learning moves among these three areas. To conclude, we make two claims concerning the design of curriculum to support teacher learning. First, based on our work thus far, we believe that curricula should be designed to provide different entry points for teacher learning. The fact that teachers using CMW could "enter" into learning through content, classroom processes, or interpreting curriculum, was a key to their using CMW as a tool for professional development. In general, teachers may feel most comfortable starting with curricula in different ways, and materials should be designed to reflect this. Second, however this initial learning takes place, teacher learning is an ongoing process. As we saw above, learning in one area can foster learning in other areas. Making connections across these different areas explicit for teachers might encourage the sort of ongoing learning that we believe was key to these teachers' success with mathematics education reform.

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TEACHING WITHOUT A NET: THE CASE OF A FIRST YEAR TEACHER USING AN NSF-FUNDED REFORM-CURRICULUM WITHOUT TRAINING

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This article describes the case of a fourth grade teacher, Steve, who chose to implement one of the NSF-funded elementary curricula without any prior knowledge of the materials and with no professional development or mentoring. The difficulties faced by this teacher, and the adjustments made to the curriculum will be discussed, along with the implications for *Standards*-based reform in general.

> *Although the student teachers were enrolled in two different teacher education programs, all of them developed the impression that if they wanted to be good teachers, they should avoid following textbooks and relying on teacher guides. They believed that good teaching means creating your own lessons and materials.* (Ball & Feiman-Nemser, 1988, p. 401)

The last two decades of the century have been active ones in the mathematics education community. The research programs of the early 1980's provided us with insights into the differences between novice and expert teachers of elementary school mathematics (Leinhardt & Putnam, 1986). During that time, the math education community was rethinking the content focus of K-12 school mathematics. Important activities in the mathematics education community during the 1980's culminated in the 1989 publication of the National Council of Teachers of Mathematics call for reform: *The Curriculum and Evaluation Standards*. These activities contributed to a general dissatisfaction with the "traditional text" as pedagogically and mathematically. Thus promoting the view expressed by the preservice teachers in the quote above: good teachers don't use the text.

In the last 20 years, there has been considerable research on the complex process of change in attitudes, beliefs and practices (e.g., Simon & Schifter, 1991; Wood, Cobb, & Yackel, 1991) and examples of professional development programs designed to facilitate change, as in the case of Cognitively Guided Instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). As promising as this work is, there is evidence that a lack of curriculum materials to support this type of instruction often served as a barrier to change (e.g., Knapp & Peterson, 1995; Laing & Meyer, 1996). The existence of curriculum projects developed and tested through National Science Foundation (NSF) funds fills this gap and thereby provides the possibility of systemic and lasting change. It also provides an opportunity to better understand the process of teacher change in an environment where one of the previously identified barriers has been removed.

Setting the Stage

This study takes place in a suburban/rural school district during a period of change in the elementary mathematics curriculum. The school district had made a decision to adopt one of the NSF-funded curricula, *Investigations in Number, Data and Space* (henceforth called *Investigations*) and implement it gradually over a three year period. Grades K-2 would implement the first year, grades 3-4 the second, and grade 5 the third. This study began in the Fall of 1997, one year *before* this implementation plan had begun.

The Participant

Steve was hired to teach fourth grade in this district starting in August 1997, before any decision was made about adopting a new program. Although Steve had graduated 18 months prior, this was his first full-time teaching position. (During those 18 months after graduation Steve did some work as a long-term substitute teacher at the middle school level.) As a new teacher, Steve sought the advice of the mathematics coordinator in making his decision about how manage the variety of resources for teaching math that were in his classroom. Steve described the coordinator as coming in one day in mid-October, dropping off the *Investigations* materials and saying "Take a look at it. We're looking at it [for potential adoption] .... just tell me what you think." Shortly thereafter Steve began to implement a program that he had never heard of before that moment.

The Researcher

In that the interaction between Steve and myself is an integral part of this case, my background is also part of the stage for this study. Although I had been involved in elementary classrooms doing constructivist mathematics units, had participated in research projects on student learning and teacher change in such environments, and had read extensively about student learning, constructivism and the NCTM Standards, I was just beginning to learn about this particular curriculum. [Since that time I have become significantly more familiar with the program, and during the second year of this study, I was one of the two leaders who provided professional development for the teachers in grades K, 1 and 2 in Steve's district during their first year of implementation.]

The Investigations Curriculum

The *Investigations* curriculum is organized around four major strands (number, geometry and measurement, data, and change) and the relationships between them. As the title suggests, the use of investigations to learn mathematics is the core of this curriculum. Students are required to explain their thinking both orally and in writing; they are encouraged to make sense of the mathematics they are learning and to use procedures that make sense to them, rather than those they may have memorized but may not fully understand. Successful implementation of this program requires a broad understanding of mathematical content and an understanding of how students' reason in such situations so that one may facilitate productive discussion of mathematical ideas (TERC, 1997, p. 13-14). With that in mind, the authors of the program built a multitude of resources to facilitate teacher learning into the curriculum itself. Examples of such resources include: descriptions of the mathematics content, questions to probe student thinking, excerpts of classroom discussions of mathematics, and samples of student reasoning and representations.

Data Collection and Analysis

The data for this study was collected during Steve's first two years of teaching. During that time I observed 10 mathematics lessons, conducted 10 interviews, and had many informal conversations (occurring via phone and e-mail). The mathematics lessons I observed covered a range of topics: fractions, multiplication, geometry and data analysis. Seven of the observations occurred during his first year teaching, the remaining observations occurred at the beginning and end of Steve's second year teaching. An initial interview was conducted before classroom observations began; once the observations began, I generally interviewed Steve after each classroom observation. Observations, interviews, and the teacher's journal were audio-taped and transcribed.

Data analysis began after the first set of initial interviews and continued throughout the data collection. This allowed for the use of a "grounded theory" (Glaser and Strauss, 1967) approach to data collection and analysis — that is, the focus of the data collection (particularly the interviews with the participant) was not completely determined a priori, but was influenced by the analysis of what had occurred up to that point.

The Case of Steve

When I first met Steve, it was early December of his first year of teaching. We met in his classroom after school and talked about his experiences as a learner and teacher of mathematics. Steve's mathematics background was not particularly strong, he mentioned the fact that he had to take the Geometry and Measurement course for elementary education majors three times before he passed it, commenting: "it took me three times to memorize all the formulas." Reflecting on his experiences in school, Steve expressed his desire to do things differently with his students:

> Ten or fifteen years from now these students will hopefully have basis of what I have taught them ... and if they don't remember, everything then that's the way it goes. But I hope they'll remember- Hopefully they'll remember that I treated them with respect and that I treated them fairly and with a kind word instead of always [being] negative all the time. Because that's what I remember most about my elementary school is always getting the negative ... so I don't want to be like that. [Int. #1, 12/97]

Finally, with regards to the way students should learn mathematics, Steve felt that the students could "discover" the mathematics even though "kids aren't used to that."

As a first year teacher, Steve was struggling with all those first year issues: wanting to be liked by the kids and thus having some difficulty maintaining order; not being sure of what he should be teaching each subject, and when he should be teaching it; everything new and nothing familiar. Yet in the midst of all that confusion and competition for his time and attention, Steve was willing to take on a mathematics program about which he knew nothing, and be observed in the process!

Steve's Teaching

[Given the limited amount of space, I have chosen discuss a brief episode at the beginning of the second class I observed. This will provide a context for discussion of Steve's teaching overall and the issues it raised.] *Ten-Minute Math* is a component of the *Investigations* curriculum designed to provide practice of important concepts. One of the ten-minute math activities in fourth grade is Guess My Number. This relatively common elementary school activity can basically be described thus: the teacher thinks of a number and provides the students with clues, the students then try to guess the number. On this particular day, Steve began the routine by saying "I'm thinking of a mystery number ... this number has four factors and it is less than 100" (2/98, p.1). The students are then to work in pairs to find numbers that fit the clues; the teacher then records all suggested solutions; the students discuss which ones do and do not fit the conditions; and then the students are allowed to ask questions to narrow down the possibilities (TERC, 1998, p.55).

Now let us turn to the way this routine was played out in Steve's class one afternoon in February. The first student asked if the number was even or odd, Steve responded that it was odd. The next three students offered answers that fit all criteria, they were told "nice guess." The next student offered 36, to which he was told: "that's an even number." This process continued—reasonable but incorrect answers received the "nice guess" response, while unreasonable answers were followed by statements or questions indicating which of the criteria were violated—until the mystery number was discovered.

Although there are certainly many variations to this routine, in fact the curriculum describes several variations, this particular variation eliminated the opportunity for students to discuss why different answers may or may not fit the criteria. As an isolated incident, this variation might just be interpreted as a missed opportunity. Unfortunately this seemed to be the case with much of Steve's teaching. The lesson that followed was designed to allow the students to explore halves, fourths and eighths and the relationships between these fractions. In this lesson, and others, Steve seemed satisfied that most children had "done the activity," which might mean that they had an example of fourths on their geoboard. He did not understand the necessity of pushing those students' thinking, individually and collectively. Steve did have some students present their fourths to the class, as recommended in the text. Unfortunately the "presentations" seemed to be a formality ... something to be done, but not discussed in substantive way. In these and other ways, all but the surface level mathematics seemed to get lost.

Although I had begun this project intending to observe only, I felt a moral and ethical obligation to not simply "research" this man's experiences, but attempt to mentor him. Given my inexperience with this particular curriculum, I found that it was not only Steve who was teaching without a net but that I, as the researcher/expert, was also forging ahead without a net. This shift in focus brought new issues for me ... how much do I guide Steve in his efforts to teach and how much do I let him stumble around. All of these issues lead to a very interesting year in Steve's classroom. We faced many of the same issues: I in my work with Steve, and Steve in his work with the students. Given the limited time I was spending with Steve, I had to seek out those "teachable" moments. Try to find productive instances and good questions to ask to force him to reflect on his actions as a teacher and grow in his abilities to teach in a constructivist manner. In one instance I even took over the classroom for a few moments to illustrate how to more productively orchestrate a whole-class discussion in which students were trying to share solutions and understand each others different strategies.

These "interventions" in my study of Steve's experiences teaching a reform curricula were, in the larger scheme of things, minor. And, after my intensive period of observing Steve during the fractions unit, our contact was sporadic and did not lend itself well to a mentoring situation. One might interpret many of Steve's difficulties as begin merely a result of his novice status as a teacher. As Leinhardt and Putnam (1986) point out, experts are much better than novices at "juggling the multiple goals of covering the mathematical content and making sure children are grasping the material" (p.29). However, I am concerned that in Steve's second year of teaching he is refining his teaching by adopting a "compromise" approach, particularly with regards to computation. In an interview in May of his second year, Steve stated that he had "*taught* them both ways" of doing multidigit multiplication: the standard algorithm and one of the alternative procedures that students using *Investigations* usually develop [Int. 5/99, emphasis mine]. This particular "compromise" is counter to the basic philosophy of the *Investigations* program, particularly the goal of having *students* develop computational procedures that make sense to them based on their understandings of number and number relationships.

Epilogue

Next year all fourth grade teachers in the district will begin using the *Investigations* curriculum. As part of the adoption process they will all attend professional development sessions throughout the school year to provide them with support during this process. The professional development experiences have the potential to help Steve learn to think about mathematics, and the teaching of mathematics, more flexibly. My concern is that these two years of experience adapting the materials to fit his conception of teaching may inhibit this process.

Final Thoughts

The NCTM *Standards* have been out for a decade. Curricula exemplifying those ideals have only recently become commercially available. Although NSF is currently funding Local Systemic Change Initiative projects designed to enable teachers to receive the professional development necessary to implement these changes, these efforts can only reach a limited number of teachers and the source of funding is not guaranteed to continue indefinitely. We need to better understand the issues teachers are faced with when using these materials, and work to build the supports necessary for successful adoption of a reform curricula into the school systems themselves if reform is to truly take hold in this country.

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A PERSPECTIVE ON THE USE OF MANIPULATIVES: MAKING SENSE OF A TEACHER'S USE OF BASE-TEN BLOCKS TO PROMOTE UNDERSTANDING OF THE LONG-DIVISION ALGORITHM

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In this paper, we demonstrate how a particular understanding of a perspective on mathematics and mathematics learning of teachers in transition can be useful in making sense of a teacher's use of manipulatives. The construct of "perception-based perspective" derives from previous research in which we analyzed teachers' participation in teacher education courses and classroom practice. In this paper, we use the construct of a perception-based perspective as a basis for analyzing Ivy's use of base-ten blocks to promote understanding of the long-division algorithm. We consider how Ivy's use of manipulatives may be common among teachers participating in current reforms and we discuss ways in which this analysis can contribute to conceptualizing ongoing teacher education.

"Teachers need to . . . make extensive and thoughtful use of physical materials to foster the learning of abstract ideas" (National Council of Teachers of Mathematics, 1989, p. 17). This statement has become commonplace and is virtually taken for granted by those participating in the current mathematics education reform. The discussion in the mathematics education community regarding the use of physical materials focuses on investigating the effect of manipulative use on student learning (Meira, 1998; Thompson, 1992), determining ways to increase the likelihood that teachers will use manipulatives in their practice (Hollingsworth, 1990; Joyner, 1990), and identifying the functions that manipulatives play in the lessons of teachers who have recently begun to reform their practice (Cohen, 1990; Peterson, 1990). Our goal is to contribute to understanding how teachers conceive of the use of manipulatives, that is, how manipulatives fit into their conceptions of practice. Towards this end, we focused on how a sixth-grade teacher, Ivy, used manipulatives (base-ten blocks) to promote understanding of the long-division algorithm. In this paper, we analyze her use of the manipulatives, present our hypothesis as to her perspectives underlying that use, and discuss ways in which this hypothesis can be useful in thinking about teacher education.

Conceptual Framework

We work from an emergent perspective (Cobb & Yackel, 1996) which has both a cognitive (constructivist) and social (symbolic interactionist) component. From this perspective, mathematics is created (learned) as students interact with and make sense of their worlds in viable ways while participating in various communities (e.g., the mathematics classroom).

A premise of the constructivist theory of learning is that individuals' current understandings enable and constrain how they interpret various aspects of their environment (including manipulatives). For example, individuals who understand multiplication as a coordination of the number of groups and the number in each group may look at an array of objects and "see" multiplication. We emphasize that multiplication is not an inherent feature of the objects; multiplication is a mental structure, a way of thinking that can be imposed on the arrangement of the objects by those who have an understanding of the structure of multiplication. Hence, being able to interpret an array as a representation of multiplication is only possible for those who have that understanding. Likewise, students may recognize and manipulate base-ten blocks as a representation of the base-ten place value system only if they have the necessary understandings of that system.

If a manipulative is a useful representation for students, they may be able to use it to carry out activities that they have developed in other contexts (e.g., dividing up a "bag full of candies" using base-ten blocks). Reflection on multiple instantiations of the activity and its effects can lead to students' identification of regularities–leading to more advanced understandings. In the case of long division, we would argue that concepts reflectively abstracted from operations on base-ten blocks are necessary, but probably not sufficient, for perceiving isomorphisms between the algorithm and the operation with blocks.

Method

This study is part of the 4.5-year Mathematics Teacher Development (MTD) Project,[1] the goal of which is to investigate the development of elementary mathematics teachers' practices. *Practice* refers to not only what a teacher does, but also how she conceives of what she does. We use a teacher development experiment methodology (Simon, in press) which combines whole-group teaching experiments in teacher education courses with case studies of individual participants. In this paper, we report on our analysis of transcribed videotapes (2 classroom observations) and audiotapes (3 interviews) that reflect one of the case studies, Ivy's, use of base-ten blocks to promote understanding of the long-division algorithm.

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Ivy's Lessons on Understanding the Long-Division Algorithm

Ivy teaches in a district that is working to promote mathematics teaching consistent with recent reform documents. The district provides a list of grade-level outcomes pertaining to specific content areas and supplies the teachers with a recommended list of resources rather than a single textbook. This gives teachers the freedom and responsibility to create or adapt learning activities and materials for their students.

Based on a pretest, Ivy determined that her students could use the long-division algorithm to solve problems with 1-digit divisors, but they could not do problems with 2-digit divisors. During an interview, Ivy explained that this meant that the students "don't have a real understanding of what we are doing" in the 1-digit divisor problems because dividing with a 2-digit divisor "is the same process–it is just a different number." For Ivy, it was not enough that her students could perform the steps of the algorithm with 1-digit divisors; she also wanted them to "see the relationship between the [base-ten] blocks and the algorithm." She wanted her students to work with the blocks so that they would know why they were doing each step of the algorithm. For example, referring to 240∏7, "I can't divide 2 hundreds [flats] with 7, so I have to trade those in and I end up with 24 tens. Here is my 24–it is coming from 24 tens. It is not: 2 slide over the 4 make 24."

To Ivy, the steps of the long-division algorithm are meaningful because they correspond to actions that one performs on base-ten blocks when using them for division. Her goal was to make those relationships apparent to her students.

Ivy began the first of the two lessons by asking her students to solve the problem 204÷4 using base-ten blocks (204 was represented using 2 flats and 4 units). Each student had a set of base-ten blocks to manipulate, and the students were arranged in groups so they could talk to their classmates about the problem as desired. Ivy then initiated a whole-class discussion in which she called on Mark to explain how he solved the problem. We include a portion of the lengthy interchange that occurred to illustrate Ivy's approach.

Ivy: Okay, so you took your 2 hundred; you put it back in the box and you got 20 tens.

Mark: I got— [Mark paused as he pointed to the four groups of 50 on his desk.] Yeah.

Ivy: Twenty tens. So each person [group] ended up with how many tens?

Mark: 51. I mean five tens.

Ivy: Five tens. [Ivy wrote the 5 at the top of the algorithm on the white board.] And how many tens did that use up? Did that use them all up?

Mark: Yeah.

Ivy: So you used up all twenty? [Ivy wrote 20 below the 20 of 204 in the algorithm.]

Mark: Yeah.

Ivy continued in this manner through the rest of the problem. Then she restated the sequence of actions performed on the blocks while pointing to the corresponding steps of the algorithm.

At the beginning of this excerpt, Mark was not describing his actions in a way in which the corresponding steps of the algorithm could be demonstrated. Ivy overcame this hurdle by loosely paraphrasing his contributions and asking him leading questions as a means of pointing out the connections between the blocks solution and the algorithm. It seems that underlying Ivy's approach was the assumption that these connections were perceivable and by pointing them out, she could assist her students in seeing them.

Ivy then repeated this instructional sequence for two more problems (413÷5 and 291÷7). During whole-class discussions on these problems, Ivy's interaction with students resembled her interaction with Mark. Additionally, if a student was having a particular amount of difficulty answering her leading questions, Ivy called on a different student to take over. After the first class, the researcher asked Ivy about her plans for the next lesson.

Ivy: Show it even better; make a connection real directly. "Hey look. This 2 is my 200, and see, I don't have enough of those to divide up, class. Put them back in the box; trade them in for tens. Now I have 29 tens." That is even really desperately clear that that was the connecting stage.

For Ivy, the connections between the actions on the blocks and the algorithm are obvious. From our perspective, her understandings of how the two systems operate enable her to see the relationship between them and that these observations cannot be made by students who do not share her understandings.

During the next lesson, Ivy repeated the instructional sequence for the problem 892÷3; however, the students still were not contributing responses that indicated they understood the relationship (that Ivy saw) between the blocks and the algorithm. During the interview after the second class, the researcher asked Ivy what she planned to do about particular students who seemed to be having difficulty.

> Ivy: I don't know, other than continue to be there to watch and plug away at it and try to get them to come along. Hopefully it clicks sometimes. We continue to describe it, and show it as the blocks. . . . so they hear where it came from, and they try it on their own, and they see other people around them.

Discussion

In Ivy's use of the base-ten blocks, we see a manifestation of a perspective on mathematics and mathematics learning that we believe characterizes the perspective of many teachers who are in transition from traditional to reform-oriented teaching. We formulated the construct, "perception-based" perspective, as a result of analyses of data on the MTD teacher education classes and classroom observations and interviews of six case-study teachers (Simon, Tzur, Heinz, Kinzel, & Smith, 1998). The characteristics of a perception-based perspective are:

1. Mathematics exists as part of the external world independent of human activity, and that external reality can be perceived by all.
2. Mathematics is a meaningful, interconnected set of phenomena. Seeing the relationships among aspects of the mathematics results in mathematical understanding.
3. Students learn mathematics by direct perception of the mathematical objects, principles, and the relationships among them (in contrast to a traditional perspective in which the perceptions of others are presented to the student through teacher lectures and textbook readings).

We find that looking at Ivy's division lesson through the lens of a perception-based perspective proves useful in understanding her use of manipulatives. Ivy believes that an understanding of the long-division algorithm derives from knowing how the steps correspond to the actions (that can be performed) with base-ten blocks. Her underlying assumption seems to be that the relationships between the two systems exist independent of human activity and are perceivable by all. Therefore, the role of the teacher is to create situations in which these connections are apparent to the students.

Ivy used the base-ten blocks in ways that differ significantly from some teachers who are first using them in their practice. Initially, some teachers use manipulatives as demonstration tools or as supplemental materials that could be (and sometimes are) omitted (Peterson, 1990). In contrast, the manipulatives were central to Ivy's lesson; she viewed the manipulatives as the source of meaning for the algorithm. However, from our perspective, using manipulatives in this way–as though the mathematical structures and relationships are in the blocks, and as such are perceivable by all, is problematic (Cobb, Yackel, & Wood, 1992). It does not acknowledge that students' abilities to see manipulatives as representations of particular mathematics and to use them as symbolic tools are enabled and constrained by their conceptions.

This study challenges us to think more deeply about the role of teacher education in promoting the use of manipulatives by teachers. Teachers have been encouraged to use manipulatives and instructed on how to incorporate them in their lessons. However, teachers' perspectives structure how they use manipulatives in their practice. Hence, it seems important that efforts aimed at promoting teachers' use of manipulatives be considered within a larger framework that includes promoting development of perspectives that underlie teachers' use of instructional materials, that is, their perspectives on mathematics, learning, and teaching.

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SHARPENING TEACHERS' ASSESSMENT SKILLS THROUGH TECHNOLOGY-SUPPORTED CLINICAL SUPERVISION

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The purpose of this study was to examine the impact of technology-supported clinical supervision provided through e-mail feedback to teachers on their reflective reports of interviews conducted with children. It was expected that the feedback would help teachers understand where they were focusing their attention during clinical assessments and that this knowledge would enable them to shift their focus from more surface aspects of mathematical thinking to the deeper structures of reasoning and logic underlying children's observable performance.

Theoretical Framework

This analysis is based on a three-tiered model of teachers' clinical interviewing levels previously developed by Kaplan and Appelbaum (Appelbaum & Kaplan, 1998; Kaplan & Appelbaum, 1998). That model related the interviewing process used by teachers to their individual attachments to mathematics as an object of self (Winnicott, 1996). This framework suggests that the form of attachment that teachers have to mathematics determines the kind of questioning that they use and the kinds of responses they attend to in the children they interview. Similarly, Davis' work on the role of listening in teaching also informs this paper in that the current data are viewed through the lens of his analysis of a middle school teacher's listening process as it was transformed from "evaluative listening" to "interpretive listening" and finally to "hermeneutic listening" (Davis, 1997). This latter view suggests the possibility that appropriate scaffolded feedback can transform teachers' perspectives about the outcomes of the interviewing process as it relates to the assessment of children's mathematical thinking.

Methods and Procedures of the Study

The data for this report were obtained from the reflective journals of the teachers enrolled in a team-taught clinical assessment course in a master's degree program specializing in teaching children mathematics. Teacher-participants in the course spent one hour of their class time for up to 10 sessions doing clinical assessments of children's mathematics conceptions. Each participant worked with one child whose past performance on standardized assessments and general classroom performance was below expectations for grade level. After each session, the teacher-participants e-mailed a reflective journal of the session along with a plan for the next week's session to the course instructors. In response, the instructors e-mailed their suggestions for further clinical interviewing to the teachers before their next meeting with the children. The feedback was intended to highlight and emphasize the children's underlying thinking processes and, consequently, were expected to be reflected in teachers' subsequent e-mail communications.

Data from one of the teachers in the course were selected to serve as an illustrative case study for presentation in this paper. The participant selected, Debra, was a female sixth grade mathematics teacher in an ethnically mixed suburban school district in northern New Jersey. Her clinic student , Kenny, was a sixth grade boy from a neighboring middle class school district who attended seven assessment sessions. Debra was both elementary and mathematics certified with 6 years of teaching experience at the middle school level in the fields of mathematics and social studies. On an open-ended survey and on a Likert-type formatted questionnaire, she had indicated that she was positively disposed toward mathematics.

Data Analysis

Three scoring categories of reflective responses to the clinical interviewing process were constructed using the Kaplan and Appelbaum (1998) object relations framework and the listening approaches of Davis' (1997) model as follows:

- Level 1: Statements indicating that the interviewer was listening for "correct answers" and judging the student based on attention to algorithms and computational procedures in terms of correctness of applications. This response is similar to Davis' evaluative listening. These comments were seen as focusing only on performance rather than on the underlying competencies that formed the basis of that performance (Flavell & Wohlwill, 1969)
- Level 2: Statements indicating a focus on how the student gets answers using concrete models, computations, and verbal explanations of procedures. This response is similar to Davis' interpretive listening. It indicated a concern with children's underlying competence in terms of the details of solution sequences.
- Level 3: Statements indicating that the interviewer was focusing on why the student used particular strategies and on the student's rationale for action, processes of reasoning, and motivational

approaches to mathematics. This response is related to Davis' hermeneutic listening in that it is based on non-evaluative access to the child's underlying competence.

Using these categories of responses, the reflective reports were first coded independently by the three authors and then by two graduate students. During the coding, reflections were broken down into discrete comments about the teacher's reactions to what occurred during each session. All independent codings were then discussed in groups until initial differences in scoring were resolved by consensus. In addition, course instructors' e-mail comments were analyzed to determine their possible contribution to shifts in the level of the teachers's reflections. Instances of support for the teacher's comments at any level were noted as were comments that encouraged the teacher to focus more on the "why" and "how" aspects of the child's performance.

Debra's Reflective Reports

Debra's comments in the reflective reports were analyzed in terms of the frequency and patterns of occurrence of each of the three categories of reflective response levels recorded across sessions. Specific comparisons were made between responses on the first two sessions and the last two sessions. Individual representative comments from the participant are presented as well.

As shown in Figure 1, Debra began her clinical experience with a strong propensity toward Level 3 responses in her structuring and interpreting of the interview with her child, Kenny (60% of all responses). Moreover, although somewhat diminished, she generally maintained the dominance of this orientation over the course of the assessment program (46% of all responses).

Focus of Response	First Sessions	Last Session
Process/Reasoning	60	46
Procedures	25	35
Correctness	15	19

Figure 1. Relative percent of reflective report responses in each category: A comparison of the first two sessions and the last two sessions.

Typical of her initial reflective reporting were statements such as:

> *Up to this point, all of his equations were solved with mental math. As he...divided 728 by 7, he initially came up with 14. Before I had time to respond, he said, "That's not right. It has to go in more than 14 times. It must be 114." Then he disagreed with himself again stating that, "There is a zero, it's 104." He asked to use paper and pencil to check himself. As he was getting his pencil I asked him why he questioned his first response of 14. Kenny told me that he knew 7 had to go into 728 at least one hundred times because 7 goes into 7 hundreds one hundred times.*

This response was considered to be at Level 3 because although it dealt with arithmetic calculations, it focused on the underlying number sense and reasoning used by the child to obtain his answers.

In fact, Debra was relatively unconcerned with the correctness or incorrectness of her child's responses and rarely focused on this aspect both initially (15% of all responses) and at the end of the assessment period (19% of all responses). Notably, though, by the last sessions she did make a slight increase in the relative frequency of her interest in Kenny's procedural routines (from 25% of all comments to 35%) while decreasing the frequency of her Level 3 observations (from 60% to 46%). This level of reporting was represented by such comments as:

> *He also explained the concept of estimating and how the decimals should be rounded to give you an accurate estimation. He showed me how when working with the decimals 0.30 and 1.50, you could round them to the nearest whole number and get 0 and 2 respectively.*

However, this line of observation was often followed by an effort to get beyond the procedures to Level 3 responses, as for example when Debra followed up on the previous remarks with:

> *It looks like you are dealing with money here. If those numbers were the prices of two items you were going to purchase in the store, would you still round them the same way? Kenny said yes and when I said it seems strange to round a price to zero cents, he interjected, stating "It balances out because the 1.50 is rounded to 2.00 and covers the amount that you rounded down."*

It is also noted that this line of inquiry followed the previous week's instructor e-mail feedback in which Debra was told:

> This demonstrates too that he knew the words to say, but perhaps did not connect those words with the numbers and with the objects....Work with problems that have words and in which he must make sense of the concepts behind the words to produce some sort of school-like problem solution.

Interestingly, too, although Debra's predominant focus remained on the underlying reasoning of her child's grasp of school mathematics throughout the assessment, some of these expressions were more related to the child's mathematical disposition than to his reasoning. For example, she commented:

> When the problem being worked on was difficult, Kenny would give up rather quickly. When I stepped in to try to encourage him to try again, he would focus for some time, but again lose attention when he could not solve the problem easily. It is evident that Kenny gives up when he does not understand a situation or find the correct answer immediately. When it requires him to think past a set format that he has memorized, his defenses kick in and he gets off track.

Instructor feedback to this type of response appeared to acknowledge the affective element, but also maintained a focus on cognitive goals and tried to direct Debra toward going beyond her observation, as for example when the instructor said:

> Good observation. Now how do you get beyond his resistance to find out how much he does know?...Ask him what remainder means. What does it stand for?

In addition to this direction, however, the instructors also focused Debra's attention more on the procedural aspects of the child's thinking, commenting, for example Debra was advised:

> You may want to go back to division and work out the structure through word problems. Provide a word story in which division is the appropriate solution. Have him model the problem, step by step as it is started. Ask him to record what he had done using numbers. If he does not come up with a division construction, suggest to him how it might look.

It appears, then, that Debra's increase in Level 2 responses over the course of the assessment might be accounted for because she was taking cues from the feedback of the instructors which actually channeled some of her observations toward less abstract reflections. Although this was not consistent with the expectation that technology-supported feedback of reflective interview reports would raise teachers' levels of reflection, this unexpected scaffolding effect was largely a function of the fact that the child being interviewed had difficulty in learning mathematics in school. Therefore, in order to uncover his weaknesses as well as strengths it was necessary that the interview process establish baseline information at all levels of knowledge. The course instructors, therefore, provided feedback to address this need, and Debra responded to it.

Conclusions

The results of this case study drawn from a larger group investigation suggest three important trends in the development of teachers' abilities to apply reflective reasoning to the clinical interviewing of children as a function of technology supported clinical supervision. The first is that, regardless of any feedback, teachers tend to have their own style of reflection based on their own knowledge and relationship with mathematics. As noted by Kaplan and Appelbaum (1998), "teachers selectively evoke and listen to those elements of students' encounters with the world of mathematics that can be recognized and acknowledged as consistent with their own perceptions and feelings about the subject matter (p. 7)." As Debra's consistent use of Level 3 responses indicates, these styles tend to remain stable during the interviewing process and are reflected in the observations and interpretations teachers make about their own interviews with children. Second, on the other hand, the case study presented suggests that technology-supported clinical supervision can shape teachers' concerns and focal points during the interviewing process and that this shaping is directly related to the direction that the feedback offers. However, it is not necessarily the case that the feedback will raise the level of teachers' reflections since the feedback is not always directed toward the analysis of more abstract conceptions during the interview. Rather the feedback in clinical supervision is directed toward the facilitation of uncovering a variety of thinking and performance levels affecting the mathematical functioning of children who have problems in learning. Consequently, of necessity, feedback is sometimes directed to clarification of procedural rather than conceptual understandings. Third, it appears that teachers, such as Debra, who are positively disposed toward mathematics, can and do vary their levels of responding to children within the same interviewing task. In fact, lower level questioning seems to serve as a starting point for subsequent expanded explorations of the child's thinking in much the way that Davis (1997) describes the teacher-student interaction of a hermeneutic listening episode.

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PRESERVICE SECONDARY TEACHERS' PORTRAYALS OF CLASSROOM DISCOURSE: ALLOWING STUDENTS TO KNOW AND TELL MATHEMATICS

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Four preservice teachers' fictional accounts of mathematics classrooms were analyzed with the goal of describing elements of their views of classroom discourse. Each teacher portrayed lecture as an inappropriate mode of instruction; students should "discover" mathematics without the teacher telling them what they should know. Although the proposed alternatives to lecture involved significant student engagement with mathematics, most examples focused on students explaining ideas and correct solutions to other students, with minimal questioning or negotiation of ideas. The preservice teachers had shifted the responsibility for "telling and correcting" from the teacher to the students. These processes appear to be resilient elements of the teachers' conceptions of classroom discourse. This finding identifies a critical area for the teachers' further professional development.

Current reform recommendations in the United States urge teachers to engage students in active and cooperative explorations so that rich understandings of mathematics as a vibrant, useful subject are developed (NCTM, 1989). However, reform poses no simple task for teachers who must personally institute changes that challenge a lasting tradition of teacher-centered, procedure-oriented mathematics instruction (Fennema & Nelson, 1997). This paper describes how four preservice teachers made sense of some of the reform themes they encountered during their teacher education experiences. In particular, the preservice teachers' views of mathematics classroom discourse are considered. The study addresses these questions: What elements of classroom discourse do preservice teachers believe most directly affect student learning of mathematics? In the preservice teachers' conceptions, what roles do students, teachers, and mathematics play in the types of discourse that lead to significant student learning?

Methods

The participants in this study were the four preservice teachers—Anne, Michelle, Sarah, and Todd (pseudonyms)—enrolled in the first year of a graduate-level secondary mathematics certification program at a large state university in the Mid-Atlantic region of the United States. Data were collected after the teachers had completed a 5-week course titled *Teaching*

Secondary Mathematics I (Aug-Sept 1998), followed by 10 weeks of "student-aiding" in local high school and middle school classrooms (Oct-Dec 1998). The data contributing to this paper were collected over a 2-week period (Dec 1998) in the context of a series of activities designed to help the preservice teachers reflect carefully about their student-aiding. Data sources include two stories and two position papers by each teacher, in addition to the following (all of which were audio-taped and transcribed): two 1-hour group discussions, two 1-hour group interviews, and two 1-hour individual interviews with each teacher.

The preservice teachers were presented with a list of approximately 50 statements extracted from the reflection papers that they and other licensure students had written during student-aiding. Each teacher chose one statement with which he or she strongly *disagreed*, and then wrote (1) an Anti-Position paper critiquing the chosen statement, and (2) a Position paper defending an alternative to the perspective of the Anti-Position. For example, Sarah chose the following statement for her Anti-Position (the one she disagreed with): "If students were allowed to discover topics for themselves, then students would either come up with the wrong answers or become too frustrated." Her Position was "Using discovery learning is an efficient and effective way to teach math." In the process of developing their two papers, the four teachers met together twice (1 hour each) to support and challenge one another's thinking, and were interviewed as a group twice. After completing their position papers, the teachers each wrote two fictional accounts (one Story and one Anti-Story) of classrooms consistent with the perspectives of their Positions and Anti-Positions.

Numerous strategies were used for analyzing the different data sources. For example, techniques of narrative analysis (Mishler, 1986; Riessman, 1993) were employed to analyze the teachers' Stories and Anti-Stories. Each story was analyzed both structurally (according to the story's complicating actions, evaluations, and resolutions) and thematically (according to the roles of teachers, students, and mathematics in the plots). Comparisons were made between characteristics of (1) a given teacher's Story and Anti-Story, (2) all four teachers' Stories, and (3) all four teachers' Anti-Stories. Major categories and patterns were organized and developed so that themes were synthesized across the different data sources.

Results

Beliefs about Lecture

The most prominent belief about classroom discourse communicated by the preservice teachers was that lecture is an inappropriate mode of instruction because it reduces opportunities for students to learn mathematics in meaningful ways. This theme is most dramatically illustrated in the Anti-Stories written by Michelle and Todd.

Michelle described a teacher named Mrs. Stringent who claimed that "the best way for people to learn was to instill the fear of not learning." This teacher "taught in a completely totalitarian style, in which each child was an island, and any talking was strictly forbidden." She structured the classroom with "desks all spread out so that no one could talk to anyone else, and each desk had sides to it, so the only way for the inhabitant to look was toward the teacher."

Todd's Anti-Story presented a teacher who "explained" from the front of the class how to label the axes of a graph. In the midst of his lecture, he noticed that students were not taking notes or paying attention. Like Mrs. Stringent, this teacher attempted to instill fear in students. He attempted to scare the students into paying attention by stating, "There *IS* a test coming up on Friday and considering more than half of you failed the last one, I'd think you'd pay more attention." He provided similar comments when students did not know answers to questions that he had just explained to them. This lecturer also had a static view of mathematics. When one student asked a "Why?" question, he stated "That's just the way it is. That's all I can say. Trust me on this."

Michelle and Todd's Anti-Stories highlight several critical components of all four teachers' dissatisfaction with lecture: (1) lecture precludes the possibility of important communication by and among students, (2) in lectures, the motivation for students to learn is primarily extrinsic (threats of tests or punishment), (3) lecture focuses on information that is known by and important to the teacher, and "transmits" it in ways that are not necessarily meaningful to students, and (4) because lecture is unidirectional communication, mathematics is treated more as rules to be memorized and applied than concepts to be explored and understood.

Alternatives to Lecture

Each of the preservice teachers suggested other classroom organizations and instructional methods as alternatives to lectures. All of the teachers' Stories included positive portrayals of more student-centered activities such as students discussing and exploring mathematics, working together in groups, and making presentations to classmates. These portrayals involved significant changes in the roles of students in classroom discourse. It is also notable that the more the teachers' Stories involved student participation, the more they also described mathematics.

In Sarah's Story, the teacher was pleased to enter her classroom one morning to find her students already "playing around" on computers and "asking each other for help and ideas." The teacher had planned "an exploratory activity" in which she "wanted students to begin to make connections between the algebraic, tabular, and graphical representations of the trig functions." In particular, the teacher had prepared the computers to display numerous graphs for which students were then to develop corresponding equations. Students could use trial and error (entering their equations into the computers to generate graphs, then checking to see if they matched the originals). During the activity, one student yelled out to the class, "Cool! When you add a number to the end of the equation, it moves the graph up and down!" Following that student's lead, the rest of the class then investigated "the idea of vertical shift" and "quickly picked up on it." Then, the teacher "began to question the students further," asking about the effects on the graphs of other changes to equations. The teacher in the story reflected, "I was extremely impressed with the students' desire and ability to take a certain problem and extend it. It was wonderful to know that I had made learning interesting enough that the students wanted to continue in their discovery of trig functions."

Sarah's Story identifies some of the positive benefits she saw of promoting extensive student discourse in the classroom: (1) students are more intrinsically motivated to learn mathematics, (2) students reach connections themselves, through activities set up by the teacher, (3) students develop new questions and insights that may go beyond the scope of what the teacher knows or planned for the student to learn, (4) through discussion and experimentation, students can decide themselves when their work is correct by reaching "consensus" about what makes sense. These themes prevailed in the other teachers' Stories as well.

Discourse Characteristics Independent of Instructional Format

The previous two sections focused on one contrast between the teachers' Anti-Stories (lecture) and Stories (alternatives to lecture) that was common to all four teachers. This section considers what images of classroom discourse were *similar* in each teacher's Story and Anti-Story. In other words, what elements appeared to be constant across stories, regardless of the form of instruction?

As described earlier, the preservice teachers made quite clear that a "good" teacher would not lecture or directly explain mathematics to students. (That is what the teachers in the Anti-Stories did.) Instead, students should "discover" mathematics without the teacher directly telling them what they should know. However, as evidenced in the Stories, it is acceptable for *students* to tell other students what they should know. In fact, most of the Stories' examples of student-centered discourse focused on students explaining ideas to other students, with minimal questioning or negotiation of ideas. In other words, it does not seem to be the *telling* that the preservice teachers viewed as inappropriate— it is *who does the telling* that may or may not be appropriate. Consider, for example, Anne's Story in which Ms. Gray placed students in small groups to solve some real-world algebra problems. When the groups finished the problems,

> Ms. Gray had each student from every team explain one of the problems, making sure that everyone participated and understood what they were doing. If anyone didn't understand a problem, she made the groups reconvene and explain it to each other. Students who grasped concepts more quickly explained it to those who were slower to understand... The students really tried to help each other understand.

Anne's description illustrates the prominence of students explaining to each other in her view of cooperative group work. When students reconvened in groups, those students who did not understand were "explained to" by other students. This example also illustrates a closely related result: In all four teachers' Stories, student mistakes were corrected by another student (not the teacher) who explained "the right way."

These themes suggest that the preservice teachers perceived a need for someone in the classroom to be *explaining*—if not the teacher, then the students. Although the preservice teachers communicated many alternatives to traditional teacher and student roles and classroom activities, their Stories and Anti-Stories (regardless of the mode of instruction) also emphasized the processes of explaining and correcting. These processes appear to be stable and resilient components of their conceptions of how students learn mathematics. These results suggest that the preservice teachers believe, at least in part, that appropriate classroom discourse prepares students to know mathematics by telling and being told.

Conclusions

As the results above suggest, this paper provides timely detail about relationships between preservice teachers' conceptions of student learning and their views of reform themes. The teachers in this study maintained "telling and correcting" as prominent themes in their classroom visions, although they shifted the responsibility for these processes (traditionally held by the teacher) to the students. This subtle finding identifies a critical area for the teachers' professional development. As teacher educators, we bear responsibility to identify experiences that might help preservice teachers extend their views of classroom discourse to more deeply integrate additional student processes and responsibilities (such as questioning, agreeing and disagreeing, negotiating, etc.) into their conceptions of how students learn mathematics.

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ELEMENTARY PRESERVICE TEACHERS AND THE *STANDARDS:* CONFLICT OR CONSENSUS?

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The purpose of this study was to describe elementary preservice teachers' perceptions of ideas expressed in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Most preservice teachers welcomed the "notion" of changing school mathematics, however, many found it difficult to endorse decreased attention to traditional topics. Findings suggest that we need to probe into preservice teachers' rationales for their views and how *they* define reform-oriented ideas.

According to McLeod, Stake, Schappelle, Mellissinos, and Gierl (1996), the writers of the *Curriculum and Evaluations Standards for School Mathematics* (National Council of Teachers of Mathematics (NCTM), 1989) intentionally chose a large "grain size" for the document, leaving the details of content up to the teacher, to remain true to "their vision of teachers as active professionals" (p. 116). Nonetheless, McLeod et al. argued that "foregoing the smaller grain size leaves a great deal of room for misunderstandings and misinterpretations of NCTM ideas about reform" (p. 116). Although it has been documented how inservice teachers and administrators have misconstrued NCTM ideas about reform (e.g., McLeod et al.; Webb, Heck, & Tate, 1996), few studies have examined elementary preservice teachers' perceptions of the *Standards*. On the eve of the publication of *Standards 2000*, it is an appropriate time for the mathematics education community to reflect upon the impression preservice teachers (PSTs) have of the *Standards*.

This paper presents the results of a study conducted in two mathematics methods courses taken by early childhood education majors. The purpose of the study was to describe elementary PSTs' perceptions of the ideas expressed in the mathematics education reform document *The Curriculum and Evaluations Standards for School Mathematics* (NCTM, 1989). In doing so, the researchers hoped to gain insight into how PSTs might interpret the content in a reform-based mathematics methods course. Furthermore, the results from this study provide the mathematics education community with a better understanding of PSTs' perceptions of the reform message as emphasized in the *Standards*.

Theoretical Perspective

Reform documents published by NCTM emphasize vastly different ways of teaching and learning mathematics than has typically been seen and experienced in most mathematics classrooms (e.g., NCTM, 1989). In concert with the recommendations to change mathematics classrooms, reform-based teacher education programs often strive to educate PSTs to teach in significantly different ways than they were taught. PSTs have typically been students in classrooms where mathematics was presented as a disconnected body of rules and procedures, handed down from the authority-teacher to students. The focus in these classrooms was getting the *one* correct answer, rather than the reasoning or strategies used in the process. As a result, preservice elementary teachers commonly view mathematics as a static body of rules and procedures. They view the teacher's role as presenting examples, demonstrating procedures, and checking assignments for accuracy, and the student's role as listening and watching attentively, in order to mimic what the teacher demonstrates to them (e.g., Ball, 1990; Wilcox, Schram, Lappan, & Lanier, 1991).

It is widely accepted that how people interpret experiences is strongly influenced by their past experiences, their existing knowledge and current beliefs (Anderson & Bird, 1994; Kagan, 1992; Pajares, 1992). This assumption is at the heart of our study. If PSTs' experiences with and beliefs about mathematics so vastly differ from what reform documents propose, how do they perceive the *Standards*?

Methodology

As part of our teacher education program, students enroll in two semesters of mathematics methods courses. The focus of these courses is to expose PSTs to the current research on teaching and learning mathematics, and mathematical content appropriate for young children. Each course includes a field component where PSTs work in elementary schools in order to experience teaching mathematics. Two cohorts of early-childhood education majors participated in this study during the 1998-1999 academic year. One cohort (n=31) was followed through both methods courses, while another cohort (n=30) was only studied during the first course. One of the researchers taught all the methods courses.

Data sources included: class assignments (such as a mathematics autobiography), class activities, and ensuing class discussions. Particular class discussions were audiotaped and transcribed, and enhanced by the researchers' field notes. Field notes were also taken throughout the semester to note any changes in the PSTs' views.

To introduce the PSTs to the *Standards* and to solicit their impressions of the document, they first participated in a scavenger-hunt activity entitled *Standards-at-A-Glance* (Hammond, 1994). In this activity we examined issues such as calculator use, manipulatives, and the summary of changes in content and emphasis in K-4 mathematics. During the class discussions, we asked students to elaborate on the sources of their perceptions, especially those that were in direct contrast to the ideas expressed in the *Standards*. Throughout the semester, the aforementioned discussions were revisited as we explored specific mathematical content and pedagogy.

The qualitative analyses of the data used the method of analytic induction (LeCompte & Preissle, 1993). Analytic induction involves scanning the data for themes and relationships, and developing and modifying hypotheses on the basis of the data. Initial data analysis consisted of grouping PSTs' written responses to the scavenger hunt and coding transcripts for issues that highlighted how PSTs' perceived the content of the *Standards* as well as their beliefs that seemed to be mediating their perceptions.

Results

Three themes, common to both cohorts, emerged as the PST's initial reactions to the *Standards*: (1) novel ideas, (2) opposing ideas, and (3) misinterpreted ideas.

Novel Ideas. Most PSTs welcomed the "notion" of changing school mathematics, however, having children justify their answers, write about mathematics, and integrating mathematics with other subjects were unfamiliar ideas. As one PST stated, "I could see discussing a lot more than writing. I mean, I don't know, just writing about mathematics is real out there." The use of calculators for anything but "checking answers" was also novel. Likewise, probability and statistics seemed "a bit much for the K-4 level."

Opposing Ideas. Most PSTs strongly disagreed with the use of calculators, especially in earlier grades. Comments such as the following abound in the data: "Calculators should only be available after simple math skills have been mastered.", "The calculator is a good tool to use to check your answers...not as a crutch.", and "[Students] rely too much on calculators instead of knowing."

Many PSTs also disagreed with the decreased use of clue words. One group of PSTs argued, "We think they should keep them because they give you an idea of which operation to use." Another PST claimed that "clue words are a best friend to someone who doesn't understand math."

Decreased attention to long division was also a source of contention. For example, one group asserted that "students need long division so that they have a good base for the next grade." Another PST commented, "I think they should keep it because students will use it in the real world, on the job or at a store. For example, if you only have $5, how many items costing $2.50 can you buy?" This comment makes one wonder how she defines long division.

Misinterpreted Ideas. Some PSTs agreed with ideas expressed in the *Standards*, but for reasons other than improving mathematical learning. Consider these PSTs' comments:

Given that it is K-4, kids that age don't know their multiplication facts well enough to do long division. My mother teaches 4th grade and she told me to focus on one and two-digit division first and leave long division to 5th and 6th grade.

I didn't use a calculator in high school and then [later] I had to have help from my teachers just to learn how to use a calculator and with technology rapidly developing, I think it is important to kids to actually have them available, because some kids aren't going to go home and have their parents buy them calculators.

Some PSTs misinterpreted reform-based ideas. For example, when one PST asked, "What does that [writing about mathematics] mean? Writing papers about math?", another PST responded, "We always had writing in math. We always had a report due or like we would have to...write the word...and as we got older...we would always write out our math problems."

Change or Retention? During the second semester, noticeable differences in the PSTs' responses indicated that they recognized the importance of children making sense of mathematical concepts and procedures. As one preservice teacher noted during the class discussion on long division, "I know like my fifth grade class, they never really understood what they were doing. Like they were just setting up like little, a rhyme thing that they learned, with the steps."

While some PSTs considered alternative approaches to teaching mathematics, others held firm to their preexisting beliefs. This difference was most evident in discussions about calculator use. Consider the following excerpt.

Hannah: After the calculator activities last semester, I do think that it should be introduced, and an early age is ok, but I think it's all in how you present it.

Megan: I don't think I would introduce it until at least mid-middle school... like after they've done multiplication then you could give them a calculator.

Jessica: If you keep it from them... it's gonna be more like ... something they're not supposed to use, instead of a tool to their learning.

Lynn: Well I think they should have it them, just not in the early grades.

Diane: I'm in a third grade right now and I wouldn't… because right now they're just doing their basic multiplication right now, and … some kids in that class are having trouble with not realizing what it is. And if they got a calculator… They'd never learn anything.

Amy: I think it would be ok earlier. I'm not saying to use it all the time, but just as another, way… Like when you teach a concept you know, every child may not do it exactly the same way but as long as they understand how they're getting the answer… I know we did one exercise showing patterns of five and tens, I thought that could be helpful.

Diane: The kids aren't getting the detailed enough stuff to need it, you know. Like their stuff is so basic… At least not in the younger grades.

Jessica: It depends on what they're doing. You said they're in a real traditional classroom, so she's not gonna give them stuff, that would pull it out

Brenda: I feel like [calculators], it should be introduced as soon as you want. And if it's kindergarten, first grade whatever. Because, it's part of technology. Just like math is a subject, technology is a subject… It shouldn't be like denying one or the other.

Discussion

The results of this study highlight the need to inform PSTs of specific areas in the mathematics curriculum that need to be reformed. It is equally important to have them reflect upon their experiences and existing beliefs about teaching and learning mathematics. In our study, the list of topics found in the *Standards* to be emphasized and de-emphasized in the mathematics curriculum served as a springboard for class discussions and activities. We used this list to provide a glimpse of a smaller grain size that is aligned with the "big ideas" expressed in the *Standards.* It has been our experience that addressing specific topics within the mathematics curriculum can provoke PSTs to begin to examine their perceptions about teaching and learning mathematics. As a result, we learned that most PSTs agreed with the "notion" of changing school mathematics, however, their view of mathematics as a set of rules and procedures interfered with their endorsing the need to change the way mathematics is taught. That is, while there was consensus among our PSTs to the "big ideas" expressed in the *Standards*, conflicts arose in their interpretation and acceptance of decreased attention to traditional mathematical topics and alternative teaching approaches. Further, these conflicts remained constant for most of the PSTs in this study.

We believe preservice teacher's perceptions about teaching and learning mathematics can only be influenced if the preservice teacher is aware of his/her perceptions and are exposed to alternative ways of thinking. Examining our PSTs' beliefs, existing knowledge, and prior experiences is important in understanding our PSTs. However, as this study suggests, we cannot stop there. We also need to probe into our PSTs' rationales for their views and how they are *defining* reform-oriented ideas. Otherwise, we, as teacher educators, may slip into believing that our PSTs are interpreting our courses in ways that we are intending, when actually this may not be the case.

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INSTRUCTIONAL DEVICES IN MIDDLE GRADES MATHEMATICS: THE EFFECTS OF FREE ACCESS ON TEACHER CONTROL AND STUDENT MOTIVATION

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This study examined the effects of a 3-month period of free access to instructional devices on: (a) 10 middle grades mathematics teachers' behaviors and (b) the motivation orientations of the 171 students in their classrooms. Placing the instructional devices in baskets on or near students' desks during free access altered teachers' instructional practices. Teachers identified as Control-Oriented exhibited fewer controlling behaviors, and their students' intrinsic motivation increased. Autonomy-Oriented teachers exhibited more controlling behaviors, and their students' intrinsic motivation decreased. The results raise several questions about the influence of control and autonomy on students' opportunities to use instructional devices in meaningful contexts and about teachers' attitudes and beliefs.

The past three decades have produced considerable research on the use of instructional devices in mathematics (Meira, 1998; Sowell, 1989; Suydam & Higgins, 1977). However, research has not fully investigated the effects on teacher control and student motivation in classroom environments where students have free access to these materials. This paper focuses on the behaviors of middle grades teachers and students during a 3-month period of free access to instructional devices. For the purposes of this study, free access was defined as the opportunity for students to select and use instructional devices that teachers placed in plastic baskets in close proximity to each student, i.e., on or near student desks during the entire mathematics class.

Theoretical Perspective

The teacher's role in classroom interactions might be characterized along a continuum ranging from Transmission (the teacher as sole authority and transmitter of knowledge) to Negotiation (the teacher orchestrates and negotiates a free articulation of ideas). Models of instruction in which teachers rely on the transmission of rules and procedures create classroom environments where mathematical activity becomes rote, making it increasingly difficult for students to develop confidence in their own mathematical thinking (Kamii, 1989). In mathematics classrooms where instructional devices are used, certain teacher verbalizations and behaviors constrain students' explorations of these materials. Meira (1998) argues that instructional devices have no intrinsic value in and of themselves, and meaning is attached to these devices only through learners' activities and the sense-making process that these tools do or do not undergo in learners' hands.

The choice to use or not use instructional devices is most often, literally out of learners' hands. Teachers' decisions to use instructional devices reflect their philosophy of classroom management, choosing to use or not use manipulatives based on their students' previous behaviors in lessons where instructional devices were used (Moyer, 1998). Researchers have identified controlling and autonomous behaviors in teachers (Deci & Ryan, 1987; Deci, Spiegel, Ryan, Koestner, & Kauffman, 1982). Deci et al. (1982) found that Control-Oriented teachers talk twice as much, give three times as many directives, make three times as many "should" type statements, ask twice as many controlling questions, make two-and-a-half times as many criticisms, and give students much less choice than their more Autonomy-Oriented counterparts. If teachers are oriented toward being controlling, the controlling aspects of their rewards or communications will be particularly salient, undermining children's intrinsic motivation. Autonomy-Oriented teachers support autonomy and independence in students and are more likely to provide their students with a choice.

It is through socially meaningful activity that higher mental processes and ideas occur (Vygotsky, 1978). It is important to consider the extent to which children's uses of instructional devices facilitate the development of culturally organized ways of mathematical thinking. Teachers have the potential to repress or enhance the development of students' enculturation into established mathematical practices. Through the use of cultural tools, such as mathematical symbol systems, or perhaps pedagogical tools like manipulatives, students may internalize the meaning of socially shared ideas and use these tools as representations of their thinking. Thus, the learning of mathematics is negotiated among those involved in these meaningful interactions.

Methods of Inquiry

The 10 teachers, 3 African-American and 7 Caucasian, all female, were selected from a group of 20 middle-grades mathematics teachers participating in a 2-week summer institute focusing on the uses of instructional devices for middle-grades mathematics. During the institute, teachers received a classroom set of instructional devices (i.e., fraction bars, base-10 blocks, geoboards, pattern blocks, color tiles, snap cubes, and tangrams) and learned to use them according to NCTM Standards-based

(NCTM, 1989) instructional strategies.

The teachers completed the *Problems in Schools Questionnaire* (Deci, Schwartz, Sheinman, & Ryan, 1981) which assesses adults' (especially teachers') controlling versus autonomy orientations toward children. Using the results of this questionnaire, 10 teachers were selected, five teachers with the highest controlling orientation scores (identified hereafter as Control-Oriented) and five teachers with the highest autonomy orientation scores (identified hereafter as Autonomy-Oriented). Teachers also completed this questionnaire prior to and following the free access phase. The 171 students in this study were from both rural and urban school settings, with minority populations ranging from 21%-65% in each of the 10 classrooms. The 10 middle school teachers selected one class section from among the various classes they taught for participation in the study. The 10 classes represented a variety of ability levels as defined by the teachers and their schools.

During the first 2 months of the academic year, teachers were asked to incorporate instructional devices into their mathematics classes using the instructional strategies learned in the summer institute. Twenty classroom observations (two per teacher) were conducted during this phase. The free access phase occurred during the next 3 months of the academic year. During this phase teachers were asked to provide students with free access to the instructional devices during mathematics classes, placing them in plastic baskets on or near students' desks allowing them access to the instructional devices during the entire mathematics class period every day during free access. Twenty additional classroom observations (two per teacher) were conducted during this phase. Teachers participated in semi-structured audio-taped interviews following the summer institute, and prior to and following the free access phase (three per teacher). Prior to and following the free access phase of the study, students in the 10 classrooms completed the *In the Classroom* survey (Harter, 1981). This self-report survey contains five main subscales, each defined by an intrinsic and an extrinsic pole and is used to identify students' motivation orientations. Fieldnotes and audio tapes of each class session and audio tapes of teacher interviews were transcribed and coded for analysis. Along with the teacher questionnaires and student surveys, 40 classroom observations and 30 teacher interviews were examined for change.

Results

Effects of Free Access on Teacher Control

The ANOVA on the *Problems in Schools Questionnaire* indicated a significant difference between Control-Oriented and Autonomy-Oriented teachers' scores throughout the study, F(1,8)= 45.958, p<.001. The results also indicated a significant interaction between the scores and time, F(2,16)=5.178, p<.05. Scores on the questionnaire are interpreted as follows: a higher score reflects a more autonomous orientation and a lower or more negative score reflects a more controlling orientation. Control-Oriented teachers' scores during the summer institute (M=0.675, SD=1.884) increased after the first 2 months of school (M=1.175, SD=1.467) and decreased following free access (M=1.075, SD=1.399). Autonomy- Oriented teachers' scores during the summer institute (M=7.900, SD=2.064) decreased after the first 2 months of school (M=7.300, SD=2.019) and decreased following free access (M=5.950, SD=2.177).

During the first 2 months of school, Control-Oriented teachers' verbalizations were highly directive, giving deadline statements, solutions, and criticisms during mathematics instruction. One Control-Oriented teacher commented, "I like patterns, routines, and a lot of structure...We have a set way we do the manipulatives." Autonomy-Oriented teachers were more likely to provide their students with choice and access to the instructional devices. One Autonomy-Oriented teacher stated, "I'm just one that believes that if they're [manipulatives] available, students should have access to them. Otherwise we're defeating our purpose of helping them to learn...I leave out the majority of the manipulatives."

During the 3-month free access phase, there were noticeable changes in the verbalizations, behaviors, and instructional practices of both groups of teachers. Control-Oriented teachers reluctantly shared control of the instructional devices with their students. They placed lists on the baskets and checked them at the end of class; they made a list of "free access rules" and posted it on the wall; and they assigned students as team leaders to oversee the instructional devices. However, their lessons included more open-ended problem solving investigations (including the creation of tessellations with the pattern blocks, or building geometric solids and comparing surface area and volume with the snap cubes), than those observed in Autonomy-Oriented teachers' classrooms. During one observation, a Control-Oriented teacher told students to "grab the geoboards and show me one square unit...two square units... five square units." In her follow-up interview the teacher reported, "I hope the kids can show me, surprise me, and find different solutions...Like yesterday with geoboards, that first class would discover things I hadn't even thought about showing and then that allowed me to show the next class...every one of my classes yesterday was completely different."

Autonomy-Oriented teachers seemed increasingly concerned with classroom management during free access. Two Autonomy-Oriented teachers reported removing the instructional devices from students' desks during some classes because of student behaviors. Other Autonomy-Oriented teachers threatened students with "book work" when the class became "too" active. In her final interview one Autonomy-Oriented teacher reflected on free access: "I think it's fine if you have already established rules in the classroom and you are a strong disciplinarian."

Effects of Free Access on Student Motivation

Scores on the *In the Classroom* instrument range from 1-4, with a score of 1 indicating that a student is extrinsically motivated and a score of 4 indicating that a student is intrinsically motivated. Prior to free access, students' motivation scores in Control-Oriented and Autonomy-Oriented teachers' classrooms were similar with M=2.640 and M=2.668, respectively. The ANOVA on the *In the Classroom* survey indicated a significant interaction between students' motivation orientation scores and time, $F(1,169)= 8.482$, $p<.01$. Student scores in Control-Oriented teachers' classrooms increased between the pretest (M=2.640, SD=0.452) and posttest (M=2.705, SD=0.431) indicating students became more intrinsically motivated. Student scores in Autonomy-Oriented teachers' classrooms decreased between the pretest (M=2.668, SD=0.427) and posttest (M=2.564, SD=0.339) indicating students became less intrinsically motivated.

Initially students responded to free access with skepticism, believing their teachers were trying to "trick" them. One teacher reported her students' reactions: "There's a catch to this. She's really not going to let us do this." Because students were apprehensive, they hesitated to use the instructional devices during the first week of free access. During the second and third weeks, students were observed using the snap cubes to build geometric solids, drawing with the circular protractors, and spontaneously using the devices for other mathematical tasks. Students increasingly used the instructional devices during basic computation and problem solving activities. During one observation, a Control-Oriented teacher posed a variety of open-ended problems. Reflecting on this observation, the teacher reported, "I can't remember the problem they were working on...I saw one group, they were using the snap cubes, and then another group, they were using the little centimeter cubes." Teachers reported that students used the instructional devices to review previously learned concepts or to explain concepts to other students. Two students were observed explaining fraction concepts to their group. In the follow-up interview, the teacher reported, "I did have these two kids that were so patient and they taught their group how to subtract mixed numbers with pattern blocks."

Discussion and Implications

It seems contradictory that students in Control-Oriented teachers' classrooms would have increasing intrinsic motivation scores. However, the classroom observations show that during free access, Control-Oriented teachers exhibited some less controlling behaviors resulting in an increase in their students' intrinsic motivation. Students may have perceived their Control-Oriented teachers to be less controlling because they were allowed choice in their use of instructional devices. On the other hand, Autonomy-Oriented teachers may have found themselves with increasing classroom management concerns which may have been perceived by their students as more controlling, resulting in a decrease in students' intrinsic motivation.

A middle grades mathematics classroom should allow students to interact with highly stimulating materials and ideas and construct their own mathematical understandings. The usefulness of instructional devices within this environment must be viewed in connection to tasks and cultural contexts in which they make sense. Providing students with a variety of tools and opportunities encourages them to use instructional devices voluntarily, invent their own problem solving strategies, and engage in higher-level mathematical processes because they see the need and value.

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CLASSROOM COACHING: A CRITICAL COMPONENT OF PROFESSIONAL DEVELOPMENT

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Abstract: The purpose of this paper is to report on an observational study of 14 high school mathematics teachers who had been involved in one to five years of professional development. The study was undertaken to determine the effects of the inservice programs on the actual classroom practices of the participants. Using a coaching model, the two authors observed over 200 classes of 17 sections. Seven categories of results were formed; this paper reports on three of those categories and discusses critical dimensions of the model of coaching.

Perspective

The professional development programs upon which this research was based endeavored to adhere to promising practices identified by the California Postsecondary Education Commission in a study of projects it had supported from 1992 to 1996 (CPEC, 1996). These included the following aspects:

- Successful projects find ways to create systemic change across entire school districts.
- While successful projects need a coherent and consistent set of goals and a reasonable theory of change, they also require strategies that allow participants the flexibility to meet their own personal needs.
- Successful grantees adopt their own system of internal assessment.
- Successful projects develop strategies that enable their teachers to achieve self-actualization.
- Successful projects do much more than explain constructivism; they model it by involving their teacher-participants in well designed constructivist learning experiences.
- Successful projects know that time-on-task is an important determinant in teacher learning as well as student learning.
- Learning about new content or pedagogy is only a necessary condition for improving one's teaching; actually employing that knowledge in the classroom requires more.

More detail about these features of successful programs can be found in Becker & Pence (in press).

The classroom coaching model we used had two main objectives as an extension of the professional development in which teachers had participated: to provide analytic and objective feedback to the teacher with regard to teacher-student behavior; and, to develop within the teacher the desire and ability to reflect upon her/his own behavior and evaluate the results of that behavior as a means of self-improvement. The coaching followed an adaptation of a clinical supervision model, providing teachers feedback that was descriptive rather than evaluative and always requested. The feedback was usually given immediately following the lesson observed, or as close to that as practical given the teacher's schedule.

Methodology

A year-long observational study was undertaken to ascertain the impact of the two professional development projects on the classroom practice of participant teachers. A sample of 24 teachers was invited to participate in this study, based upon the number of years of involvement in the inservice, which varied from one to five, and demographic factors such as gender, ethnicity, school district, and teaching experience. Fourteen teachers agreed to participate, and 17 different classes of the 14 teachers were observed by the authors for a combined total of 210 classroom observations over a six-month period. The authors alternated weekly visits of each teacher.

Courses observed varied from an algebra restart [for students who were unable to succeed in algebra the first semester] to algebra 2/integrated course 3. We observed one algebra restart, six algebra 1 classes, five geometry classes, one algebra 2, one integrated course 1, two integrated course 2, and one integrated course 3. The textbooks used varied from the traditional (3 classes) to "transitional" (10 classes), to integrated (four classes). The sample of teachers included five males and nine females from nine schools and four different school districts. Three teachers were Asian-American and the rest European-American. The participants varied in teaching experience from under five to thirty years, with inservice participation from one to five years.

In addition to the classroom visits, both informal and formal interviews were held periodically with each teacher to probe issues which arose in the lesson.

Description of Coaching

The classroom observations primarily used a clinical supervision model, in which the teacher and observer discussed before the lesson specific items on which the observer would focus. The observer then endeavored to collect data to help inform the teacher about the teaching behavior(s) of interest to him/her. After the observation, the teacher and observer discussed the lesson; the observer shared information about the focus of the lesson with the teacher in a non-evaluative way. That is, the observer reported data on the behaviors of interest without assessing the merit of the teacher's classroom practices.

During each visit, the researcher acted as a participant-observer, taking an active part in the class activities, helping students as they worked individually or in groups and asking questions of students to determine student understanding of new concepts. The observer acted as a true collaborator in the classroom. The debriefing sessions freqently included questioning and reflection on the part of the teacher, as we exchanged ideas for:

- further developing new mathematical concepts;
- reforming the curriculum;
- infusing technology into their teaching;
- ways to teach upcoming mathematical concepts;
- assessment techniques;
- equity issues in teacher-student interactions; and
- ways to help recalcitrant or struggling students.

Results

In analyzing the data from observations and both informal and formal interviews, we formed seven categories on which we partitioned the classroom practices of the fourteen teachers. These categories included equity, multiple representations, and the use of technology, the foci of the observers. The additional four categories - student understandings, use of cooperative groups, alternative assessment, and reform curriculum - arose either from the observations or from questions raised by the teachers in debriefing sessions. Table 1 delineates the partitioning formed, with "yes" indicating strong evidence of practices consistent with the emphasis of the

We will discuss three categories in this paper: multiple representations, technology, and grouping. Additional categories appear in Becker & Pence (in press).

Multiple Representations

Bella is a prime example of the teachers who were using multiple representations. For example, in a first year algebra class, Bella very carefully placed on the chalkboard a table, a graph of the function, and the algebraic equation. However, the symbolic representation was the primary focus and as such was placed in a prominent place on the board, while the

Table 1

Observation Categories

Category	Yes	No
Equity	6	8
Muliple representations	5	9
Technology	5	9
Student understandings	8	6
Use of cooperative groups	8	6
Alternative assessment	5	9
Curriculum	10	4

table and the graph were put in an ancillary position across the room. Also, the table and the graph were never connected to each other and were never an integral part of the discussion of the function. Bella never raised with us the connections among the multiple representations; these connections were clearly not a conscious part of her instruction.

Technology

Overall, we subdivided the use of technology into five categories: use as a tool for concept development; use for data collection and analysis; use for drill and practice; use for computation; and no use observed. Of the 14 teachers, six used technology in relationship to data. One of these used the graphing calculator and the CBL to collect and analyze data; all six used technology to analyze data. Seven teachers were observed on multiple ocassions using technology to aid in development of mathematical concepts, such as functions and geometrical generalizations; the number of teachers who fell into the intersection of these two categories was five. Thus five of the 14 teachers accounted for all observations of these two higher level uses of technology. All 14 of the teachers allowed calculator use for computational purposes. In addition, two of the teachers used computers for drill and practice in arithmetic and algebraic skills.

Bella displayed a rather complex set of classroom practices relative to use of technology. She was quite adept at use of graphing calculators, including incorporation of the Calculator Based Laboratory (temperature probe), which she used to collect data, represent them in a table, and generate a graphical representation. Bella also used the internet to investigate fractals, which was a new topic in her curriculum and one in which she lacked confidence. She used the internet to expand her knowledge of this topic and extend the textbook treatment of it. Although Bella asked questions specific to the use of the technology, she did not expand those questions to investigate its impact on student understanding.

Grouping

Most of the teachers had made a concerted effort to incorporate cooperative grouping, a strategy on which we spent considerable time in the inservice programs, into their instruction. Connor, however, never made use of cooperative grouping in his intermediate algebra class during the course of this research. He did discuss with us his deep concerns about student learning and student understanding, especially after the class visits ended. At this time Connor expressed a desire to change his instructional strategies to seek further student involvement. The following academic year, Connor requested a change of furniture in his classroom, from individual desks to tables and chairs to facilitate group instruction. While observing a student teacher who worked with Connor that year, it was clear than cooperative groups had become an integral part of his instruction. At the end of that year, Connor commented that he would not return to a more traditional classroom configuration, as the grouping seemed to aid student understanding and increase student interactions. Although we cannot claim that the coaching caused this instructional change, the change did seem to be stimulated by Connor's deeper reflections on his teaching which the coaching facilitated.

Discussion

The intensive inservice program over three to five years impacted the teachers in a number of ways. The programs established a common base of content and pedagogical content knowledge (Cooney, 1994) and a common language between teachers and researchers which allowed for a challenge of traditional beliefs about the teaching and learning of mathematics. In addition, the teachers received a considerable amount of resource materials and technology, such as graphing calculators and computer software, essential for implementing new knowledge in the classroom setting. Perhaps most important to the teachers was the strong network of peers formed over the years. This network supported teachers as they attempted to implement change in their curriculum and instructional practices (Becker & Pence, 1996).

Past evaluations of the inservice documented these positive effects of the programs (Becker & Pence, 1996). In this work we used a variety of self-report measures to determine the effect of our inservice programs. However, we are well aware of the limitations of such data. Therefore, we initiated this observational study to ascertain the impact of the two professional development projects on actual classroom practices of participant teachers.

But, we were surprised that many of the 14 teachers reported that the coaching itself was a critical extension of the professional development program. The two-to-one contact over six months with each teacher served several purposes. First, it provided the opportunity for both observers to experience the development of full units of mathematical content. As the content developed, we were able to see how student work progressed and how student understandings grew. Weekly visits enabled the teachers to identify a conflict or concern, ask questions about student experiences and glean insight into their teaching from our feedback. Second, although we had a good working relationship with these teachers before they assented to participate in the study, the coaching helped establish a stronger rapport and true collegial collaboration. We were not the authorities, but rather sounding boards with whom teachers could formulate instructional questions, extend those questions, and work out solutions for themselves.

Finally, our presence supported the teachers who were trying to make real change in their curriculum, instruction, and assessment. At some school sites, parents, other teachers, or administrators hindered reform efforts. Observer feedback and encouragement ameliorated such challenges. We acted as another voice, counteracting negativity these risk-takers sometimes faced.

Summary

We should point out that the classroom coaching described in this paper requires a great deal of time and thus financial resources to effectuate. However, the results of this work indicate that coaching may well be a critical component to consolidate changes in classroom practice as a result of inservice programs.

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TEACHERS' EVOLVING WAYS OF THINKING ABOUT THEIR STUDENTS' WORK

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This paper will focus on changes that occur, over time, in upper elementary school teachers' knowledge about the nature of mathematics and children's thinking, and how this impacts both the learning environments they provide and their approach to assessment. In particular, we will document one teacher's approach to helping students reflect on and assess their own work. This will be done by analyzing the relationship between the solutions developed by two groups of students to a thought revealing mathematical investigations, and their responses to the self assessment form developed by the teacher.

Research Design and Theoretical Framework

This study focuses on one teacher within a group of upper-elementary teachers involved in a three year teacher development intervention. The teachers had been selected by their principals to form a task force representing four small regional schools. The purpose of the intervention was to help the teachers consider and then implement, more meaningful forms of mathematics instruction and assessment, first with their own students and then with their colleagues. An underlying premise of this intervention was that teachers need appropriate experiences and materials from which to build new models of instruction, learning, and assessment. Teachers must also be provided with opportunities to develop a deeper understanding of the mathematical concepts they are expected to teach and an increased awareness of the ways in which children learn (Schorr, Maher, & Davis, 1997; Janvier, 1996; Alston, Davis, Maher, & Schorr, 1995; Cobb, P., Wood. T., Yackel, E. & McNeal, B. 1993; and Davis, 1984). To this end, teachers met regularly with researchers in workshop sessions where teachers were presented with challenging problem tasks and asked to work together to produce solutions that represent important mathematical ideas. After sharing their own ideas and representations, they agreed to use these or similar tasks in their own classrooms. During classroom implementation, often with researchers present, teachers were encouraged to recognize and analyze their students' evolving ways of thinking about these mathematical ideas. They studied and assessed their students' work, and selected particularly interesting products to share in subsequent workshop sessions. Studying these samples of student work, as a group, provided the opportunity to both consider the development of these ideas in students, and to discuss the pedagogical implications of using this approach in their own classrooms.

In this research design, the researchers' goal was to simultaneously stimulate and document changes in student and teacher knowledge over time. This was accomplished through the use of "thought-revealing activities" in which teachers and students were repeatedly challenged to reveal, refine, revise, and extend important aspects of their ways of thinking. The thought-revealing activities that were emphasized were intended to promote learning, and simultaneously produce a trail of documentation that revealed important aspects about the nature of the constructs that developed. So, for example, the activities that were chosen for students were specifically designed to document and reveal the development of mathematically significant models. As such, these activities could then be used to help teachers in their assessment of students' developing knowledge. Teachers were also encouraged to revise and refine their methods for assessing their students' products and mathematical thinking. This provided the researchers with an opportunity to stimulate changes in the teachers' practice by helping them to become more familiar with student-generated ways of thinking (Schorr & Lesh, 1998).

Data

The data for the study include: (1) The teachers' responses to questions asked periodically about what they perceive to be the main ideas in particular problem activities, and important student behaviors to observe and document during classroom problem-solving sessions; (2) Reflections of researchers based upon their observations, both of teachers and students, during classroom problem-solving sessions; (3) Student's work from thought revealing activities that were analyzed during working group sessions; (4) Student's reflections on their own and other students work; (5) The teachers' criteria for analyzing and assessing the children's work for each task, and their continuing development of generalized criteria for assessing individual children's developing knowledge; and, (6) Researcher field notes taken while working with teachers in classrooms and workshop sessions.

Discussion of Results

To summarize, results of the whole study indicate that over the course of the project, all of the teachers: (1) Changed their perceptions regarding the most important behaviors to observe when students are engaged in problem activities; (2) Redefined their roles during classroom problem-solving sessions; (3) Changed their views on what they considered to be strengths and weaknesses of student responses; (4) Changed their views on how to help students reflect on, and assess their own work; (5) Reconsidered their notions regarding the use of the assessment information gathered from these activities; and (6) Redefined their needs regarding both their own and their students' mathematical content knowledge.

For the purposes of this paper, we will focus on how one teacher began to collect and use assessment information in different ways to inform instruction, and how she helped students to reflect on and assess their own work. As a context, it is important to understand where the teachers were at the beginning of the intervention. Early in the intervention, our data indicates that the teachers often chose student products as exemplary based upon the appearance of the product rather than its content. They also tended to focus primarily on a student's ability to work well within his or her group rather than on the mathematical ideas that were emerging (see Schorr & Lesh, 1998). Since space restraints do not allow us to document the growth of all of the teachers, we will focus on representative examples taken from one of the teachers, Jane, a fifth grade teacher.

At the end of the second year of the intervention, Jane designed an "Assessing Myself" sheet for her students to use as a means of reflection about their own problem solving. This sheet was filled out by each student after all the groups had shared their solutions with the class. It included the following questions:

- What were you able to contribute to the solution of this problem?
- What kind of outside information, math, tools, or materials helped your group solve this problem?
- What kind of math skills and ideas did you use working on this problem?
- If you could change your product, what would you do to make it better? Explain how these revisions might improve your products - feel free to use ideas from other groups, or ideas that you heard in your group that didn't get used.
- What did you learn while working on this project?

The problem activity, "Olympic Proportions", was a pilot version of one of the PACKETS Investigations for Upper Elementary Grades, developed at ETS with support from the National Science Foundation. The teachers had worked on the investigation themselves during one of the workshops and Jane had chosen to implement it with her 5th grade students because it dealt with ideas about fractions and ratio in a context that required the analysis and interpretation of data. The problem materials provided to each group included photographs of 5 athletes and a drawing of a "Greek Ideal Athlete". The students were asked to compare various body parts of each to the Greek Ideal Athlete, using the athlete's head length as the unit of measure. Children worked in groups to complete their solutions and write a report describing and justifying their results and explaining their method. Group 1's solution included the following:

> We judged the athletes in the following way. First we made a ruler using the heads of the athletes. After we had done that for each athlete we divided each of the heads into five equal sections. Then we scored each athlete the following way. Each fifth of a head that the athlete is away from the ideal is a point. So, for example if the ideal was five heads and the athlete is four and three-fifths away from the ideal he would receive two points because he was two-fifths away from the ideal. After all the ideals and all the athletes had been scored we added up all the points. The athlete with the lowest score is *Classic Man*. This is a good way to decide who is *Classic Man* because it uses a fairly simple scoring system. The only problem we encountered was when the ideals were three and three-fourths and one and one-half because the denominators aren't five. For three and three-fourths we decided, since the difference is so small it wouldn't make much of a difference, to consider it three and four-fifths. For one and one-half we considered it one and two-fifths and half of a fifth which it is equal to. To use this method on other photos just follow the steps we followed while we were working on this problem.

Their solution was accompanied by two charts, one indicating the actual measurements, and the second indicating the actual scores.

The solution of a second group of students included the following:

> 1) We took one of the strips of paper, lined it up with the persons head, and made a mark at the chin. 2) Then we moved the #1 mark to the top of the head, and made a second mark at the chin. 3) We did this until we had 8 marks. Last, we measured all the athletes, then compared their results to the Greek measurements....You can do this with other photos by doing steps 1-3, and then measuring the body parts as shown on the recording paper. Example: Lower leg. Then see how close each person came to the Greek expectations. You can figure this out by seeing how many of the persons numbers match the Greek numbers. If some don't match, see how

far away it was, then add up the numbers of fractions to see who is closest without being to far away. Example: 1/2 + 1.

This product also included a bar graph which showed the total accumulated difference from the ideal for each of the athletes. The athlete with the smallest accumulated difference was considered to be the closest to the ideal.

In analyzing the reflections of these students, two episodes are particularly interesting. One member of the first group noticed that the measures of various body parts in relation to the head, in most cases, included fractions that were closer to fourths than to fifths. As she considered the fourth question in the Assessing Myself sheet, she wrote the following: "If I could change our product I would have used a 4 for our scoring system because all the fractional ideals were or could easily be converted to fourths. This would have made our product more accurate because we wouldn't have had to estimate." In responding to the third question, she had identified the math skills and ideas used in the following way "The math skills and ideas we used were fractions converting fractions, comparing fractions, and estamation (sic)".

This student originally divided her unit of measure (the head of the athletes body) into fifths, which required her to "estimate" her measurements to the closest fifth. Her "estimates" appear to be carefully considered, however. She in fact appeared to invent a symbolic representation for one half using fifths. Upon reflection, she realized that dividing the head into fourths would have been a more efficient, or certainly easier, approach to the problem.

One of the students from the second group, in response to question 4 of the "Assessing Myself" sheet, stated that "I would see how much each person missed each cadigory (sic). I would see by how they missed it and if it was by to much I would eliminate that person". His comments reflect the fact that his original solution had been based on accumulated differences. His group product did not account for major differences from the ideal for any one body part within the total accumulation. He is now suggesting that he would consider eliminating a person if one body part missed the ideal "by to much" regardless of the accumulated difference.

The two responses to the "Assessing Myself" sheet gives strong evidence that these questions provided the students an opportunity to reflect on their strategies for solving the problem. It also gave them an opportunity to reflect on the strategies offered by other students, and consider how they might revise and refine their work accordingly. This, along with the written products and classroom observations, provided the teacher with useful information about their mathematical thinking. In particular, the first case provides important insights regarding the student's knowledge of fractions. The second provides a basis to build ideas about range, measures of central tendency, and variance.

Conclusion

There are several main points that should be made. The continuing cycle of studying, implementing, and assessing the students results of "thought revealing" activities was itself a "thought revealing" activity for both teachers and researchers. One "product" of this activity for teachers included Jane's "Assessing Myself" sheet. The value of this product was an outgrowth of Jane's thoughtfulness about her teaching, and her ability to create a classroom environment where meaningful mathematics could take place, and students could use tools like the "Assessing Myself" sheet in a meaningful manner. Jane's continuing growth is evidenced by additional data from the third year of the project that includes an additional assessment form for her students to use in reflecting on each of the groups solutions. A final piece of documentation is her response letter written to each group including comments and questions about the mathematics that they had done.

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MATHEMATICAL AND PEDAGOGICAL AUTHORITY: DESCRIBING THE CONCEPTIONS OF DEVELOPING SECONDARY MATHEMATICS TEACHERS

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Two important components of mathematics teacher development: mathematical authority and pedagogical authority, are presented. Examples from four studies about teacher change are used to describe these categories. Mathematical authority deals with teachers' knowledge about content and the manner in which they share it with their students. pedagogical authority characterizes teachers' use of their own voices and conceptions in determining classroom activities and events.

This paper focuses on authority in mathematics teacher development. Results of four empirical studies and several theoretical discussions highlight a discussion about *mathematical authority* and *pedagogical authority* (Wilson and Lloyd, in press). It is important for teachers to focus on mathematical content and how to make that content more accessible to students. An important component of teacher development involves the realization of the power of moving away from an emphasis on procedures toward a greater emphasis on relational or conceptual understanding. Acknowledging and honoring diverse ways of knowing, and allowing students to build their own understandings through cooperative exploration, require even more substantial changes. The difficulty of this kind of change may be related to teachers' underlying pedagogical orientations: many mathematics teachers see themselves as the ultimate arbiters of mathematical correctness and find it extremely difficult to share responsibility with their students.

Methods and Data Sources

The ideas described in this paper build upon themes my colleagues and I have developed and discussed previously (e.g., Wilson & Goldenberg, 1998; Wilson & Lloyd, in press). Examples are taken from data I collected in four different studies (each study focused on secondary teachers interested in teaching mathematics in innovative ways). The first study investigated 18 high and middle school mathematics teachers who each completed 2 written surveys and were interviewed once in 1990. The focus on these teachers was their conceptions about assessment. The second study investigated 3 prospective secondary teachers during the final stages of an undergraduate teacher certification program (1991). These individuals were interviewed 7 times each and observed throughout their participation in a teacher education class. The third data source was a study of 4 middle school mathematics teachers attempting to reform their teaching practice. These teachers were interviewed between 10 and 25 times each over a 3-year period (1993-96), and they were observed in numerous meetings during this period. The final study investigated 3 veteran high school teachers implementing a reform-oriented mathematics curriculum for the first time in a ninth grade class (1995-96). Teachers were observed daily 25 times and interviewed 8 or 9 times each. Although these four studies involved relatively few participants because each focused on detail, there were several themes prominent in the experiences of most of the teachers. Two of these themes are discussed in the following sections.

The Model

Mathematical Authority

The first common theme is related to the writings of Paul Ernest (1991), and Alba Thompson (1992) who suggested that teachers' orientations to the subject of mathematics are intimately connected to how they teach. *Mathematical authority* deals with the manner in which individuals understand and come to understand mathematical ideas. An important component of this category is what Skemp (1987) referred to as relational understanding and Hiebert and Lefevre (1986) described as conceptual knowledge. The positive impact on students of secondary teachers possessing and sharing rich, flexible, and connected understandings of mathematical concepts is illustrated, for example, in the work of Sowder, Phillip, Armstrong, and Schappelle (1998). It is essential for teachers to appreciate that they cannot simply explain all important connections and concepts to students, they must be willing at times to *share* control or responsibility of the mathematical ideas. This sharing allows students to accept responsibility for their own learning. Such sharing is also an important part of the mathematical authority category.

Mr. Burt's experiences illustrate this category. Mr. Burt attempted to implement a highly pictorial and conceptual approach to fractions (Wilson & Goldenberg, 1998). He adjusted his instruction to emphasize deep understandings, shifting his focus from correct procedures to correct concepts. However, he maintained his role as the primary arbiter of correctness, carefully explaining every idea. In contrast to Mr. Burt, Molly was a preservice teacher who claimed that she would use many "hands-on" activities in her future teaching to emphasize understanding of mathematical relationships (Wilson, 1994). In other words, she wanted to share responsibility with her students. However, her own image and understanding of mathematics primarily as a set of procedures to be mastered, conflicted with this claimed acceptance of more innovative teaching techniques. Molly's ability to share responsibility with her future students was inhibited by her own understanding of mathematics.

The other sixth grade mathematics teachers in Mr. Burt's school communicated concerns about their ability to teach conceptually even though all of them allowed students to cooperate and explore on a regular basis. One recurring theme in their attempts to teach about fractions was their own learning experience emphasizing rules rather than concepts. Mr. Burt had difficulty sharing mathematical responsibility with his students even though he understood the topic very deeply. The learning experiences of Molly and the other teachers at Mr. Burt's school prevented them from understanding the content conceptually. All were unable to share control or responsibility of the mathematical ideas with their students. These examples emphasize the importance of both deep content knowledge and a desire (and ability) to share mathematical responsibility with students.

A final illustration of mathematical authority comes from a study of veteran high school mathematics teachers attempting to implement a reformed curriculum (Wilson & Lloyd, in press). Ms. Gifford possessed a deep understanding of the mathematical ideas she wanted her students to learn. She typically brought examples to whole-class discussions that were not in the students' text materials but illustrated connections between the current topic and other mathematical ideas or the real world. However, she also understood the importance of allowing her students to investigate and explore mathematical ideas. Her goal of emphasizing conceptual understandings and problem-solving is reflected in her comments as she began the school year:

> My hope is that they'll be able to problem solve better. They'll be able to look at real-world situations and know what to do with them. . . . And I'm hoping that Core-Plus gives them more tools to investigate different situations and say, "Well maybe I could analyze it this way or I could analyze it this way." They'll know that they don't have to always do it the same way and yet they can come up with the same answer or answer that's comparable. I think they'll have more confidence in their problem-solving skills.

This statement evidences Ms. Gifford's understanding of the potential value of students' developing multiple perspectives on and solutions to meaningful mathematics problems. She believed that it was important for students to develop connections through cooperation and exploration. She continued:

> [The mathematics is] something they can get out of their seats and put their hands on and have some ownership of the data that's being used. Rather than my giving them whatever equation, *they* come up with the relationship.

Along with student ownership of the mathematics in the activities, for Ms. Gifford the role of the teacher also changed. As she explained, "In theory the teacher goes from being ruler or dictator who is disseminating all the information, to a facilitator. . . . You're there to assist." Ms. Gifford not only possessed a deep understanding of the mathematical ideas that she wanted her students to understand, but believed strongly that she should share mathematical responsibility with them.

Pedagogical Authority

In addition to teachers possessing deep mathematical understandings and sharing these understandings with students in responsible ways, the development of teachers' voices of authority in deciding *how* to share mathematical responsibility with students is critical. Teachers' experiences with reform recommendations are influenced by the nature of their views about *pedagogical authority*, that is, the strength of their *own* voices and conceptions for determining classroom activities and content (Wilson & Lloyd, in press). For teachers to effectively apply pedagogical authority it is important for them to view the classroom as a contextual domain in which "correct" or "best" depends upon circumstances. Through critical reflection on practice, using their own pedagogical authority, teachers determine for themselves the value of particular innovations. There is a fine but extremely important distinction between mathematical authority and pedagogical authority. Teachers, like their students, must gain ownership of new ideas in their own ways. It is easy for reformers to simply tell teachers the best or most appropriate ways to do things, and expect teacher compliance. The reflective judgement model developed by Kitchener and King (1994) describes how adolescents and adults develop in relation to their orientations to authority. This model is central to the ideas proposed here because it emphasizes the importance of reflection and the need for the source of correctness to be internal to the individual.

Mr. Burt is an excellent example of one mathematics teacher's struggle to develop pedagogical authority (Wilson & Goldenberg, 1998). As illustrated in the previous section, Mr. Burt did not always share mathematical responsibility with his students. However, his ability in certain contexts to do so was related to his own acceptance of various beliefs and behaviors. His incorporation of an *experimentalist* (Dewey, 1958) teaching approach, in which he based decisions about what do on circumstances, illustrates this idea. His own critical reflection enabled him to make sense of and at times use reform ideas. However, it was not just Mr. Burt's compliance with reform ideas that illustrated his development and use of pedagogical authority. Mr. Burt's reflection about his own practice and consequent decision to use more traditional methods because they made sense to his circumstances also illustrates this idea.

Secondary teachers discussing their assessment practices provide other examples of the importance of reflection. It was common among the 18 teachers interviewed to elaborate about using open-ended, innovative assessment items during class discussions but not in formal evaluations. Reflection about this inconsistency—teachers wanted their students to explore rich problem situations but did not feel an obligation to make them responsible for such connections in formal assessment activities—might enable teachers to adopt alternative assessment strategies. The focus of these teachers' experiences (as well as those of Mr. Johnson, which are described below), is on *why* teachers make decisions, not just what those decisions are.

Mr. Johnson's desire to promote student autonomy was based on him thinking critically about his own past experiences (Wilson & Lloyd, in press). A few students were discussing why some of the graphs they had drawn were continuous while others consisted of isolated points. Students agreed that the best reason was that connecting points helps one to see the pattern or relationship depicted by the graph. When Mr. Johnson walked past the group, the students asked him why they should connect the dots. Instead of explaining, Mr. Johnson replied, "You want to discover that on your own." When the group explained their reasoning, Mr. Johnson replied, "That's as good a reason as any." Mr. Johnson wanted students to rely on their own ideas and not depend on him. This dialogue illustrates Mr. Johnson's genuine willingness to encourage student independence and within-group cooperation. Mr. Johnson's indicated to me that his decision to share mathematical responsibility was based on his reflection about the relative ineffectiveness of more traditional, teacher-centered instructional approaches. According to Mr. Johnson, many students thought teacher lectures were "boring and confusing" but that group explorations were "refreshing." He claimed that he lost "half his students when he talk[ed]." On another occasion, Mr. Johnson explained that in past years most of his students were so turned off by his lectures that he could "do anything and they would just sit there." In such situations, he would plead, "You guys are dead!" and students would respond, "But it's boring." He agreed that traditional, teacher-directed classrooms are very boring. As these comments illustrate, based on his reflections on past experiences, Mr. Johnson was using his own voice in deciding what to do in the classroom.

Concluding Comments

As Goldsmith and Schifter (1997) indicated, the reconceptualization of teaching and learning roles often requires an extended period of time for teachers. Perhaps this extended time requirement is related to teachers' need to explore and develop their own strategies. It is important that the source of correctness be internal for teachers. Reflection on practice seems to be a key factor in the development by teachers of pedagogical authority. The two categories illustrated in this paper are intimately connected. However, the process of recognizing both mathematical and pedagogical authority will help teacher educators better assist mathematics teachers who are reforming their practice, as well as researchers who are attempting to characterize teachers' experiences. There is an acknowledgement of the need for rich content knowledge by teachers, but also a recognition of the importance of teachers developing their own ways of adapting and using successful teaching strategies.

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META-LEARNING IN MATHEMATICS: HOW CAN TEACHERS HELP STUDENTS LEARN HOW TO LEARN?

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Nationwide figures suggest that 1.1 million college students are currently studying mathematics at the remedial level and the numbers are increasing (AMATYC, 1995). Many of these students struggle through their remedial courses to no avail. This report describes the birth, design, and implementation of a study based on the assumption that many students enrolled in remedial mathematics courses are unsuccessful because they simply do not know how to study and learn mathematics. A remedial mathematics course was developed at the Ohio State University for the purpose of addressing this need. Nineteen students enrolled in this strategy-embedded remedial mathematics course during the fall of 1997 and received mathematics-specific learning strategy instruction as part of the content course. Data indicate that these students' mathematics-specific self-regulatory skills improved throughout the course and these improvements were sustained during subsequent mathematics courses.

The direction for this research investigation grew out of a seven-year observational/interview study with underprepared college mathematics students. The researcher/teacher found that contrary to instructor opinion, the majority of college students enrolled in developmental mathematics courses were not purposefully engaging in behaviors to undermine their success. It was discovered that the majority of students who were enrolled in remedial mathematics courses were actually devoting a significant amount of time and effort to their mathematical studies, but often to no avail.

As a result of these observations, the researcher/teacher began to question the directions of the students' efforts and discovered that many were misguided. Most students enrolled in college developmental mathematics courses simply did not know how to productively approach studying and learning in a mathematics-specific context. Furthermore, students were often unaware that their *approaches* to learning mathematics were ineffective. For example, after course examinations, students who were dissatisfied with their performances were asked how they would approach learning differently the next time. The majority of students reported that they would continue to do the same kinds of things that they had done before, but increase the frequency of these activities.

In an attempt to help underprepared college mathematics students analyze, evaluate, and re-direct their efforts, a *Strategy-Embedded Remedial Mathematics* course was developed at The Ohio State University. This course differed from the typical developmental mathematics course in one significant way. That is, mathematics-specific learning and organizational strategies were taught side-by-side with the mathematical content. This research report focuses on the implementation and results of this course.

Perspectives and Guiding Frameworks

Both practical and theoretical lenses were used to build multiple *conceptual frameworks* (Eisenhart, 1991) for this investigation. The practical base for the project was grounded in the work of Dr. Claire Ellen Weinstein and in the details of her Learning-to-Learn course (Weinstein, 1987, 1994, 1996a, 1996b). Data collection and analysis were guided by theories of self-regulated learning (Schunk, 1990, 1991; Zimmerman, 1990a, 1990b, 1995, 1998).

Theories of self-regulated learning attempt to explain students' personal initiative in acquiring knowledge and skill (Zimmerman, 1990a). There are three interrelated components of most theories of self-regulated learning in academic settings: strategy use, metacognitive awareness, and motivational control.

Self-regulated learning strategies are the "actions and processes directed at acquisition of information or skills that involve agency, purpose, and instrumentality perceptions by learners" (Zimmerman, 1990a, p. 5). Strategies that are employed when a student attempts to complete a homework assignment in mathematics, for example, may include: (a) reviewing related notes from lecture; (b) reading and monitoring understanding in the text; (c) checking homework solutions with answers found in the back of the text; (d) generating questions to clarify understanding; and (e) seeking assistance from others (classmates, instructors).

Good strategy use involves more than knowing *what* to do. It also involves knowing *how*, *why*, and *when* to do it. For this reason, metacognitive awareness is recognized as one of the key components of self-regulated learning theory. According to Brown (1987), metacognition is bi-dimensional—it includes both knowledge of cognitions and regulation of cognition. The *knowledge of cognitions* dimension of metacognition includes knowledge about ourselves as learner and the factors that influence our performance (declarative knowledge); knowledge about learning strategies (procedural knowledge); and knowledge about when and why to use a strategy (conditional knowledge) (Bruning, Schraw, & Ronning, 1995).

The *regulation of cognition* dimension of metacognition includes planning, regulation, and evaluation. Selecting appropriate learning strategies and allocating available resources are included in the *planning* component. *Regulation* activities include: making predictions, selecting appropriate repair strategies, and strategy sequencing. *Evaluation* refers to one's self-appraisals of products and regulatory processes.

The *motivational control* component of self-regulation addresses a student's ability to set goals, evoke positive self-beliefs, and emotionally balance the demands of the learning process.

Students displaying motivational control set realistic goals, regulate their anxieties and frustrations, and recognize the sources of their own strengths and weaknesses. They do not allow themselves to engage in self-defeating thought—"I don't know why I even bother doing this homework, I'll still fail the test." They make accurate performance-related judgments—"I'm not surprised that I got a 'D' on the exam since I didn't go to class at all last week." And, they understand themselves as learners—"I know that if I wait until the last minute to study then I get completely stressed out."

Taken together, these inseparable components offer an explanation of learning in academic settings. According to theories of self-regulated learning, effective learners are aware of a variety of learning strategies. They know how to choose appropriate strategies and when to modify their approaches. They are able to control their frustrations and anxieties as well as any self-defeating behaviors that may impede learning. Research suggests that learner self-regulation can be improved through instruction (Winne, 1995, Zimmerman, 1990a).

Zimmerman (1998) identified three major cyclical phases that self-regulated learning theorists have analyzed with respect to the learning process. These phases are forethought, performance, and self-reflection. The forethought phase includes goal setting, strategic planning, self-efficacy beliefs, and goal orientation. Forethought processes prepare the learner for the performance phase. Performance subprocesses include attention focusing, self-instruction, and self-monitoring. These performance subprocesses, in turn, affect self-reflection. Self-evaluation, self-reactions, and attributions of success and failure are examples of self-reflection processes.

These phases are cyclical in that self-reflection ultimately influences forethought and hence the cycle begins again. Data reported herein were analyzed in terms of these three cyclical phases—forethought, performance, and self-reflection.

Methods and Data Sources

This qualitative investigation examined underprepared college mathematics students' approaches to learning mathematics while enrolled in a strategy-embedded developmental mathematics course and then subsequently enrolled in a college-level mathematics course that did not purposefully embed mathematics-specific strategy instruction. Data were collected in the form of students' written work, journal entries, classroom observations, and questionnaire responses. Additionally, focus-group and individual interviews were conducted and course-related grades were collected.

As part of the strategy-embedded developmental course, the students (n = 19) received instruction on reading a mathematical text, learning from a lecture, taking notes, and studying mathematics. Students also engaged in course-related activities that required them to investigate available resources, monitor their understanding, and analyze their own mathematical errors. Transfer of the use of learning strategy skills and activities from the developmental course to subsequent college level courses was explicitly and regularly addressed throughout the term. Additionally, strategy-specific course requirements (but not recommendations) decreased as the semester progressed. Students were encouraged to take control of their own learning by selecting their own strategies and reflecting on the effectiveness of their choices.

Eight of the students (primary participants) who completed the strategy-embedded developmental mathematics course were interviewed as they progressed through their first college-level mathematics course (College Algebra). Four additional students (secondary participants) who had not enrolled in the strategy-embedded course—but were previously enrolled in a developmental mathematics course—also participated in this phase of the investigation.

Results

The 19 students enrolled in the strategy-embedded developmental course: (a) reported feeling less teacher driven; (b) took responsibility for their own successes and failures; (c) attributed failures to actions rather than abilities; (d) became familiar with a variety of different types of resources; (e) made decisions about when and why to use these resources; (f) learned to reflect on the outcomes of their decisions; (g) gained a sense of control over their mathematical experiences; (h) began to see mathematics as a connected whole; and (i) earned higher grades on common course examinations than students who were not enrolled in the strategy-embedded course.

Data indicate that the students who participated in the strategy-embedded developmental mathematics course continued to take responsibility for their own successes and failures and to exhibit control over their learning during subsequent college-level courses. They were empowered as a result of strategy instruction. Change occurred and was sustained as a result of this newly acquired knowledge.

With respect to the *forethought phase* of self-regulation, the primary participants continued to set goals and devise plans of actions for navigating their college-level courses. They also maintained a high sense of self-efficacy for learning throughout the subsequent term. With respect to the *performance phase*, the primary participants were found engaging in a variety of different learning strategies. Self-monitoring, a performance-related activity, was difficult for all the participants. The lack of explicit emphasis on the structure and inter-relatedness of mathematical content in the college-level course contributed to the students' difficulties in this area. The primary participants used the knowledge they acquired during the *self-reflective phase of self-regulated learning* to develop new goals and devise new courses of action. These students, unlike the secondary participants, continued to refine their approaches to studying and learning throughout the term as the cycles continued.

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DEVELOPING PRESERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF SLOPE

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The curriculum and pedagogy of school mathematics have come under much scrutiny in recent years. The National Council of Teachers of Mathematics (NCTM), for example, suggests that the emphasis of mathematics curricula should move away from rote memorization of facts and procedures to the development of mathematical concepts. It is also recommended that connections among various representations of those concepts be investigated by students through problem solving (NCTM, 1989). In order to facilitate these reforms in mathematics education, teachers must have a strong knowledge base including knowledge of mathematics, knowledge of student learning, and knowledge of mathematics pedagogy (NCTM, 1991). It is the task of teacher educators to help preservice and inservice teachers develop these types of knowledge, yet research has indicated that the task is often difficult (Brown, Cooney, & Jones, 1990).

One important type of knowledge for both preservice and inservice teachers is pedagogical content knowledge. Shulman (1986) has described pedagogical content knowledge as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Two important components of pedagogical content knowledge are understanding of representations for particular mathematical topics, and insight into students' potential misconceptions of these topics.

This investigation focuses on the development of preservice secondary mathematics teachers' pedagogical content knowledge of slope. Slope appears in various forms within the secondary mathematics curriculum. Geometrically, the slope of a line is a measure of its steepness, determined by the ratio rise/run. Algebraically, the formula $(y_2 - y_1)/(x_2 - x_1)$ represents slope. Slope also appears parametrically in the equation $y = mx + b$.

It is believed that the use of real-world representations help students develop understanding of abstract mathematics (Fennema & Franke, 1992). Real-world representations of slope appear in two forms: *physical* situations such as mountain roads, ski slopes, and wheelchair ramps, involving slope as steepness; and *functional* situations such as time versus distance or quantity versus cost, involving slope as rate of change. Considering the emphasis on the study of functions in high school (NCTM, 1989), functional representations of slope are especially important.

A previous investigation of secondary mathematics teachers' knowledge of slope revealed that both preservice and inservice teachers were more likely to include physical representations of slope than functional representations in their descriptions of classroom instruction. Some teachers failed to mention either type of representation. Furthermore, teachers expressed concern with students' understanding of the meaning of slope, but the specific student difficulties they identified focused on procedural aspects (Stump, 1996).

The purpose of this investigation was to see if experiences in a mathematics methods course can help preservice secondary mathematics teachers develop the two components of their pedagogical content knowledge for the concept of slope. The investigation focused on the following questions: (1) What do preservice teachers learn about students' difficulties with slope, and how is this knowledge reflected in their teaching? (2) What do preservice teachers learn about various representations for teaching slope, and how do they use these representations in their teaching?

Methodology

This study employed the perspective of "practitioner research," as defined by Liston and Zeichner (1991), who used the label to refer to "inquiries that are conducted into one's own practice in teaching or teacher education" (p. 147). The participants were six mathematics teaching majors enrolled for one semester in a secondary mathematics methods course at a mid-sized midwestern university. After the course, five of the preservice teachers worked in pairs or individually to teach MATHS 107, a basic algebra course at the university, for the entire semester. As a practitioner researcher, my various roles included instructor of the methods course, supervisor of MATHS 107, and researcher for this study.

The methods course adopted a constructivist perspective similar to that of a classroom devoted to the development of mathematics content knowledge. That is, the course was based on the assumption that the preservice teachers' learning was contingent on their own activity and involvement in the various readings, discussions, and projects (Maher & Alston, 1990).

One goal of the methods course was to help preservice teachers develop their knowledge about students' difficulties with slope. Thus, they completed an Interview and Analysis Assignment, in which they interviewed one high school chemistry student and one college student enrolled in MATHS 107 to learn about their understanding of slope, and then compared the interviews through written analysis.

Another goal was to expand their repertoires of representations for teaching the concept of slope. Functional representations were emphasized with the expectation that the preservice teachers would develop an appreciation for the importance of slope as rate of change. Toward that aim, the class participated in activities using graphing calculators, designed to focus on the interpretation of slope within the context of functional situations involving linear relationships.

In the methods course, each preservice teacher wrote a series of Slope Lesson Plans for a hypothetical middle school or high school algebra class. In MATHS 107, each preservice teacher taught a lesson involving slope. These lessons were video-taped and the tapes were transcribed. The preservice teachers were interviewed toward the end of the methods course, and again after they taught their slope lessons in MATHS 107. The interviews were audio-taped and the tapes were transcribed.

The data sources for this investigation included an Initial Survey, the Interview and Analysis Assignment, the Slope Lesson Plans, the transcripts of the audio-taped and video-taped sessions, and the handouts prepared by the preservice teachers for their MATHS 107 lessons. The data were analyzed using the process of analytic induction described by Erickson (1986).

Results

The data analysis revealed various patterns in the development of individual preservice teachers' pedagogical content knowledge of slope. Two examples are presented here. Accounts of two preservice teachers illustrate the development of knowledge of students' difficulties with slope and knowledge of representations for teaching slope.

Joe: Knowledge of Students' Difficulties with Slope

The preservice teachers came to the methods course with specific beliefs about students' knowledge of slope, as evidenced by their responses to the question, "What difficulties do you think students might have with slope?" on the Initial Survey. These specific beliefs then appeared as themes throughout the preservice teachers' work in the methods class and later in their teaching.

For example, Joe, a returning graduate seeking a teaching license, speculated that students "do not follow the formula Dy/D*x*: (a) they don't understand it, (b) they switch the order, or (c) they make computation errors." As we can see, his focus was procedural. Early in the methods class, however, we discussed differences between conceptual and procedural knowledge (Hiebert & Lefevre, 1986), and later, when Joe completed his Interview and Analysis Assignment, his responses reflected this discussion.

He began each interview by asking, "In your own words, describe what slope is." One student, Bill, answered, "Relation in the change between x and y coordinates," while the other student, Brad, answered, "Given two points on a graph, slope is $(x_1 - x_2)/(y_1 - y_2)$." In his analysis, Joe wrote, "It was interesting to see Bill's memory of slope to be more conceptual and Brad's memory of slope to be related to the formula (even if it was an incorrect answer)."

It was encouraging to see evidence of Joe incorporating a new idea into his knowledge for teaching. However, the following comments, also from his Interview and Analysis Assignment, suggested a misconception about the goals of mathematics instruction:

> The area in which both of the students were weak was working with equations concerning an algebraic representation of slope. They both had trouble coming up with an equation for a line that was already graphed, and vice versa. This is something that I found very surprising, *as well as encouraging* [emphasis added]. What we have discussed so far in this course is the need to focus more on conceptual understanding, and not so much on memorizing formulas. Well, these students exhibited these characteristics almost to a fault. They were very good on the conceptual questions, and not so good with algebraic representations and formulas.

Fortunately, Joe did not completely dismiss the importance of algebraic representations and formulas. He developed these representations in both his Slope Lesson Plans and his MATHS 107 lessons. However, despite his initial propensity toward procedural aspects of slope, Joe displayed a willingness to try teaching with a focus on the development of conceptual understanding. When he introduced the concept of slope to his MATHS 107 class, he spent the first few minutes "philosophizing" about slope, rambling a bit, setting the stage for a discussion about the *meaning* of slope. One of his students became impatient with his verbal treatise, devoid of symbols, and interrupted Joe by saying, "I have a question. What exactly is slope? Can you define it?"

Joe eventually defined slope as "vertical change/horizontal change," and presented a graph of the line passing through the points (0, 0) and (3, 2). He emphasized that the slope was a fraction, 2/3, up 2, over 3. Later, while students were working at their seats, one student was having difficulty understanding how the two fractions 5/-6 and -5/6 could both represent the same slope. Although at the time Joe struggled in vain to help her understand, he later described her difficulty with the following: "They think you are describing a movement as opposed to you describing a number, a measurement."

Tracie: Representations for Teaching Slope

The Initial Survey asked the question, "What analogies, illustrations, examples, or explanations do you think are most useful or helpful for teaching the concept of slope?" In response to this question, Tracie replied, "Something tangible. Examples of common uses are the slope or pitch of a roof and the slope or grade of a mountain road." Later, these physical representations for slope appeared in both her Slope Lesson Plans and her MATHS 107 lesson.

Tracie did, however, began to develop a repertoire of other representations for slope. Her Slope Lesson Plans included an activity using *Geometer's Sketchpad* to construct lines and calculate slopes. She asked the students to make conjectures about the appearance of the graphed lines and the slope measurements. She also presented five different problems involving real-world representations of slope. One problem, entitled "Carpentry," asked students to determine how to fit a staircase into a given space. The other four problems involved linear functions. "Class Fund Raiser" was about selling pizzas. "Population Growth" compared the rate of growth of two high schools. "Communications" asked for a comparison of the rates of three major long-distance companies. "Financing the Prom" involved magazine sales. For each of these linear functions, Tracie asked for a graph of the data. However, she did not ask students to determine the slope of the line or interpret its meaning within the context of the situation.

When I interviewed Tracie later, I asked her, "What representations are most important when teaching the concept of slope?" She said, "In the beginning, I think it would have to be very concrete and the idea of just what slope is, as in steepness and stuff like that. And the further along you get, things like rate and talking about speed and acceleration and distance and finding what slope means when you graph those things."

The following semester, Tracie's aspirations for her MATHS 107 class were evidently not "further along" on her continuum, because she barely touched on the notion of rate of change. She presented the following situation and series of questions:

Anissa is reading a book for her English class. She decides to read two chapters each day in order to avoid having to read it all at the last minute. Graph Anissa's reading progress below.

> Which equation below represents the graph of Anissa's reading? (Use ordered pairs to see which equation they are solutions to.) $y = x + 2$, $y = 2x + 0$, $y = 2x + 2$, $y = -2x + 0$
>
> How do you know that this is the correct graph?

> Graph the other three equations.
>
> When an equation is in the form $y = mx + b$, the numbers m and b tell us something about the graph. What do you notice about each graph above and its equation?

Although she did ask the students to make connections between graphs and equations, she did not ask them to connect the real-world situation to the abstract notion of slope. In other words, she included a functional representation in her lesson, but she did not ask students to interpret the slope in the context of the situation.

Discussion

Joe's encounters with students revealed evidence of him carefully pondering various aspects of students' difficulties with slope. The methods class provided ideas about conceptual and procedural knowledge to frame his thinking. He attempted to address conceptual knowledge in his teaching, but these attempts were awkward and both he and his students were more comfortable discussing procedures. Joe, however, continued to be thoughtful, and would most likely have benefited from some more discussion on the topic.

Tracie was predisposed to using physical representations of slope. In fact, she mentioned that she had first learned about slope from her father, a carpenter. She did expand her repertoire to include functional representations, but she never emphasized connections between slope as the property of a line and slope as rate of change. She remained focused on developing slope as a measure of slant, believing that functional representations were more abstract and should be saved for later.

In conclusion, it was apparent that the methods course provided a space in which their initial set of ideas about slope could be developed and even challenged. Later, their evolving ideas about slope appeared in their lessons, suggesting a transformation in their pedagogical content knowledge about slope and how they would apply this knowledge.

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POSTULATING RELATIONSHIPS BETWEEN STAGES OF KNOWING AND TYPES OF TASKS IN MATHEMATICS TEACHING: A CONSTRUCTIVIST PERSPECTIVE

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Abstract: In this paper, we address the problem of how one might think about and carry out mathematics teaching that is consistent with a constructivist learning theory. Building on the idea that learning proceeds through reflection on activity-effect relationships, we formulate three stages in constructing a new mathematical conception—initial, reflective, and anticipatory. We then apply this three-stage process to thinking about teaching. In particular, we propose three types of tasks—initial, reflective, and anticipatory—that the mathematics teacher can generate, or modify, to promote students' development through the three stages.

The problem that this paper addresses can be illustrated with a situation that is commonplace in the classrooms of many teachers endeavoring to implement current approaches to mathematics teaching. After a few lessons in which students successfully solve mathematical problems with manipulatives, the teacher opens the next lesson with a question to recap learning from the previous lessons. The teacher expects the students to answer the question easily, but the students' responses are troubling. Consider a teacher who for a few lessons engaged students in using paper strips of various unit fractions (1/2, 1/3, through 1/20) to solve various problems such as who gets a bigger portion of a candy bar. During this activity, the teacher required the students to explain their answers in small groups. In *reflecting* on their activities, most students show and argue that Lee's pieces (e.g., sevenths) are smaller than Pat's (e.g., fourths) because more of Lee's pieces are needed to make the same one whole. At the beginning of the next lesson, ostensibly as a review of what they learned, the teacher writes two fractions on the board (1/4 and 1/7) and asks the students which one is larger. Much to the teacher's surprise and frustration, most of the students respond by identifying 1/7 as the larger because 7 is larger than 4.

How is the teacher to think about this situation and what would be a useful pedagogical response? In this paper we propose a way of thinking about a relationship between mathematics learning and teaching that can help to explain such situations and to conceptualize plausible interventions. This work contributes to the formulation of specific models of teaching

(Simon, 1995) based on a constructivist epistemology and theory of learning (cf., Piaget, 1970; von Glasersfeld, 1991). Such models of teaching must provide an approach to promoting specific conceptual advancements in learners, that is, learners' construction of abstract mathematical ideas they do not yet have.

In recent years, several scholars adhering to constructivist principles proposed ways of thinking about teaching. Brousseau (1997) and Balacheff (1990) proposed two types of didactical situations that capitalize on Piaget's fundamental way of explaining knowledge construction, namely, reflecting on actions. The first type—situations for actions—fosters learners' use of specific activities intended by the teacher. The second type—situations for communication and for decision—foster learners' reflections on those activities. Likewise, Cobb, Boufi, McClain, & Whitenack (1997) proposed two major teaching activities, engaging students in problem solving processes and using classroom discourse to encourage individual and collective reflection on those processes. We concur with the idea that reflection is a key process in learning, but we think that further distinctions are needed to explain why, in situations like the one above, the students cannot solve a problem they could solve and reason about on the basis of reflection on the previous day.

In this paper, we distinguish three stages in the process of developing a new conception. Then, we apply these distinctions to the teacher's creation and adjustment of appropriate mathematical tasks for students.

Three Stages of Knowing in the Process of Learning

To discuss the stages of knowing that comprise the focus of this article, we begin by articulating key aspects of our conceptualization of mathematics learning. Piaget's fundamental notion of assimilation implies that an individual's conceptions determine her or his recognition—perception and interpretation—of events and processes in which she or he participates (i.e., the mathematics one "sees" in the world). This implies the "learning paradox"—how to explain the advancement of conceptions without attributing to the learner prior conceptions that are as advanced as those to be learned (Bereiter, 1985). We consider learning to be a process of transformation in learner's current conceptions. Building on the work of Steffe & Cobb (1988) and von Glasersfeld (1991), we explain the transformational process in terms of reflection on activity-effect relationship in the context of goal-directed activity (Simon & Tzur, 1999; Tzur, in press). That is, using current conceptions and abilities, the learner can set a goal and carry out activities—mental operations and physical actions—to accomplish the goal. The learner might differentiate (often without awareness) among the effects of engaging in the activity. Reflection on the relationship between the activity and its newly differentiated effects leads to new levels of understanding. Bickhard (1991) noted that such an explanation of the transformational process does not require learners' use of an advanced conception to assimilate a situation prior to constructing that conception; it only requires the use of and reflection on established activities that bring about differentiation of activity-effect relationship.

Distinguishing Stages of Knowing

Building on von Glasersfeld's (1991) analysis of types of reflection and on Steffe & Cobb's (1988) analysis of types of accommodation, we formulate a distinction among three stages in the process of constructing new conceptions—initial, reflective, and anticipatory. We illustrate these stages by considering a 7-month old child who has no knowledge (discrimination) of geometric shapes, playing the game of putting objects into a box through holes in the cover shaped as in Figure 1. The child knows to put an object above the hole and push it down, but she does not yet know to rotate the objects. Therefore, she has a better chance of success with circular objects.

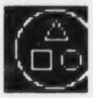

Figure 1

An initial stage of knowing is marked by the learner's use of activities allowed by her or his established conceptions. The learner differentiates the effects of her activity in terms of their contribution to advancing her goal for engaging in the activity. In our example, the child randomly picks up objects, attempts to push them into different holes (the activity), and differentiates results; the objects either go through or do not. (Note that the different affective experience associated with the child's successes and failures are part of the effects and thus contribute to the differentiation.)

A reflective stage of knowing is marked by the learner's reorganization (abstraction), through reflection, on activity-effect relationships *while engaging in the activity*. The knowing may be implicit or explicit. An example is when the child purposely attempts to put the next object in the hole with which she just had success (the circular hole). A day later, the child would again try holes at random until she observes a regularity with respect to a particular hole, then she will try that hole consistently. (This parallels the classroom example at the beginning of this article.) This illustrates that at a reflective stage of knowing, the learner can only recognize (and maybe explain) the activity-effect relationship while engaged in and reflecting on the activity and its effects.

An anticipatory stage of knowing is marked by the learner's abstraction of an activity and its effects without or prior to carrying out the activity. This is developed by reflecting on the relationship between intended effects (one's goal) and the activity-effect relationship abstracted at the reflective level. An example of an anticipatory stage of knowing is when, from her first attempt, the child goes directly to the circular hole. The three stages can be considered as the components of a cyclical process. Anticipatory conceptions developed through the three stages described can provide the basis for setting a goal and carrying out an activity to accomplish it, which can bring about a new initial stage.

Relating Teaching Activities with Stages of Knowing

To postulate a relationship between the three stages of knowing and teaching activities, we extend and refine Simon's (1995) model of teaching. The two currencies of Simon's model are (a) posing and solving mathematical tasks (problem situations) and (b) facilitating learners' activities and reflections through interactive discourse. We specify three types of tasks (and related reflection) that the teacher can use to promote intended advancements in learners' mathematics, from initial to reflective to anticipatory knowing. When creating or modifying tasks, the teacher must work from an understanding of the learners' conceptions.

Initial, reflective, and anticipatory tasks are tasks designed to promote transition to initial, reflective, and anticipatory stages respectively. These categorization of tasks should be understood as relative to the conceptions of the learner and *not as a property of the tasks*. Initial tasks foster learners' use of established activities that potentially create a differentiating experience on which to reflect. Based on the teacher's understanding of students' conceptions, he or she creates or adapts a task to motivate the students' setting of a particular goal and use of particular activities.

Reflective tasks foster learners' identification of regularities in activity-effect relationships in the context of the activity that generates them. The teacher identifies a task (or reflective question) intended to promote students' reflection on the relationship between the activity and the differentiated effects of the activity, leading to observation of regularities within the context of that experience. Reflective tasks can build on initial tasks posed by the teacher or on spontaneously occurring activity of the learners.

Anticipatory tasks foster learners' reflection on the relationship between intended effects and the activity-effect relationship abstracted at the reflective level. The learners' interpretation of the task includes setting a goal similar to the one that motivated, in the initial stage, the use of an established activity. Because the learner has abstracted a new relationship between that activity and its effects, that relationship is available once the learner begins to mentally run the activity. Sufficient iterations of the anticipatory task allow the learner to reflect on the resulting mental activity and the consequence is the development of anticipatory knowledge.

Illustrating the Stages of Knowing and Related Tasks

We use what we have articulated to reexamine the situation presented at the beginning of this article. Let us assume that to figure out which fraction is larger while acting on manipulatives, the students used a combination of two key activities, iteration—replicating a single unit fraction to produce a whole equal in size to the reference whole, and adjustment—increasing or decreasing the size of the unit to be iterated in the next trial (Tzur, in press). In reflecting on the effects of these activities, the students abstracted the inverse relationship between adjusting the size of the iterated unit and the resulting number of units in the whole. In the context of the activity, they could use this relationship to figure out which of two unit fractions is bigger. However, they were not yet able to identify the appropriate activity when given the symbolized results. In short, the children's conception of unit fractions only reached the reflective stage.

According to our model, the students' inability to solve the problem (Which is bigger, 1/7 or 1/4?) indicates the need to advance their conception of the relative size of unit fractions from reflective to anticipatory. For example, the teacher can ask: "Pretend that you were given 1/8 of a pizza. Think what would you do to cut another, same-size pizza so that your little brother would get a smaller piece?" Students who have previously developed reflective knowing through iteration and adjustment might think about iterating a smaller piece, the activities undertaken at the previous stage, which would prompt the abstracted relationship between the activity (adjustment and iteration) and its effects. Reflection on iterations of the anticipatory task and its (mental) effects could lead to anticipatory knowledge of the inverse relationship between size of pieces and the number of pieces in the whole. Key to this example is that the teacher does not indicate the activity to be used in solving the task.

Conclusion

The conceptual framework that we have presented provides distinctions about learners' knowing and ways for teachers to coordinate instructional tasks with stages of knowing. The framework can serve both to organize thinking about instructional design (whether by teachers or curriculum developers) and to analyze and adjust lessons that do not accomplish the goals for which they were intended. Further, researchers can make use of the framework to organize their interpretations of classroom learning and teaching. The framework addresses two of the greatest challenges in mathematics education, promoting the development of new conceptions and supporting the development of a level of abstraction that Sfard (1991) characterizes as a "structural conception."

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GIVING VOICE TO MENTOR TEACHERS

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In an effort to understand the gap between theory and practice, mentoring roles of teachers are explored using their voices. A continuum of how the mentor teachers approached sharing their expertise was developed and is described. We found that not only did mentor teachers approach how they shared their knowledge and experience differently, but what they hoped to accomplish through mentoring varied as well.

One of the most influential periods in a teacher's development is the transition from student to student teacher to novice teacher (e.g., Peterson & Williams, 1998). Many participants are suggesting the links between the university programs and field experiences during this critical period need rethinking (Wideen, Mayer-Smith, & Moon, 1998; Zeichner, 1985). One of those links involves classroom teachers' roles in working with student teachers. This paper uses the mentors' voices to describe the roles of mentor teachers working with secondary preservice teachers in mathematics.

The purpose of our research is to understand the mentoring experience from the viewpoint of the mentor teacher. By focusing on the mentor's voice, we gain a deeper understanding of the psychological dimensions involved in the mentoring process. We believe that a better understanding of the dynamics of mentoring will allow mathematics educators to build better school-university links and consequently create better coordination between preparation of mathematics teachers on a university campus and in the field.

Review of the Literature and Theoretical Perspectives

The literature on mentoring tends to be prescriptive in nature, informing the classroom teacher of the "important" roles for mentors. Peterson and Williams (1998) discussed the "bewildering array of possible roles" (p. 730) of mentors mentioned in the literature. They argued that listing such descriptions of mentoring affords little help because of the idiosyncrasies of mentoring relationships. Sudzina, Giebelhaus, and Coolican (1997) studied the expectations and perceptions of the mentoring relationships between cooperating teachers and student teachers. They found that cooperating teachers expected student teachers to either follow their lead or engage in collaboration.

The literature related to the preparation of teachers suggests a tension between focusing on practical or utilitarian issues of teaching versus encouraging an analytic or reflective stance (Goodman, 1984; Peterson & Williams, 1998). There is a call for balance, but there is a lack of clarity about the criteria for an effective balance. In order to understand how people interpret and understand this balance, researchers need to consider the perspectives that guide people. Charon (1998) explained that perspectives "are not perceptions but are guides to our perceptions; they influence what we see and how we interpret what we see. They are our 'eye-glasses' that we put on to see" (p. 8). To understand how teachers see mentoring we need to gain insights into their perspectives.

Methodology

Twenty-five teachers from the Partnerships in Reform in Mathematics Education (PRIME) project, representing six high schools and one middle school, participated in this study. The placement of student teachers, dates and goals of the field experiences, and criteria for being a mentor were collaboratively developed with an advisory board of mentor teachers.

Twenty-four teachers completed an initial survey and 23 completed a year-end PRIME evaluation. The initial survey questions, based on research on student teaching, asked for teachers' perceptions of the purposes of student teaching, and the value of various topics that teachers might discuss with their student teacher. The year-end evaluation included questions related to the roles of a mentor teacher. Four teachers (Betty, Deborah, Jackie, and Rhoda) were chosen to be focus mentors based on their responses to the initial survey, their enthusiasm, and their willingness to provide additional data. The data for each focus mentor consisted of transcripts from two interviews and one mentor teacher/student teacher conference as well as a year-end evaluation. We used researcher notes from school visitations during the student teaching experience, PRIME meetings, and individual communications. These notes were used to understand individual characteristics such as a mentor's classroom and school environment.

We tallied the initial survey, creating graphs and tables to help us understand the diversity of the mentors' perspectives on student teaching and their roles. On a weekly basis, we reviewed the surveys and transcripts from the interviews, discussing emerging themes as they related to the research questions. These analyses informed further investigation in the field and helped frame subsequent interview questions, conversations, and the year-end evaluation. We engaged in peer debriefing (Guba and Lincoln,

1985) as a means to share ideas and explore hypotheses about each case study. We analyzed data about mentor roles by putting the precise name or description of each role provided by each participant on a card. As we sorted the cards into similar groups of mentor roles, we recognized two major categories, expert and comforter.

Using the two categories, developed through analysis of the full group, we perused the focus mentors' data to identify confirming and disconfirming evidence of actions, thinking, or comments related to their roles of sharing expertise or creating a comfortable situation. This analysis suggested that merely naming a role was inadequate in describing the essence of that role. It was necessary to understand how mentors approached the role and the ultimate purpose of fulfilling that particular role. We generated a continuum to describe the various mentor approaches to their roles. As a method of validation, we returned to the focus mentors' data to see if mentor actions and ideas could be positioned on the continuum. The continuum helped to capture the variation in perspectives among mentors who chose the same role to describe their work.

Results

"Expert" was one of major categories representing roles where mentors shared their knowledge and experience (e.g., coach, guide, counselor, model, leader, teacher, observer, expert). "Comforter" was the other major category representing roles that helped mentors maintain a comfortable environment (e.g., encourager, listener, supporter, helper, friend). We noted how two mentors could use the same role, with different approaches, to address different goals. In addition, our analysis helped to identify instances where mentors used different roles or approaches to address comparable goals.

The continuum represents mentor's approaches based on how directive they were in sharing their knowledge and experience. The positions on the continuum, from left to right, are non-directive, inquisitive, negotiative, suggestive, and directive. Since the continuum is based on data from our four focus mentors, data from other mentors may identify additional points.

Rhoda was inquisitive in her approach to sharing her expertise. She asked questions without appearing to have preconceived an answer. Her

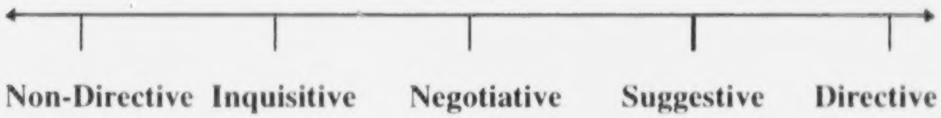

Figure 1. How Mentor Teachers Approach Sharing Knowledge and Experience.

feedback was consistently positive; she did not give suggestions on how to handle a situation. She felt responsible to "encourage him to look for his strengths and weaknesses" (Interview 1) as opposed to pointing them out herself. She elaborated, "I'm the mom that's not going to see anything wrong with it" (Interview 1). For Rhoda, the mentor's approval is unconditional.

Deborah's responses generally reflected the negotiative position on the continuum. She raised issues and gave her point of view but valued negotiation. She offered invitations to discussion such as "Can you think of anything that you might have been able to do that might have pulled them in at the end?", "How would you compare the effectiveness of that?" (Conference). Deborah's role became more directive when she discussed "the logistics of classroom management." She described her role as "running interference for her [student teacher]." She commented, "I tried to help pace instruction…I wrote out how many days she could spend on a topic and which problems I would suggest that she assign" (Interview 2).

Betty's approach lies near suggestive on the continuum. She thought that the most difficult role for a mentor would be telling "them that what they did didn't work…you don't want to be too critical…you want them to feel good about what they're doing, but at the same time, you can't just let stuff that's not working continue to not work" (Interview 1). She was not looking forward to this role because she also wanted the student teacher to "feel comfortable coming to me to ask questions" (Interview 1). Betty also believed that the main role of a mentor teacher is to be "a guidance person" and "a support person." Thus, the way Betty approached her role was to "soften" her direction, inundating her comments during a conference with her student teacher with "might," "maybe," and "perhaps." Betty suggested, "You might want to even go over that one tomorrow. That might be a good one to do" (Conference). Yet, there seemed to be an assumption that the student teacher would follow her suggestions.

Jackie's approach lies near directive on the continuum. When speaking about important roles of a mentor teacher, she commented, "I am a biggy on organization... you set a good example for students...[and] you don't end up repeating.... It's easier to do it right the first time than to do it over" (Interview 1). Jackie explained, "anytime you are doing a test always [ask]... where can they mess up and how is that going to affect what it is I am looking for?" (Conference). In regards to assigning student groups, Jackie advised, "you need to always have an idea of why you want to put certain people with certain people" (Conference).

Not only were their approaches to their roles different, but what they were trying to accomplish with their roles was also different. Rhoda wanted her student teacher to be comfortable in the classroom. She allowed him to make decisions and live with the consequences: "He's going to be the one that's presenting the lesson and he's going to be responsible for the outcome of the students' knowledge of that. And so if he's comfortable, I'm comfortable with it" (Interview 1). Rhoda ended one conference with her student teacher with the statement, "I'm pleased. You're becoming much more comfortable" (Conference). Rhoda encouraged reflection because for Rhoda, "every day, teaching is reflection" (Interview 1). Although she could have shared her opinion of a lesson, it was more important that the student teacher form his own opinion.

For Deborah, essential to the mentoring process was creating an experience that would provide opportunities for her student teacher to enjoy teaching and feel confident in her role as a teacher. When asked what she wished her student teacher would gain from the mentoring experience, Deborah answered, "Confidence — that when they go out on their own that they have the skills necessary to do things well and to enjoy what they do" (Interview 1).

According to Betty, one of the most important roles of a mentor is to "make sure, absolutely, that the mathematics is correct; that is very important" (Interview 1). Betty was concerned about maintaining a comfort level, but her students' learning of the mathematics was paramount. Her comments were primarily suggestions of topics and problems for the student teacher to incorporate into her lessons. Betty consistently offered advice such as, "I'd like to suggest that maybe tomorrow you go over one that looks like this" (Conference).

Jackie emphasized dedication to teaching and love for the profession. Jackie said, "Hey, you need to love this to do it and do it well.... You can't do a good job if you don't love it!" She later said, "I want people to see how important the job is but how much fun and just how exciting it is." (Interview 1). Jackie wants her student teacher to develop "a love for the job" from the student teaching experience. After her student teacher asked her a question, Jackie replied, "That's okay - that is what you are here for, is to learn what to do" (Conference). She asserted that she is "here to teach them how to teach" (Interview 1).

Discussion

We agree with Charon (1998) that understanding perspectives is essential in working with teachers. As researchers, we must look beyond the nominal role of mentors and understand how they approach their roles and the objectives they wish to accomplish by those roles. Identifying the roles of mentor teachers is not sufficient for understanding the mentoring process. Sudzina, Giebelhaus, and Coolican (1997) concluded that "Explicit conversations about respective roles and expectations would assist in clearing up existing misconceptions and open the door for effective mentoring to occur" (p. 33). The continuum provides a way for researchers, mentors, and student teachers to explicitly discuss roles of mentors as well as the approach to those roles. Our work expands the work of Sudzina et al. by identifying additional approaches and documenting the approaches in the teachers' voices.

As we studied teachers perspectives on mentoring, we gained insight into the delicate balance between practical and reflective issues of teaching (Goodman, 1984). Teachers may address the practical because it is easier to discuss, easier to model, and easier to recognize. Teachers were more directive when addressing practical issues. However, we found teachers addressed more than the practical. Rhoda wanted to ensure her student teacher's comfort and to encourage him to be reflective. Deborah wanted to provide opportunities for her student teacher to develop confidence in herself as a mathematics teacher and to enable her to develop her own teaching style. Betty wanted to ensure that her student teacher focused on students' understanding of the material when she taught mathematics. Jackie wanted to instill in her student teacher a love for teaching.

We are interested in determining whether teachers' positions on the continuum tend to be stable. For example, would a different student teacher change a mentor's role? Would discussion with peers or university personnel alter a position? We are also interested in exploring how teachers' roles may vary depending on whether they are addressing practical issues, affective issues, analytical issues, or promoting reflection.

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Note: PRIME is a teacher enhancement project at the University of Georgia, supported by national Eisenhower funding.

ORGANIZING FOR LEARNING THE BIG IDEAS IN UNDERGRADUATE MATHEMATICS FOR ELEMENTARY TEACHERS: TWO APPROACHES

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In the reform of undergraduate mathematics courses for elementary teachers (UMET), a common component of projects is experimentation with different course materials and sequences. Educators in these collaboratives have been asking, "What content can we do really well, and how can we organize and sequence a course that that the big ideas stand out?"

This study investigated two approaches. The Number Subset Focus (NSF) represented the traditional organization of material which expands one number subset at a time from natural numbers to real numbers. We were interested in an Operation Focus (OF) which presents all real numbers up front and organizes additive group structure first and then multiplicative group structure. We collected data in two Number Systems classes for elementary education majors. Pretest data indicated that the classes had similar attributes, attitudes, and conceptions of mathematics. Different group personalities emerged as the term progressed. The NSF group would show hostility while the OF group would be more open to new models. Attitude and sequencing need more study.

The two groups had identical finals. The only real difference in performance was that the OF group was better able to create word problems and identify operation actions. We were encouraged that the OF group did not perform worse than the NSF group on the number subset items. We were discouraged that they did not perform better on all items that assessed understanding of operation. We will be adding the Necessary Arithmetic Operation test to the pre/post assessment and will rework the OF approach so that students can better organize and communicate their understanding of operation.

WRITING ABOUT MATHEMATICS IN A SECONDARY METHODS COURSE

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The mathematics reform movement encourages the conceptual understanding of mathematics. This, in turn, results in the need for a change in the instructional practices a teacher uses to help students build their own mathematical knowledge and understanding. The teacher must have the conceptual knowledge that will allow them to "adapt and improvise in the face of what happens as learning unfolds" (Ball, 1996, p. 503). Writing about mathematics was used in a mathematics methods course to help preservice teachers develop their conceptual knowledge. Writing gave the preservice teachers the opportunity to organize their thoughts, synthesize the material, and obtain a better understanding of the mathematics they will be asking their secondary students to do.

The context of the study is the preservice secondary mathematics methods course at the University of Colorado at Boulder. During the course the preservice teachers were to identify and analyze the concepts and connections within and between particular mathematical areas; generate possible questions and (mis)conceptions that secondary students may have, along with possible discussion points that would help students understand the concepts; choose and solve a variety of problems related to a particular concept or set of concepts; and, finally, discuss approaches that could be used in teaching the concept(s) in a secondary mathematics class. The preservice teachers explored problems from the major areas of Functions, Geometry, and Data Analysis as defined by the 1989 NCTM Standards.

The focus of the short oral report examines the students writing with respect to its effectiveness developing conceptual knowledge, the adaptations that were made to the writing assignments, and proposed changes for future mathematics methods course.

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UNDERSTANDINGS OF MATHEMATICAL ARGUMENT

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While children seem to make an implicit distinction between inductive and deductive inference by age 8 or so, the majority of adolescents do not seem to make the deductive-inductive distinction in the domain of mathematical argument (Fischbein & Kedem, 1982; Galotti et al., 1997; Morris & Sloutsky, 1998). At least through adolescence, people often demand empirical support for deductively valid mathematical proofs, and accept inductive arguments as proofs. In the current calls for reform, teachers are expected to develop students' abilities to make and prove conjectures, and to understand the relationships between these processes. However, Martin and Harel (1989) found that, even after taking a college course that dealt with mathematical proof, more than half of 101 preservice teachers accepted an inductive argument as a "proof," while a third accepted both inductive and deductive "proofs." Their study, however, was based on written responses; and participants responded to arguments provided by the investigators. Perhaps it was simply the case that participants did not understand the relevant mathematics sufficiently well to judge necessity.

Two studies examined preservice teachers' understandings of mathematical argument. In interviews, 46 participants formulated arguments for problems that required little mathematical content knowledge (e.g., "The primes 2 and 3 are consecutive counting numbers. Is there another pair of consecutive primes? Prove your answer is correct"); judged whether their own arguments "established the conclusion was *necessarily true* for each and every counting number"; ranked inductive and deductive solutions to the same problems that were provided by the investigator from "most logical" to "least logical"; and judged whether each of the solutions established the conclusion was necessarily true for all counting numbers. Understandings about inductive and deductive argument were probed. In Study 2, preservice teachers were interviewed prior to, and after taking a course that dealt with the nature of a mathematical system, nature of proof, role of axioms, and the concept of logical necessity. Study 2 attempted to relate the instruction to changes in the set of metalogical beliefs identified in Study 1.

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THE INTERACTIONS OF CONTEXT, SYMBOLS, AND NUMBER LINE DIAGRAMS IN DEVELOPING UNDERSTANDING

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An analysis is presented of the mathematics embodied in a sequence of three activities from a mathematics course for preservice elementary majors. The activities use interactions between contextual problems, number sentences, and number line diagrams to study the structure of operations on negative integers. The sequence of activities is intended to move students from an instrumental to a relational understanding of the symbol systems and underlying mathematical concepts (Skemp, 1987). Preservice students' work illustrates the complex nature of the mathematics under study.

In the first two activities, students write and solve word problems, and then examine how well their solutions, written as number sentences and number line diagrams, model the given quantitative relationships. A four-step modeling process is introduced as a framework to help students examine their representational systems and uses of negative integers. In the third activity, students study expanded number sentences, statements of number facts, and number line diagrams to understand the structure of multiplication and division of positive and negative numbers. Initially, number sentences and number line diagrams are used as computational tools to find solutions to contextual problems. As these representations are examined for their ability to accurately represent the given context, they become *models of* particular situations. Finally, arithmetic operations with negative integers become objects of study, and number sentences and number lines become *models for* thinking about the mathematical properties of operations on integers (Gravemeijer, McClain, & Stephan 1998).

TEACHERS' KNOWLEDGE OF STUDENTS' NOVEL STRATEGIES FOR WHOLE-NUMBER OPERATIONS

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The reform movement in mathematics education has put teachers in the predicament of having to "teach more than they understand" (Floden, 1997, p. 13). Ball and Cohen (1996) have suggested that a partial solution may exist in the design of curriculum materials to support teacher learning in practice. Research on children's thinking about multidigit operations provided a backdrop against which to pose questions about the nature and extent of what teachers learn about students' novel strategies as they use curriculum materials designed to support teacher learning.

The 13 participants in this interview study comprised all the teachers in grades three, four and five at a single elementary school, located in a district that has adopted *Investigations in Number, Data, and Space* (TERC, 1995-1998) as its elementary mathematics program.

Results indicate that all of the teachers acquired a predisposition to elicit strategies from children, and to expect a variety of responses. Teachers believed quite strongly that they as well as their children were benefiting mathematically from the new curriculum. Further, their knowledge of alternative algorithms for multidigit multiplication was especially strong, although many teachers were limited in their interpretations of the mathematical and developmental significance of novel strategies for division and subtraction.

A WEB-BASED DATABASE OF PROBLEMS AND PRACTICES AND REAL COMMUNITIES OF ESP TEACHERS AND STUDENTS

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The Emerging Scholars Program (ESP) set up at U.T. Austin by Uri Treisman has dramatically improved the academic performance and retention of African-Americans and Latinos. This success had a wide influence, inspiring numerous similar programs around the country. At the core of the ESP is an intensive workshop in which students practice with challenging problems.

As a way of supporting the classroom activities of teachers involved in ESP-style programs, and as a promising component of teacher development, we have developed a database of calculus problems hosted on the web. The problems are indexed by many keywords and teachers are encouraged to annotate problems or to respond to other teachers' annotations. Problems and comments are also hyperlinked to teaching manuals and essays about pedagogical and mathematical topics. Furthermore, there is a mechanism for formatting selected problems for printing for direct classroom use.

We believe this will not only be a useful tool for teachers, but also contribute towards the development of reflective practice, a sense of collegiality, and a thriving community of practitioners. Preliminary results will be displayed indicating the social and pedagogical implications of such a tool for teacher's classroom practice and professional development.

Of course such a web database may be useful for students, in particular advanced students or teachers wanting a review material. For a general student audience, great care must be taken to carefully and thoughtfully guide their experience in the web site. We have begun preliminary work on a web database of problems that would have an interface and organization more suitable for students wishing to use it as an electronic companion to a classroom calculus course. Such a student presentation will be enhanced with multimedia and access to problems from standardized tests such as AP Calculus tests. If ready, we would like to simultaneously present results from an early version of such a student-tailored site.

Technology

THE INCORPORATION OF NEW TECHNOLOGIES TO SCHOOL CULTURE:THE TEACHING OF MATHEMATICS IN SECONDARY SCHOOL

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Abstract: This paper describes a project of applied research aimed at articulating computational technologies with the implementation of a new mathematical curriculum in secondary school. From this project, we intend to build a viable model for describing the impact of new information technologies in school practices that can lead to research developments and to the transformation of the educational system in the near future.

Background and Aims of the Project

A study carried out in Mexico and England (Rojano-Sutherland, 1997) involving mathematical practices in the Science classes, revealed that in Mexico few students are able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggests that it is necessary to replace the formal approach in the new curriculum, with a "down-up" approach capable of fostering the students' explorative, manipulative, and communication skills. Thus, it is necessary to choose the topics to be taught according to criteria of pertinence and relevance. These criteria are necessary, as we are not trying to subordinate technology to the curriculum but to study its impact on a new curriculum, which has not yet produced rejection mechanisms to the technology.

Hence, our main aim is to study the impact that the blending of technology into the new curriculum has on the mathematical practices at school.

Theoretical Framework

One of the main assumptions of this project, considering the aims described above, is that of Mediated Action (Wertsch, 1991). Thus, assuming

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that mediation modifies the process of knowledge construction, we should regard the importance of articulating a rationale on the cognitive and epistemological effects of computational instruments (Balacheff-Kaput, 1996) conceived as mediational tools.

An Educational Development Project

More than a year ago, the Ministry of Education in Mexico, decided to incorporate new technologies into Basic Education (from 4 to 16 years olds) in order to provide access to modern scientific and advanced mathematical ideas.

Our project began with a pilot phase (in 1997) during which technology-based educational models were put to trial using relevant results from previous computer-based educational studies carried out in different countries. To achieve this goal, human resources were trained, and 15 secondary schools from all around the country were equipped with new technology. During the first stage of the project the technology included: Spreadsheets (Excel), Cabri-Géomètre, SimcCalc-MathWorlds, Stella (research version) and the TI-92 algebraic calculator; all aimed at covering curricular topics such as arithmetic, pre-algebra, algebra, geometry, variation and modeling. It is important to point out that class work was mainly based on a collaborative-learning model.

In the following pages, we will discuss unfolding research results from the educational development project mentioned earlier, as well as the methodology used to obtain these results.

Methodology

The methodological design incorporates both a global and a local level of assessment. The global level focuses on understanding the educational system as a complex model. Its goal is to regulate the evolution of educational processes including teachers, headmasters and parents as essential elements. On the other hand, the local level concentrates mainly on longitudinal case studies. The latter is sought to provide useful feedback for improving dissemination and implementation, as well as to produce auditable trails of documentation that can reveal the nature of obtained achievements.

Preliminary results (local study)

Data coming from the local level of assessment (written questionnaires and pupil interviews) was used to analyze the evolution of skills and specific knowledge within the mathematics curriculum. The worksheets for the experimental tasks were designed taking into account the mathematical curriculum contents intending to promote a model of collaborative work in the classroom. The sequences of activities were developed according to evolving lines in the different curriculum contents. For instance, from arithmetic to algebra; from intuitive to exploratory dynamic geometry; from static descriptions to variation model; from solving closed problems to modeling. Different technology was implemented in different sites; the first generation groups used different technological tools, except for the calculator that was used by all groups. Pupils collaborated in pairs and groups of three when working in front of the computer and, worked both individually and collaboratively when using the calculator. The teacher's role consisted of i) giving support to students as they worked out the activities described in the worksheets and ii) organizing collective discussions to enhance individual experiences and problem-solving abilities. Thus, the teacher's role was twofold: Besides being a mediator during the classroom activities, he was also a mediator between the student and the tool during the process of appropriation of the latter by the formers.

The results reported below were taken from the first implementation semester of Spreadsheet and Cabri-Géomètre, and focus mainly on the role of the tools as shapers of the school mathematical culture.

Spreadsheet's results

In the case of the spreadsheets the activity sequences were designed to help pupils evolve from using numeric and natural language to using algebraic language for expressing themselves in problem solving and generalization tasks. The spreadsheet's results account for the use and evolution of symbolic codes placed in between the natural and the algebraic languages. This is illustrated by the analysis of the pupils' responses to the items in Problem 3a taken from the questionnaires (applied twice to the first generation students).

Problem 3a. Complete the sequence and give a formula to express it.

5, 10, 15, _____, _____,   Formula: _________________

Types of responses

Sequence	Examples
Correct	20, 25
Incorrect	20, 30; 25, 35
No answer	

Formula	**Examples**
III Additive Symbolic	9 + 5 ; C1 + 5
III Multiplicative Symbolic	B x 5 ; A1 x 5
II Expressing the Operation	To add five, plus five, + 5
I Word description (Correct)	It goes in steps of five
I Word description (Incorrect)	Subtracting five; adding ten
I Performing the operation	15 + 5 = 20 ; 20 + 5 = 25
No answer	
Others	

Note. Categories: I Non-algebraic, II Pre-algebraic, III Algebraic

Table I
Pupils' responses to the items in Problem 3a taken from the questionnaires applied to the 1st generation in school years 1997-1998 and 1998-1999

School year	**1997/1998**		**1998-1999**	
Categories	Fre. (239)	%	Fre. (151)	%
a) Correct sequence	185	77.40	146	96.69
Incorrect sequence	45	18.83	2	1.32
No answer	9	3.76	3	2.00
III Algebraic	48	20.08	67	44.37
II Pre-algebraic	0	0.00	49	32.45
I Arithmetic (verbal explanation)	110	46.02	25	9.93
I Arithmetic	46	19.25	10	6.62
No answer	35	14.23	0	0.00
Others	0	0.00	10	6.62

The numbers in Table I indicate that pre-algebraic expressions increase significantly (from zero to 32.45 % in the population) while arithmetic utterances, such as performing the operation or expressing it in words, tend to disappear. This can be interpreted as the product of using a numeric tool with expressions based on symbolic codes (the spreadsheet). Th effect of using the calculator can also be observed in other items. These tendencies were cross-confirmed with interviews. For instance, Fernando (one of the participants) showed high skills for solving problems and expressing generality while using a spreadsheet. However, he did not show any need to verbalize or express numerically his findings e.g., with paper and pencil. Fernando gives =B1*x as an answer to the interviewer when he was questioned about how to translate to algebraic language a particular situation expressed in natural language.

In the case of Cabri-Géomètre, the use of the dragging capability of the software (direct manipulation) contributed to the distinction between geometrical constructions and drawings. The geometrical phenomenology, as it is revealed on the screen, helps to develop verbal descriptions and argumentation. These arguments will be developed in a forthcoming paper.

Final Remarks

The results discussed above provide evidence on the impact of learning environments on the ways in which children express their mathematical thinking. This is not only the product of using learning tools, but also of the close interaction occurring between the students *and* the tools. Although no evidence is here provided, the following observations within the global level of assessment can also be added: a) Parents value technology as it brings better work opportunities to their children, b) Teachers point out that technology helps to build a new learning milieu within the classroom in which new strategies for problem solving and new ways of introducing teaching materials can emerge. These observations will be part of the assessment of the project's collective ways of thinking about the project itself.

The project also tries to answer questions like the following:

What new insights are productive teachers developing?

Are the teachers and parents expectations evolving together with the project?

Is the evolution of values manifested accordingly to regional cultures?

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A PROPOSAL FOR AN INTELLIGENT TUTORING SYSTEM IN A CONSTRUCTIVIST ENVIRONMENT

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By considering some characteristics of the Intelligent Tutoring Systems and the Intelligent Learning Environments a model of Intelligent Tutoring System (ITS) with an underlying constructivist approach of learning is proposed. The type of ITS proposed promotes a constructive learning from the student through the activities that the tutor requests the student to do. The activities presented to the student follow a didactical plan.

The most relevant premises in which is based the proposed model are the following:

- The type of ITS proposed was created to support the teaching and learning of mathematics.
- The main component of the system is a didactical plan that follows a personal adaptation of one of the authors of the didactics of Aebli (1995) and the psychology of Piaget.
- The type of ITS proposed does not pretend to be a replacement of the teacher but a partner or an assistant, relieving the teacher of many tasks that personalized teaching demands.
- The didactical plan is organized in lessons. Each lesson states a problem. The tutor guides the student through a series of suggested activities designed for the student to know the object of study, to solve the problem, and to acquire a particular operation in the Piagetian sense.
- The activities are organized so that after an operation is done the execution of its inverse is requested. No specific method of solution is imposed. The solution of a problem for different ways is encouraged.
- The help provided by the tutor is given upon request. The suggested order of the lessons is optional.

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LIREC, AN ALTERNATIVE FOR THE TEACHING OF THE STRAIGHT LINE: AN EXPERIMENTAL EDUCATIONAL RESEARCH

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This work describes the process followed to perform an investigation in educational experimentation to test the effectiveness of the teaching of the straight line, based on the intelligent tutorial system "LIREC" versus the traditional teaching methods. An experiment was designed in order to achieve a reliable analysis. Two stages were considered during the research process:

- During the first stage, two groups were taken into consideration, the experimental and the control groups.
- During the second stage, the first-stage control group, for which only traditional teaching methods had been applied, was then taught at the computer laboratory using the LIREC software program for a length-time similar to that of the first-stage experimental group.

In both cases, the problem lay on the statistical inference field, that is to say considering the **null hypothesis** that states that "there is no difference between the teaching of the straight line using the LIREC software-program and using the traditional teaching methods", versus the **alternative hypothesis** that states that "LIREC Software-program- based teaching is more efficient that traditional teaching".

After defining and controling the pertinent educational variables, when processing the data in both stages, the rejection of the null hypothesis over the alternative hypothesis was determined. It was concluded that teaching using the LIREC software-program is more efficient than the traditional teaching methods.

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MODELING IN BIOLOGY AND CHEMISTRY WITH SPREADSHEETS: AN EXPERIMENTAL STUDY WITH HIGH SCHOOL PUPILS

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Purpose of the study. To investigate the role of spreadsheets as a modeling tool of phenomena of science subject areas such as chemistry and biology with high school pupils. More specifically, it is investigated to what extent, having access to the process of building up a mathematical model of a certain phenomenon (and not only having access to it's simulation) pupils of this school level can develop contextual meanings for relationships between the different elements of the model and in this way can get a better understanding of the phenomenon itself.

Theoretical background, methodology and results. Issues on modeling with spreadsheets arising from Rojano & Sutherland's work are immediate antecedents of this study. Basic aspects of modeling and it's implications for education analyzed by J. Ogborn and H. Mellar were taken into account for modeling activity design and data analysis. Vygotsky's theoretical ideas about the role of mediation in learning processes influenced the design of the experimental work in the classroom. The study was carried out with two generations of high school pupils in the State of Morelos. Both groups worked with spreadsheet modeling activities for one school year, one in the chemistry lessons and the other in the biology lessons. Results from a written pre-test allowed to choose subjects to be case-studied. Initial and final interviews together with classroom observation provided the material to develop the case studies. The results suggest that spreadsheets modeling activities help students to bridge the gap between school mathematics and science when making sense of phenomena behavior through dealing with the different systems of representation available in this computing environment.

Whole Numbers

THE IMPACT OF DEVELOPING WRITTEN COMPUTATION AS A REPRESENTATION OF MENTAL COMPUTATION

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This study investigated the similarities and differences of 137 fourth graders' understanding of and reasoning about multiplication, division and proportion tasks. All students were administered a pretest mid-year and a posttest at the end of year. A representative sample of 18 students were individually interviewed on tasks involving reasoning about multiplicative and proportional relationships. The study provides evidence that students who are encouraged to use reasoning procedures for multiplication and division based on number relationships have a better understanding of the meaning of those operations and are more successful in extending their knowledge to proportional reasoning tasks than are students who are taught conventional procedures exclusively.

Background and Statement of Problem

For years researchers have documented the difficulties young children have learning the standard algorithms for computation with whole numbers. It has been shown that many children do not fully understand these algorithms and, therefore, make errors when carrying out the procedures that oftentimes lead to nonsensical answers (Ashlock, 1972, 1976, 1982; Brown & Burton, 1978). In addition, many students who are able to memorize the standard algorithms and use them accurately are unable to apply their knowledge or provide an adequate explanation of what the operations mean (Carpenter, et al., 1998; Clark & Kamii, 1996).

Typical problems children have with multiplication, in particular, have been well-documented. The findings have shown that when presented with a multiplicative situation children often add instead of multiplying (Hart, 1981). They often base their choice of operation on the size of the numbers in the problem rather than thinking carefully about the situation being described. In addition, when children do not know a product, many are unable to use other products that they do know to figure it out. This difficulty is in contrast to children's facility in figuring out a sum they do not know (Kamii & Livingston, 1994). Finally, it has also been found that even children who have little difficulty with computation often have problems describing the meaning of multiplication (Lindquist, 1989). For example, when children were asked to write a story problem for 6 x 3 = 18, 37% of fourth graders and 44% of fifth graders formulated problems such as "There are six ducks swimming in the pond. Then a while later three more ducks come, so how many are there?" (O'Brien & Casey, 1983).

Similar issues seem to arise when dealing with more complex multiplicative situations, such as those requiring proportional reasoning. Research has shown that (a) children often view ratio as an additive operation and, as a result, resist using multiplication strategies (Hart, 1981); (b) some children use multiplication strategies periodically but avoid those methods if multiplying by a fraction is necessary (Hart, 1981; Luke, 1990); and (c) even when multiplication strategies are part of a child's problem solving repertoire, there is a tendency to revert to more simplistic methods when problem situations are unfamiliar or numerically complex (Tourniaire, 1983).

Some have suggested that the reason for this undeveloped understanding of multiplicative situations and lack of facility with written computation may be a result of the pedagogical constraints of traditional curricula. Standard textbooks have focused almost exclusively on the repeated addition model of multiplication and then rely on continual practice of the memorized standard algorithm. The standard algorithms for multiplication and division often lead to a fragmentation of numbers into digits, which in turn presents a major obstacle to the development of the kind of reasoning about numbers necessary for more advanced multiplicative thinking (Fischbein, et al., 1985; Luke, 1988).

Proponents of a constructivist approach to teaching and learning argue that students must develop their own strategies for solving mathematical problems. Kamii (1994) found that children who developed their own procedures for solving computational problems performed substantially better on than children who memorized the standard algorithm for doing those types of problems. Carpenter, et al. (1998) found in a longitudinal 3-year study that students who initially used invented procedures for addition and subtraction developed place value understanding before students who primarily used standard algorithms. In addition, these students were also more successful in solving extension problems than students in the standard algorithms group.

These results suggest the value of encouraging children to invent procedures for addition and subtraction and warrant an investigation of the operations of multiplication and division. This study analyzed the kinds of numerical reasoning children displayed in multiplicative situations and how this reasoning transferred to more complex proportion problems.

Methodology

This study occurred over a 6 month period and involved 137 fourth grade students from a suburban school district. Six intact classrooms from three schools participated. A pretest-posttest was administered to all students in the study followed by a one-on-one post-interview with a representative sample of students. Schools were chosen to represent a variety of instructional approaches to teaching multiplication and division. This was done not to compare instructional programs per se, but to ensure a range of approaches to the multiplicative tasks from the students. One group of 2 classrooms (Reasoning Procedures group) was encouraged to develop written computation procedures based on their mental computation strategies. These procedures will be referred to as *reasoning procedures* rather than the more commonly used term, *invented procedures*, to indicate the requisite thinking involved in using these procedures. The second group of 2 classrooms (Conventional Procedures group) was taught to memorize a variety of conventional procedures, such as partial products and the lattice method. The third group of 2 classrooms (Standard Algorithms group) was taught to memorize the standard algorithms for multiplication and division.

The pretest was administered to all students by classroom teachers mid-year, while the posttest was administered at the end of the year. It consisted of 24 questions organized into the following three sections: computation (written and mental), word problems, and conceptual understanding. The computation section included standard multiplication and division problems of varying degrees of difficulty. The word problem section contained standard word problems representing equal groups and area interpretations of multiplication and measurement and partitive interpretations for division. The conceptual understanding section included items requiring explanations of the meaning of multiplication and division, second solutions of problems, and explanations of how to use a known multiplication fact to figure out another. Strategies used on a selection of items from the posttest were coded and categorized. Explanations of the meaning of multiplication and division from the posttest were also coded and categorized.

A sample of students (3 from each classroom) that were closest to the 25th, 50th and 75th percentile on the posttest were interviewed individually by the researchers. The interview tasks assessed students' ability to: (1) reason multiplicatively by using known multiplication facts to figure out another; and (2) transfer their reasoning to solve a variety of word problems requiring more advanced multiplicative reasoning. The word problems were taken from published research studies (Clark & Kamii, 1996; Lamon, 1989; Scarano & Confrey, 1996) on proportional reasoning. The researcher read each task to the child and, as the child was working on the task, if the task appeared to be misinterpreted, the question was read again. Students were prompted to explain their thinking during each step of their solution process. All interviews were tape recorded and transcribed to analyze solution strategies. Strategies used on the interview tasks were categorized as productive or unproductive. Productive strategies lead to correct solutions, assuming no computational errors. They varied in sophistication from inefficient, intuitive methods to those that were refined. Unproductive strategies, on the other hand, were incomplete or erroneous methods.

Results and Discussion

The Reasoning Procedures group had the highest mean percent correct on the posttest, followed by the Conventional Procedures group and the Standard Algorithms group. The means of the scores from the pretest were also used to partition the students into low, middle and high incoming quantitative background (IQB) groups and no significant differences existed among any of the groups on the pretest. When analyzing posttest scores, no significant differences were found among groups on the computation and word problems subsections. However, significant differences ($p < .001$) on the posttest occurred on the conceptual subsection for all three IQB groups. This section contained tasks related to the most common misconceptions about multiplication. On one item on the posttest requiring use of a known fact, 74% of the students in the Reasoning Procedures group provided an appropriate answer, while 64% of the students in Conventional Procedures group and only 45% of the students in the Standard Algorithms group were able to do so.

Two items that showed large differences among groups were those that required an explanation of what 6 x 7 and 28 ∏ 4 mean. Repeated addition was the predominant interpretation for multiplication for all groups. Almost a third of the Standard Algorithms group provided no meaningful explanation. Rather, they either provided no explanation or described 6 x 7 as "6 times 7," which did not indicate whether or not the meaning of the operation was understood. The Conventional Procedures and Standard Algorithms groups also struggled with interpreting division. Nearly one half of all students in both of those groups could not describe division adequately. Less than a third of the students in those groups described division as the inverse of multiplication, whereas almost half of the Reasoning Procedures group did so. The Reasoning Procedures group provided a measurement interpretation of division 16% in comparison to 8% for the Conventional Procedures group and 3% for the Standard

Algorithms group. The partitive interpretation was used by all the groups at close to the same frequency.

There were differences between the multiplication and division strategies used by each of the groups. The predominant method for the Standard Algorithms group was to use conventional procedures. In addition, the Standard Algorithms group provided no work or meaningless responses 21% of the time for multiplication and 43% of the time for division. The Conventional Procedures group also relied heavily on conventional procedures, but displayed some flexibility in using other strategies, especially for division problems. However, they too provided no work or meaningless explanations at a high frequency, especially for division. When asked to multiply a second way, over half of the Standard Algorithms group and 40% of the Conventional Procedures group did not respond. It appears that, if the standard algorithm could not be recalled, then students in the Conventional Procedures group and the Standard Algorithms groups were more likely to give up than those in the Reasoning Procedures group. The Reasoning Procedures group, on the other hand, was more varied in their usage of strategies, and predominantly relied on procedures that displayed their conceptual understanding of problems.

On the interview, the three most difficult proportional reasoning items were identified based on combined scores. The Reasoning Procedures group performed the best on all three of the most difficult items, followed by the Conventional Procedures group and the Standard Algorithms group. The most difficult item asked students to think about how many miles a person rode her bike in 4 hours if she rode at a constant speed during her trip and road 20 miles in 3 hours. A correct solution was provided by a third of the students in the Reasoning Procedures group, whereas not one student was able to provide a correct solution in the Conventional Procedures or Standard Algorithms groups.

Conclusion

We began this study to investigate the different ways students think about multiplicative and proportional situations. The evidence suggests that students who are encouraged to develop written procedures for multiplication and division based on mental computation strategies had a qualitatively different way to think about those operations, to make sense of what they mean and how they are related, and were better able to transfer that understanding to proportional reasoning tasks. In addition, the fact that students from every level of incoming quantitative background in the Reasoning Procedures group had significantly more conceptual understanding than the comparable Conventional Procedures and Standard Algorithms groups suggests that learning reasoning procedures benefits all students.

While the focus of this study was not on instructional approaches and no systematic analysis of instruction was undertaken, the patterns of strategy usage in each of the groups suggest adherence to the different instructional approaches. Therefore, the findings suggest the benefits of instruction that encourages children to develop reasoning procedures based upon well-understood mental computation strategies. Viewing written computation as a paper record or representation of mental computation is a novel way to think about paper-and-pencil computation. It ensures that students will necessarily understand each and every step of the written procedure they are using because they are mentally carrying out each step first. And this has positive and far-reaching effects on their ability to think flexibly about numbers in a variety of situations and to be more successful in making sense of the mathematics they are doing.

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FREE CHAIN: A GEOMETRICAL REPRESENTATION OF THE ADDITIVE OPERATORS ON THE "ONE HUNDRED CHART"

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Summary: This paper is framed within the line of investigation of Numerical Thinking, and more concretely in connection with systems of representation. We try to construct and formalize a geometrical representation of an arithmetic concept, as is the additive operator, within the well-known numerical table: the one hundred chart. This representation constitutes a new visualization of the additive operator, and it permits the study of such an operator from a figurative and geometrical point of view.

Representations are worked with assiduity in the Numerical Thinking field. The works of Hiebert and Carpenter (1992) with regard to the notion of representation are significant, as well as the semiotic focus contributed by Duval (1993). The role of the representations in the concept of function is dealt with the studies of Janvier in 1978 producing sophisticated materials in the *Shell Center of the University of Nottingham* (1986) for the teaching of functions based on graphical representations. Researchers like Behr, Lesh, Post and Silver (1983) study and analyze different representations of the rational number, and Goldin, Janvier and Vergnaud coordinate the *Working Group on Representations* within the *International Group for the Psychology of Mathematics Education*, until 1995. The notion of representation and the mathematical systems of signs are being studied in the *Department of Mathematical Educational of the Center of Investigation and Advanced Studies of the IPN* in Mexico by researchers like Hitt (1998), Filloy and Rojano (1993). In the group of Numerical Thinking of the Department of Didactics of the Mathematics of the University of Granada (Spain) some works about numerical representations and punctual configurations (Castro, 1994) and the introduction of the real numbers (Romero, 1995) have been carried out.

1. The one hundred chart or Table-100

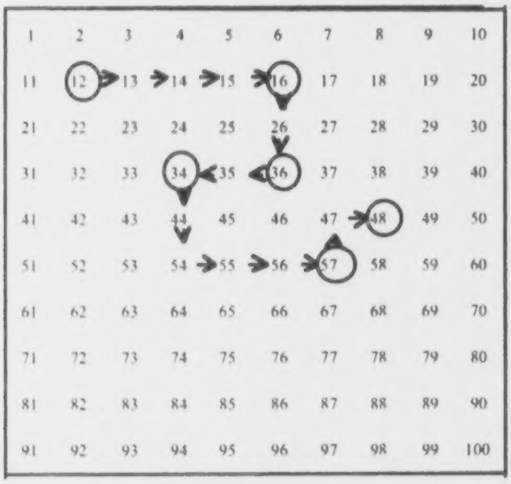
A numerical table with the numbers 1 to 100 arranged in rows and columns, (figure 1), is a structured system of the representation of the set of the first hundred natural numbers. Diverse authors like Litwiller and Duncan (1980, 1986), Thornton et al. (1985), Ajose (1991) have studied this board.

The numbers on each row (respectively, column) are in arithmetic progression and they are related additively by means of the addition or

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1: Table-100 (T_{100})

subtraction of units (respectively, tens). These additive characteristics allow us to consider simple operations on the table with a different focus from the conventional algorithmical one. So, the addition and subtraction of natural numbers is visualized by means of movements on the table; displacing k cells to the right/left from a determined number is equal to adding/subtracting k units to/from that number, and displacing k cells down/up from a number is equal to adding/subtracting k tens to/from such a number. For example, if 12 is the starting point, and we want to successively carry out the operations "adding 4, adding 20, subtracting 2, adding 23 and subtracting 9," whose decimal expression could be +4 +20 -2 +23 -9, this process can be visualized by means of the following displacements starting in 12: "right 4; down 2; left 2; down 2 and right 3; up 1 and right 1," producing the corresponding path in the table (see figure 2).



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 2: Path on the T_{100}

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1.1. Table-100

Considering the numerical line as a usual representation of the natural numbers (Figure 3), we can contemplate ten segments with ten numbers, starting from 1, spaced equally and placed parallel to one another, by way of a grid or a 10x10 geoboard. With this graphic organization the lineal representation is enlarged to two dimensions of the plane and new geometric elements are added to the study of numbers.

Figure 3: Numerical line

The formal characterization of T_{100} takes into account:

a) The ordered set C of the hundred first natural numbers C={ 1, 2, 3, 4., 99, 100}.

b) A subset discreet G of the Euclidean plane: the 10x10 geoboard (Figure 4).

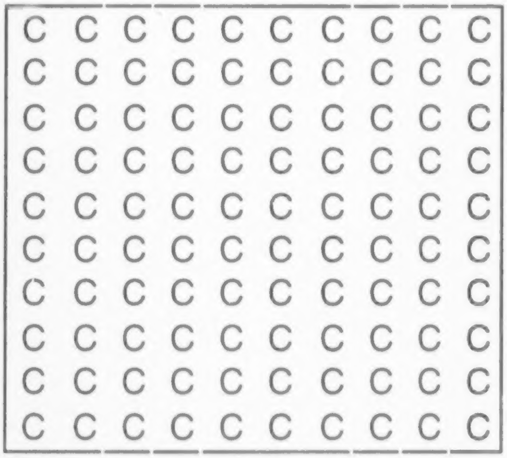


Figure 4: Geoboard 10x10

c) The grid, or square reticle (Figure 5).

Each point of the geoboard G is identified with a number of the set C located in the center of each cell of the grid. A characterization of T_{100} is established so. We could consider the elements of T_{100} both as a points and numbers. This format helps to visualize the transformations in the board by means of *paths* formed by adjoining cells (Figure 6).

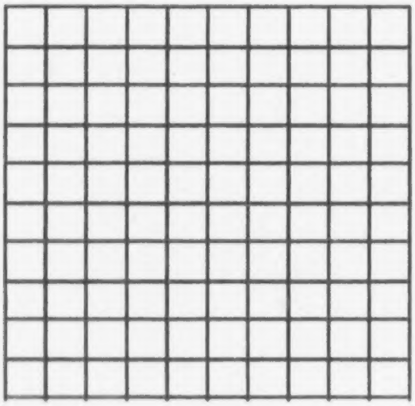


Figure 5: Square reticle

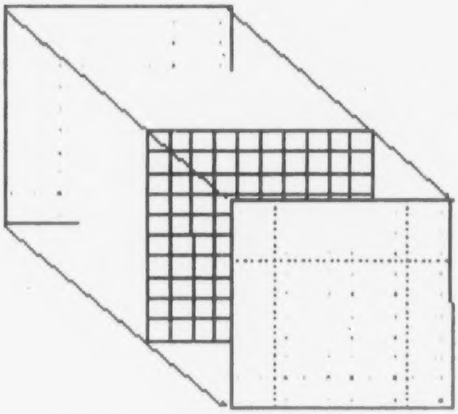


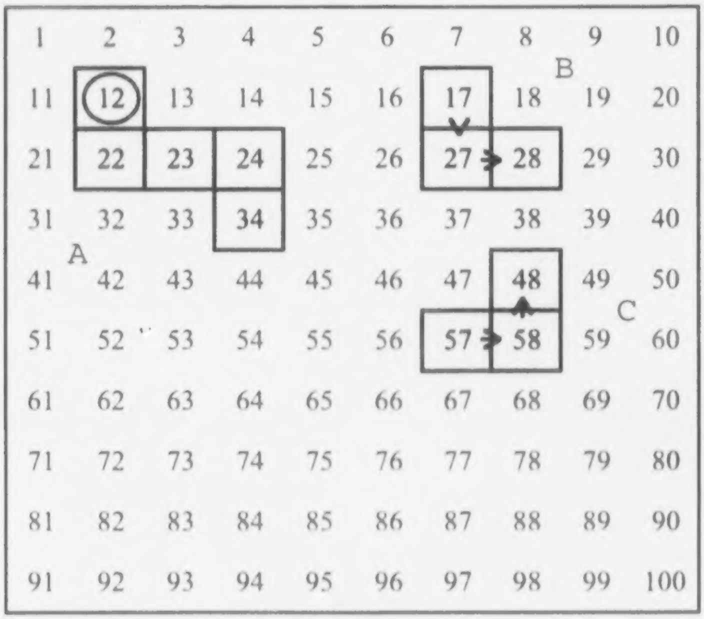
Figure 6: Table-100 with its components

2. Additive operations in T_{100}

The double condition of each element of T_{100} as a point and as a number permits the consideration of the operations and relationships between these elements at least with two different interpretations: arithmetical and geometrical. An additive operation in T_{100} can be interpreted as an arithmetical algorithm or as a geometrical transformation. On one hand we consider the addition/subtraction of two natural numbers with the usual arithmetical rules and, on the other hand we consider the displacements to the right/left with those which identify the addition /subtraction of units, and down/up for the addition/subtraction of tens. The displacements on the table highlight the usual meaning of the addition and subtraction of operators (Vergnaud 1983; Fuson 1992) since they show a state or initial cell (first addend or minuend), a state or final cell (result or final state), and a transformation or path through the cells (second addend or subtrahend).

2.1 Fixed chains

Starting from the cell corresponding to an initial number (origin cell) it is possible to move until the all corresponding to another number (final cell) following several paths by considering the geometric interpretation assigned to the addition/subtraction as a displacement within the board (see figure 2). Each path expresses a sequence of sums or subtractions that leads from the initial number to the result. The vertical and horizontal cells of each part of the path indicate, respectively, the tens and units of the successive additions or subtractions. We denominate each of these paths on T_{100} described as a concatenation of cells with a common side as *fixed chain*, in the way of an orientated polimino (Figure 7).



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 7: Fixed chains on T_{100}

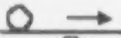
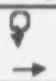
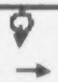
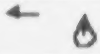
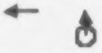
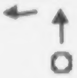
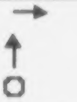
2.2 Arithmetic expressions of the paths. Reduced expression and fixed simple chain.

The path of figure 2 corresponds to a fixed chain, that is arithmetically interpreted as the successive application, starting from cell 12, of additive operations such as (+4), (+20), (-2), (+23), (-9). If we consider only the horizontal and vertical part of the chain, we conserve the order of the displacements, we assigned the sign (+) to advance and (-) to backward movements and we note the vertical parts by means of ($^{\text{superscript}}$) and the horizontal parts as ($_{\text{subscript}}$), such a chain is described by $4_+ 2^+ 2_- 2^+ 3_+ 1^- 1_+$ and we call it *arithmetic expression* of the path. These expressions associated to the fixed chains are reduced to an orderly pair; it is enough to add algebraically, on one hand, the components of the units and, on the other hand, the components of the tens. The simplified expression of the chain $4_+ 2^+ 2_- 2^+ 3_+ 1^- 1_+$ is $3^+ 6_+$, that corresponds to "adding 3 tens and to adding 6 units."

Given two different fixed chains, if their origin and final cells match up respectively, both chains have the same reduced arithmetic expression. Therefore, the correspondence between the fixed chains and the reduced arithmetic expressions is not bijective, since there are different fixed chains with the same reduced arithmetic expressions; this is the case of the chains

$2^+3_+1^-$ and $1^+1_+1^+2_+1^-$; the pair 1^+3_+ corresponds to both chains. The reduced arithmetic expressions correspond to a sort of very simple fixed chains that we call *simple fixed chains*, since they are made up at the most by only one vertical part and only one horizontal part, and in this order.

Table 1

Operator in decimal form	Reduced arithmetic expression	Fixed simple chain	Minimal fixed simple chain (canonical representative)	Arithmetic expression of canonical representative
[+9]	$0\,9_+$			$C(1^+1_-)$
[+21]	2^+1_+			$C(2^+1_+)$
[-14]	1^-4_-			$C(1^-4_-)$
[-28]	2^-8_-			$C(3^-2_+)$

2.3 Polynomial expression of a fixed chain.

Given the initial cell and a fixed chain, the base ten expression of the addend to which the chain is reduced is the *polynomial expression* of the chain. For whatever fixed chain whose reduced arithmetic expression is $a^\pm b_\pm$ (a and b take values between 0 and 9) we write briefly their polynomial expression as $10(\pm a) \pm b$.

2.4 Equivalence of fixed chains.

Only one reduced arithmetic expression corresponds to each fixed chain, and therefore only one additive operator. However, for a given additive operator diverse fixed chains that represent it exist, and therefore the correspondence between fixed chains and additive operators on T_{100} is not bijective (Figure 8). We introduce an equivalence relation $\neg$ in the set C of the fixed chains on T_{100} as follows: "two fixed chains are equivalent if and only if they correspond to the same additive".

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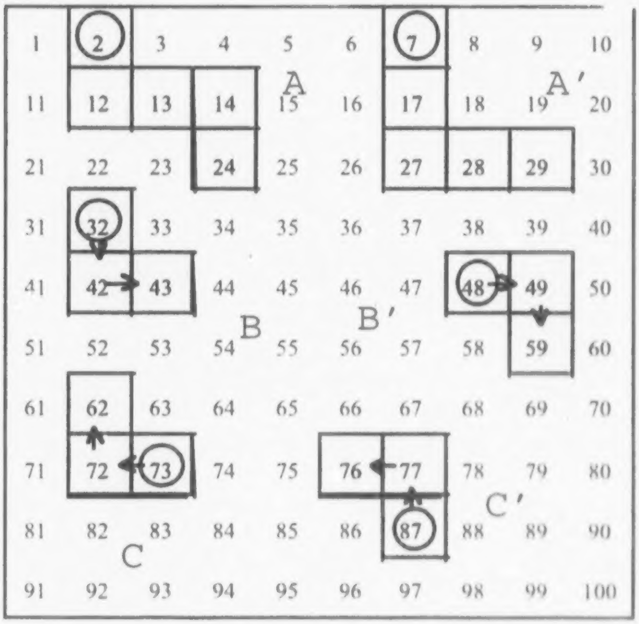

Figure 8: Pairs of equivalent chains

2.5 Free chains and canonical representatives.

Each class of equivalence obtained due to the previous relation R we call *free chain*; all the fixed chains of a same class represent the same additive operator, and reciprocally.

This identification permits the definition of the set of the additive operators or free chains on T_{100} as the quotient set W = C / ¬_

The canonical representatives of the free chains or *minimal simple fixed chains*, are those simple chains with the least number of cells (Figure 9). Table 1 shows an example of each one of the four possible types of canonical representatives.

Formally an additive operator on the set Z of the whole numbers is an application:

$f_k: Z \rightarrow\_Z; \quad x \rightarrow x + k,$ (k is a fixed element of Z).

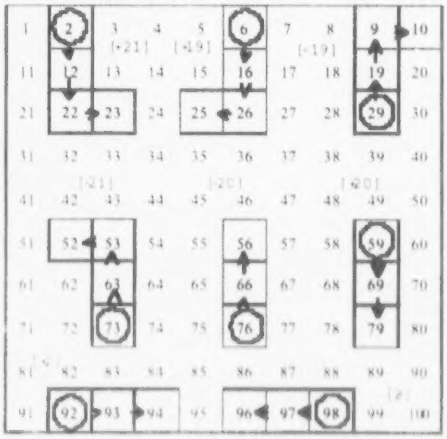

Figure 9: Canonical representatives of the additive operators [+21], [-21], [+19], [-19], [+20], [-20], [+2], [-2]

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The free chains as a geometrical representation of the additive operators on T_{100} allows the study of them using also some isometries on the plane.

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THE CONVENTIONAL ADDITION ALGORITHM USED AS A WORKING TOOL AND NUMERICAL DIAGRAMS USED AS CONCEPTUALIZING TOYS

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Abstract. The paper analyzes the mathematical activity of two fourth graders when they solved two arithmetical tasks posed to the whole class at the beginning and at the midpoint of the school year (the fourth grade class participated in a yearlong teaching experiment). In solving the first task the children proceeded to use the conventional addition algorithm as an instrumental tool. Though, when solving the second task, the children introduced their own diagrams as conceptualizing toys to operate with numbers. The children's solutions to these two tasks indicate a sharp contrast in the nature of their mathematical activity. On the one hand the working activity was guided by the goals pre-scribed by the algorithm and on the other the playful activity was guided by self-scribed goals.

Theoretical Rationale

This paper examines the mathematical activity of two fourth graders in terms of the notions of *work* and *play*. Anthropologists have characterized these two activities according to whether the ends and means are chosen by others or whether they are self-generated. Anderson (1998) considers work to be an activity that is scripted by someone else in advance (pre-scripted). In this case, tools (conceptual or physical) are used in a pre-determined fashion to achieve particular goals. In contrast, she considers play to be a free activity in which toys/tools (conceptual or physical) are used in an open-ended fashion to achieve emergent self-generated goals.

Conventional algorithms in arithmetic are efficient and pre-scripted ways of functioning with numbers. When a child performs an arithmetical operation using a conventional algorithm in an instrumental manner in Skemp's (1987) sense, this activity becomes a *working activity* in Anderson's sense. This is because the child follows, without much thought, the rule pre-scripted in each step of the algorithm. In contrast, when a child performs a numerical operation by means of his/her own conceptual devices, this activity becomes a *playful activity* in Anderson's sense. This is because the child generates self-scripted goals and the strategies to achieve them.

When children are allowed the freedom to engage in conceptual play, they not only introduce new vehicles, such as diagrams, for expressing numerical reasoning but they also generate new ways of speaking mathematically. More often than not, how children operate with numbers resembles the mathematical underpinnings of the conventional algorithms (Kamii and Livingston, 1994). Teachers, who honor both the ingenuity of children and the contributions of different cultures when the algorithms were established, are able to balance their guidance between the teaching of these algorithms and the encouragement to construct personal reinventions (Wilder, 1986; Freudenthal, 1991).

Charles Sanders Peirce (1906), both mathematician and semiotician, considers reasoning to be a deliberate action that finds its way out of the mind through physical forms that resemble, in one way or another, the thoughts of the individual. In this regard, he says, "Reasoning has to make its conclusion manifest. Therefore, it must be chiefly concerned with forms, which are the chief objects of rational insight. Accordingly, icons are specially requisite for reasoning. A diagram is mainly an icon, and an icon of intelligible relations" (Hoopes, 1991, p. 252). Diagrams, from Peirce's theoretical perspective, are the products of mental imagery, subordinated to mental transformations and reinterpretations, and serving as vehicles for conveying the meanings ascribed to them by the individual. Then it is not surprising to find that children generate numerical diagrams to represent self-scripted numerical strategies.

The purpose of this paper is to analyze the contrasting nature of the mathematical activity of two fourth graders as they solved two arithmetical tasks. The first task was solved using the conventional addition algorithm by following the pre-scripted steps (work) they knew. The second task was solved using numerical diagrams generated by the students as they scripted their own goals and strategies (play).

Methodology

Teaching experiment. The teaching experiment consisted of daily classroom teaching episodes. These episodes were characterized by the teacher-student and student-student dialogical interactions mediated by the children's numerical diagram. In each teaching episode, the teacher tried to interpret the children's mathematical activity, taking into account their perspectives so that dialogical interaction could be sustained. The children were also interviewed, once a week, in small groups.

Subjects. The fourth grade class that participated in the teaching experiment was characterized, by the school administrators, as low achievers with a long history of low performance in the state standardized tests. The two children analyzed in this paper were typical students in the class of sixteen fourth graders, not only in terms of their initial mathematical level but also the conceptual achievements throughout the school year.

Data collection. Along with maintaining records of the mathematical activity and the changes in numerical understanding, the daily lessons were videotaped and field notes of the children's solutions were kept. Task pages and scrap papers were also collected.

Expectation of the mathematical activity. An explicit agreement among the teacher and the children was established at the beginning of the school year. The children were given the responsibility to listen carefully to the solution of others and the obligation to express their agreement or disagreement with justifications. This agreement was maintained throughout the school year. This strategy was modeled after other socioconstructivist classroom teaching experiments (Cobb, Yakel, and Wood, 1992).

Instructional routine. Some arithmetical tasks were posed verbally and the children were expected to solve them mentally. Other tasks were presented on paper and the children were expected to solve them individually before interacting with other children. After allowing enough time for individual work, the teacher went around collecting answers that children whispered to her. While collecting the answers, the teacher encouraged the children to validate their strategies. Then the teacher proceeded to whole-group discussion in which children explained their answers and justified their solutions.

Instructional tasks. The instructional tasks for the teaching experiment were prepared beforehand. Nonetheless, they were subject to changes according to the mathematical needs of the children. Classroom numerical arguments often led to the generation of new tasks that facilitated conceptual shifts. In general, it could be said that the generation of instructional tasks and the teaching activity co-evolved in a synergistic manner.

Analysis

The fourth graders, at the beginning of the school year, were given the task of adding mentally two two-digit numbers. The numbers were chosen in such a way that the sums of the one-units and ten-units digits were larger than ten to inquire about the children's understanding of place-value.

Teacher: Add 85 and 37 in your head.

Randy: That's hard without paper.

Richard: Uh-huh!

Teacher: Try it.

Randy: One twenty-two.

Teacher: How did you get the answer?

Randy: In my mind I put 85 on top of 37. 5 and 7 is 12. Put down the 2 and carry the 1. 1 and 8 is 9; 9 and 5 is 12.

Teacher: When you carry the 1, you carry 1 what?

Randy: You carry 1.

Teacher: OK. Richard, how did you do it?

Richard: I put 37 on top of 85. 7 and 5 is...8-9-10-11-12 [showing five fingers]. Write down the 2 and carry the 1. 1 and 3 is 4. 4 and 8 is...5-6-7-8-9-10-11-12 [showing eight fingers]. One two two.

Teacher: When you added 1, and 3, and 8 you got 12. This 12 is 12 what.

Richard: 12 ones.

The above dialogue indicates that, even in the absence of paper and pencil, the only way Randy and Richard knew how to add two numbers was by using the conventional addition algorithm. Their explanations of how they added these numbers indicate that: (a) Richard operated with units of one when adding any two digits, while it was not clear how Randy added the digits; (b) neither Richard nor Randy had a clear concept of place-value; and (c) in the context of the addition algorithm, neither Richard nor Randy could differentiate units of one from units of ten and units of a hundred.

Close observation of the counting activity of these children was done in small group interviews. These interviews indicated that they understood "counting" as a memorized pattern of number words. It seems that this was the main reason for which the children had not developed more sophisticated counting patterns. Counting patterns are base on different types of composite units (Stefee, 1991). Richard, for example, understood five only as five ones rather than as a composite unit of one five. This lack of composite units led him to an adding strategy based on counting by ones.

Richard was not the only child in the class lacking conceptualization of composite units and place-value. Hence it was clear that our first instructional priority should be guided toward helping the children develop their conceptualization of different units of ten and their understanding of the role of these units in the place-value notation. Money was used as an instructional tool and several money related tasks were generated using bills of denomination 1, 10, 100, and 1000 dollars. These mathematical tasks advanced the children's constructions of different units of ten and their understanding of the place-value notation as indicated by the following classroom event.

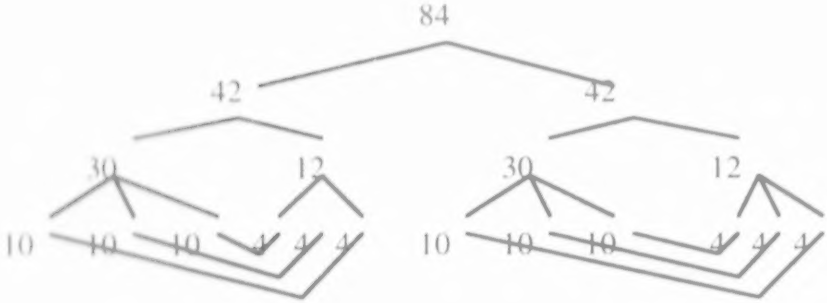
The children were asked in November (three months after the teaching experiment began) to represent $374 in different ways. In the whole group discussion, the class came up with the following representations: 3 hundreds, 7 tens, and 4 ones; 3 hundreds and 74 ones; 30 tens and 74 ones; 37 tens and 4 ones; 2 hundreds, 15 tens, and 24 ones; 1 hundred, 25 tens, and 24 ones; 1 hundred, 27 tens, and 4 ones. In addition, several children were willing to discuss the numerical pattern when they saw the units of ten increasing while the units of a hundred were decreasing.

With the construction of composite units children began to generate different strategies to operate with numbers. Among those strategies were the diagrams (conceptualizing toys) used by the children to represent numerical operations. The following task was presented to the children in February of the school year: "Mrs. Walgamuth wants to share $84 dollars among 6 students, how many dollars will each student get?" The children were allowed paper and pencil to solve the task. In the whole group discussion, Richard and Randy presented the following solutions to the class.

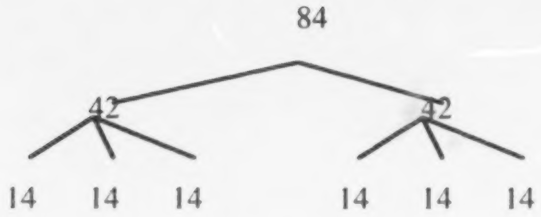
Richard: I split 84 in 42 and 42. Then I split 42 into 30 and 12. I know 30 is 3 tens, and 12 is 3 fours. Then I put together 10 and 4. 14 dollars each.

Teacher: Would you like to show us what you have on your paper?

Richard: [goes to the board and reproduces the diagram he had on paper]. 1-2-3-4-5-6 [counting the lower lines of the diagram]. 14 each.



Randy: I did it some other way. Look [he makes this diagram on the board].



> 84 is 42 and 42. I know that 14, 14, and 14 is 42. 1-2-3-4-5-6 [counting the lower lines of the diagram]. Each will get 14 dollars.

These diagrams indicate that Richard and Randy have started to conceptualize numbers in terms of units other than units of ten. They did not decompose 84 only into 8 tens and 4 ones as they have done in the past. Instead, they decomposed 84 into two 42's and six 14's. The diagrams clearly indicate that they self-scripted two goals. First, to have two equal amounts of money. Second, to decompose each of these two equal amounts of money into three equal amounts. This second goal was accomplish by Randy in only one step while Richard did it in two steps. The number of steps to achieve the second goal was linked to the strategies that each child generated to decomposed 42.

The diagrams served the children as conceptualizing toys, which mediated the decomposition of the numbers into smaller units. The branches at the first level of the diagram seem to represent the first goal that Randy and Richard intended to achieve. Similarly, the branches at the second and third level of Richard's diagram, and at the second level of Randy's diagram seem to represent their intended second goal. These two goals were generated independently and achieved in different ways. This mathematical activity of the children was a self-scripted activity, in other words, a playful activity, in contrast to the working activity these children displayed when adding 85 and 37 using the pre-scripted steps of the addition algorithm.

Conclusion

The analysis indicates that Richard and Randy's progressive conceptualization of composite units shifted the nature of their mathematical activity. They went from using the addition algorithm as an instrumental tool (working activity) to using diagrams as conceptualizing toys (playful activity) to solve a task that could have been performed using the division algorithm. The playful activity of these two children indicates that there is pedagogical value in allowing freedom in the classroom because it may lead to the generation of self-scripted conceptualizing toys.

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STRATEGIES FOR THE ESTIMATION OF DISCREET QUANTITIES: A FIRST STUDY ON ABILITIES

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Abstract: This work is summed up in the study of the strategies and components that the children use from 6 to 14 years when they estimate discreet quantities (numerosity). Also, and in the mark of a development theory, studies how these strategies evolve with the age.

Introduction

This work is framed within the researches on Measurement Estimation and more specifically it is focused on the study of estimation of discreet quantities. We introduce a part of a research related to the estimation of a number of elements of a discreet quantity of items. This paper has been done by the Research Group on **Numerical Thinking** of the Department of Didactics of Mathematics from the University of Granada (Spain). Such a Research Group works on the different processes of teaching, learning and communication of numerical concepts within the Spanish Education System and social sectors; moreover, it studies the different cognitive and cultural processes through which the human beings, by making use of different numerical structures, can give and share the subsequent meanings obtained (Rico, 1996).

Aims and background

The general aim of our research is to describe and characterize the strategies for estimating discreet quantities among children aged 6-14. Within the framework of a Cognitive Development theory, we intend to supply new information to interpret the evolution of the estimation abilities of the pupils aged 6 to 14, by means of several indicators obtained by doing tasks designed for that purpose. From this point of view, we are interested in identifying abilities that characterize the strategies such pupils use to solve tasks on discreet quantities estimation. Within the backgroun on discret quantities estimation, we can mention the following research works:

- Siegel, Goldsmit and Madson's work (1982). They analyze those strategies used by children from the second to the eighth grade of Primary Education and by adults.

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- Crites research (1989) establishes a database of strategies for estimation of numerosity used by pupils attending the third, fith and seventh grades.
- Markovits and Mershkovitz employ in their research (1992) different groups of objects -between 10 and 40- with different forms and a limited display time. They identify four strategies used by children attending the third grade of primary education.

The theoretical framework on cognitive development we have considered is focused on Case's development theory. Case (1989) states that intellectual development is carried out through four stages, each one related to a basic intellectual operation. These stages are called: **Sensorymotor, Relational, Dimensional and Vectorial.** An operation, in whatever stage, is the result of the assembly of those components consolidated in the previous stage. Each stage is made up of four substages: Substage 0 or **Operational Consolidation**, substage 1 or **Unifocal Coordination** and substage 3 or **Elaborated Coordination.**

For the analysis of strategies used to conduct the assessment of discreet quantities tasks, we consider the following congnitive abilities:

Counting and using the cardinality rule.

This strategy is necessarily related to the **cardinality rule or principle.** Only children from the first and the second dimensional substage may have some problems regarding the acquisition of this ability.

Mental computation.

Given that children have no time to count each quantity of our experience, most of the time they are obliged to carry out mental computation. Case and Sowder (1990) also describe the development of mental computation which can be employed in problems like those of our research.

Decomposition and recomposition of a quantity.

In most estimations of quantities, the strategy is based on decomposing the quantity into parts, determining the value of its components and uniting the total quantity.

To Carter (1986), this technique depends on the development of conservation and on the **Part-Whole Scheme**. Moreover, **Arithmetic Recomposition** of a quantity is a procedural strategy.

The Part-Whole Scheme

A child has acquired the Part-Whole scheme if he considers that a "quantity (the whole) can be divided (into parts), in the way that the

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combination of its parts does not exceed nor be less than the whole " (Resnic, 1983 p.114). The assimilation of the Part-Whole Scheme is from 7 or 8 years on (Piaget and Szeminska, 1964).

Arithmetic Recomposition

When children have determined the number of elements within a part of the quantity, they have to recompose the total. Recomposing is based, then, on establishing a relation between the part and the whole, and then apply that relation to obtain the number corresponding to the total quantity. This relation can simply be a comparison of the kind "bigger than" and according to that relation, give a bigger number; but it can also be the establishment of a rate between the part and the whole. Anghileri (1989) and Kouba (1989) establish different levels of development for simple rate situations.

Regarding the development of discreet quantities estimation, under the conditions of our experience and within the framework of Case's theory, our hypothesis is that during the estimation of discreet quantities, besides the already mentioned components, the number and size dimensions also take part initially. From a certain substage on, when number and size have been integrated in a mental numerical line, other elements take part in the process such as the number given to the part, the number determined for the whole, the size of the part and the size of the whole.

Methodology

With this work we intend to describe and characterize the estimation of discreet quantities made by children aged 6-14; with that purpose, we have designed sixteen estimation problems, which are displayed on a computer screen, and have the following characteristics:

a) In each problem and during a limited time -between 6 and 8 seconds- children are shown a number of objects in order to estimate the whole.

b) The objects of each quantity are small circles, with the same diameter, which are organized according to different kinds of lines.

c) The shown quantities range between 20 and 70 circles. Quantities with four different cardinals were proposed -the first one around 25, the second one around 40, the third one around 55 and the fourth one around 70 circles. This variable is called **size of quantity**, and takes the four mentioned values.

d) Quantities are also presented under four different lines, which correspond to two noticeable characteristics. The first one is topological and is based on the opened-closed dichotomy; the second one is geometrical and is based on the "composed-non-composed" dichotomy when the parts on a line are considered as different. The four kinds of lines chosen to organize the quantity of circles are: Straight line, Square, Sinusoid and Circle. This variable is called **Structure of quantity**, which also takes four values, as shown in Figure 1.

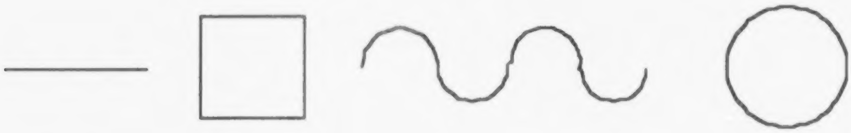

Figure 1: Lineal Structures of Quantities

When mixing the four values of the size variable with the four values of the structure variable, we obtain 16 different problems, which are posed to the students.

Each child has a limited display time which is not enough to count one by one all the circles of the proposed quantity, thus being obliged to make an estimation.

In four of the sixteen problems solved by children, they will be asked to explain the procedure carried out in the estimation of the quantity.

Sampling

Our experience, this took place in a State School. This school consists of eight levels, comprising the Compulsory Basic Education (from 6 up 14 years old), having 192 students with an average of 24 students per grade. We selected for our study 12 students from every grade. Each student must explain his/her criteria for assessing more than 4 times, because solving and explaining all the problems could be tiring for them. Moreover, in order to minimize a likely learning effect, the explanations given by the children about the followed procedure, referred to the last four problems of the sixteen comprising the test.

Design

Our research is a cross-sectional study and it compares groups of different ages in a precise moment. This study differs from lineal design research in which the same subjects are studied in different age levels (Baltes, Reese and Nesselroade, 1981).

Results

The answers given by chikdren when they were asked: *Can you explain how you have found this result, in other words, how have you calculated it*? Were recorded on audio tape, and then a transcription was made to analyze such answers. 384 explanations were found; those explanations with a similar content, from a total of 384, were eliminated; in this way, in this way, based on the children's explanations we obtained 68 staments corresponding to different procedures.

Within a second phase and following Goetz and LeCompte (1988), the common features of the above pieces of information were determined in order to be characterized, classified and simplified. We used the following structural aspects, regarding estimation, as classification criteria:

- Overall or partial consideration of the quantity to be estimated.
- Quantification technique used: counting, direct assignation, use of numerical sequency.
- Use or not of Size as quantification criteria.
- Use or not of any kind of decomposition.
- Way of recomposing the decomposed quantity: repetition, addition, duplication and multiplication.

Taking these criteria into account, we obtained twenty different kinds of procedures described by children: each kind of procedure was called, then, a strategy.

The procedure classification through the twenty strategies theoretically obtained gave rise to a new reduction. Some strategies had no related statement, while others had to be modified in order to fit to the procedures stated by children. Finally, the strategies were reduced to twelve, through the procedure of theoretical definition and empirical control.

These strategies were gathered in four general categories:

I) *Non-justified categories.*

Strategy 1: **Non-justified (NJ)**

Those procedures which children were not able to explain or just said "I do not know" were included here.

II) *Overall assessment without referent*

These characteristics were defined by the fact that children considered the quantity as a whole, i.e., they did not consider a part of it to assess the total. We had the different strategies taking into account if children used a numerical sequency or not, if they counted mentally or over a real image, and if they bore in mind size as an assessment criterion.

Strategy 2: **Say the numerical sequency without considering quantity (GA)**

This procedure was focused on determining the numerical sequency without relating the numbers to the elements of the quantity and stopping at one without any criterion related to the quantity.

Strategy 3: **Say the numerical sequency according to size (GB)**

This strategy was based on saying the numerical sequency stopping at a number that children related to numerical or spatial size.

Strategy 4: **Assign a number without considering the quantity (GC)**

Children explained a criterium which did not imply a delibered cardination procedure: "I have invented" "It came to my mind", "I have thought about it", etc.

Strategy 5: **Assign a number according to Size (GD)**

Children assigned a big or small number according to the numerical or spatial size of the quantity: "because it was very big", "because it is small", "because there are many", etc.

Strategy 6: **Count the real or mental quantity (GE)**

Children counted the number of circles while the figure was on the screen. If they did not have enough time, they went on counting a mental image.

III. ***Strategies which implied an overall assessment of the quantity through comparison with a referent .***

Strategy 7: **Assign a number by comparison (GF)**

In this case, children assigned a number to the quantity by comparison with another quantity they had seen in any of the previous problems.

IV. *Strategies which implied a partial assessment of the quantity.*

These strategies can be classified in two subgroups:

IV.1) *Without previous decomposition of the quantity:*

Strategy 8: **Count a part and estimate according to size (PA)**

Children counted while the image was on the screen and guessed the total without any explanation or just with simple criteria such as "because it is very big", "because it is small", etc.

Strategy 9: **Count a part, estimate the rest and sum (PB)**

Children counted while the image was on the screen, estimated the rest by comparison with the counted part and made a sum. For example, a child said "I have said 23 because I have counted 13 and I think there are 10 more"

Strategy 10: **Repeat a part on the total**

Children counted, estimated a quantity of elements, 5 for example, repited their length on the total and through partial additions (..(((5+5)+5)+5)+...)+5, they obtained the result.

IV. 2) *With a previous decomposition of the quantity*

In this case, procedures were more complex, and were made up in general of three components: definition of a part and its relation with the total, determination of the number of elements in that part, and recomposition of the total.

Strategy 11: **Determine half the quantity and duplicate (PE)**

Children counted half of the quantity or tried to count while the image was on the screen, determined the remaining quantity through comparisons like "more or less the same", "a few more", etc, and obtained the total by addition or by multiplication by two.

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Strategy 12: **Count a part, multiply or sum (PF)**

Children decomposed the quantity into three or more equal parts and recomposed the total by addition or by multiplication of the number obtained in the determination of the part by the number of parts.

The statistical analysis made shows the relation among the strategies described above and the substages stated by Case, such as can be seen in table 1.

Table 1.
Substages characterization

Substages	**Strategies**
Substage 1 (5-7 years)	GA, NJ
Substage 2 (7-9 years)	PA (GC, GD, NJ)
Substage 3 (9-11 years)	(GD, PC, PE)
Subsatge 4 (11-13 years)	PF, PE
Subsatge 5 (13-15 years)	PF, PE

(*) The strategies non significantly associated are in brackets

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THE NOTION OF QUANTITY AND THE INITIAL LEARNING OF THE ARITHMETIC

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This work forms part of a theoretical development of an investigation from the perspective of the genetic epistemology on the difference between quantity and number, the meaning that in the daily life is given to the word quantity, the notion of quantity and its evolution in relation with the count and number and the relation between these notions in the elementary of scholastic arithmetic. The reflection about of the didactic meaning of the principle of conservation of quantity and the epistemology positions of Piaget with respect to the number genesis, drives to investigate on the quantity and number. In the experimentation with teachers was found that daily expressions in which is used the word quantity allude to a subjective quantification not numeric of "quantities" in the one which the referring of comparison are of different nature what is quantified. It is said for example: what quantity of people there is in the classroom! because is much in relation with the size of this or because it is more than the people that used to be there daily. In different circumstances the same quantity of people is not considered. The appreciation is made taking into account the space, or the personal expectations, or what customarily is observed and therefore changes are made according to each individual and environment, something that contradicts the principle of conservation of the quantity.

In a historic analysis the emergence of symbols to represent quantities was related to the equivalence establishment and hierarchies between quantities.

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